Zeckendorf family identities generalized

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Abstract. In [WooZei09], Philip Matchett Wood and Doron Zeilberger have constructed identities for the Fibonacci numbers f_n of the form

$$\begin{aligned} 1f_n &= f_n \text{ for all } n \geq 1; \\ 2f_n &= f_{n-2} + f_{n+1} \text{ for all } n \geq 3; \\ 3f_n &= f_{n-2} + f_{n+2} \text{ for all } n \geq 3; \\ 4f_n &= f_{n-2} + f_n + f_{n+2} \text{ for all } n \geq 3; \\ &\text{etc.}; \\ kf_n &= \sum_{s \in S_k} f_{n+s} \text{ for all } n > \max\{-s \mid s \in S_k\}, \end{aligned}$$

where S_k is a fixed "lacunar" set of integers ("lacunar" means that no two elements of this set are consecutive integers) depending only on k. (The condition $n > \max\{-s \mid s \in S_k\}$ is only to make sure that all addends f_{n+s} are well-defined. If the Fibonacci sequence is properly continued to the negative, this condition drops out.)
In this note we prove a generalization of these identities: For any family $(a_1, a_2, \ldots, a_p)$ of integers, there exists one and only one finite lacunar set S of integers such that every n (high enough to make the Fibonacci numbers in the equation below well-defined) satisfies

$$f_{n+a_1} + f_{n+a_2} + \cdots + f_{n+a_p} = \sum_{s \in S} f_{n+s}.$$

The proof uses the Fibonacci-approximating properties of the golden ratio. It would be interesting to find a purely combinatorial proof.

This is a brief version of my note [Grinbe11]. For a long version, which gives more details in the proofs, see [Grinbe11].

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1. The main result

The purpose of this note is to establish a generalization of the so-called *Zeckendorf family identities* which were discussed in [WooZei09]. Before we can state it, we need a few definitions:

Definition 1.1. A subset S of $\mathbb{Z}$ is called *lacunar* if it satisfies $(s+1 \notin S$ for every $s \in S)$.

In other words, a subset S of $\mathbb{Z}$ is lacunar if and only if it contains no two consecutive integers.

Definition 1.2. The *Fibonacci sequence* $(f_1, f_2, f_3, \ldots)$ is a sequence of positive integers defined recursively by the initial values $f_1 = 1$ and $f_2 = 1$ and the recurrence relation $(f_n = f_{n-1} + f_{n-2}$ for all $n \in \mathbb{N}$ satisfying $n \geq 3)$.

(Here and in the following, $\mathbb{N}$ denotes the set $\{0, 1, 2, \ldots\}$.)

Remark 1.3. Many authors define the Fibonacci sequence slightly differently: They define it as a sequence $(f_0, f_1, f_2, \ldots)$ of nonnegative integers defined recursively by the initial values $f_0 = 0$ and $f_1 = 1$ and the recurrence relation $(f_n = f_{n-1} + f_{n-2}$ for all $n \in \mathbb{N}$ satisfying $n \geq 2)$. Thus, this sequence begins with a 0, unlike the Fibonacci sequence defined in our Definition 1.2. However, starting at its second term $f_1 = 1$, this sequence takes precisely the same values as the Fibonacci sequence defined in our Definition 1.2 (because both sequences satisfy $f_1 = 1$ and $f_2 = 1$, and from here on the recurrence relation ensures that their values remain equal). So it differs from the latter sequence only in the presence of one extra term $f_0 = 0$ at the front.

The Fibonacci sequence is one of the best known integer sequences from combinatorics. It has had conferences, books and a journal devoted to it. By way of example, let us only mention Vorobiev's book [Vorobi02], which is entirely concerned with Fibonacci numbers, and Benjamin's and Quinn's text [BenQui03] on bijective proofs, which includes many identities for Fibonacci numbers.

In [WooZei09], Wood and Zeilberger discuss bijective proofs of the so-called *Zeckendorf family identities*. These identities are a family of identities for Fibonacci numbers (one for each positive integer); the first seven of these identities are

$$\begin{aligned}
1f_n &= f_n \text{ for all } n \geq 1; \\
2f_n &= f_{n-2} + f_{n+1} \text{ for all } n \geq 3; \\
3f_n &= f_{n-2} + f_{n+2} \text{ for all } n \geq 3; \\
4f_n &= f_{n-2} + f_n + f_{n+2} \text{ for all } n \geq 3; \\
5f_n &= f_{n-4} + f_{n-1} + f_{n+3} \text{ for all } n \geq 5; \\
6f_n &= f_{n-4} + f_{n+1} + f_{n+3} \text{ for all } n \geq 5; \\
7f_n &= f_{n-4} + f_{n+4} \text{ for all } n \geq 5.
\end{aligned}$$

In general, for each positive integer k, the k-th Zeckendorf family identity expresses kf_n (for each sufficiently large integer n) as a sum of the form $\sum\limits_{s\in S} f_{n+s}$, where S is a finite lacunar subset of $\mathbb{Z}$. Of course, the subset S does not depend on n.

Our main theorem is the following:

Theorem 1.4 (generalized Zeckendorf family identities)**.** Let T be a finite set, and let a_t be an integer for every $t \in T$.

Then, there exists one and only one finite lacunar subset S of $\mathbb{Z}$ such that[1]

$$\left(\begin{array}{c} \sum\limits_{t\in T} f_{n+a_t} = \sum\limits_{s\in S} f_{n+s} \text{ for every } n \in \mathbb{Z} \text{ which} \\ \text{satisfies } n > \max\left(\{-a_t \mid t \in T\} \cup \{-s \mid s \in S\}\right) \end{array} \right).$$

Remark 1.5.

1. The *Zeckendorf family identities* from [WooZei09] are the result of applying Theorem 1.4 to the case when all a_t are $= 0$.
2. The condition $n > \max\left(\{-a_t \mid t \in T\} \cup \{-s \mid s \in S\}\right)$ in Theorem 1.4 serves only to ensure that the Fibonacci numbers f_{n+a_t} for all $t \in T$ and f_{n+s} for all $s \in S$ are well-defined. (If we would define the Fibonacci numbers f_n for integers $n \leq 0$ by extending the recurrence relation $f_n = f_{n-1} + f_{n-2}$ "to the left", then we could drop this condition.)

The proof we shall give for Theorem 1.4 is not combinatorial. It will use inequalities and the properties of the golden ratio; in a sense, its underlying ideas come from analysis (although it will not actually use any results from analysis).

2. Basics on the Fibonacci sequence

We begin with some lemmas and notations:

We denote by $\mathbb{N}$ the set $\{0, 1, 2, \ldots\}$. Also, we denote by $\mathbb{N}_2$ the set $\{2, 3, 4, \ldots\} = \mathbb{N} \setminus \{0, 1\}$.

Also, let $\phi = \dfrac{1+\sqrt{5}}{2}$. This number ϕ is the famous *golden ratio*. It satisfies $\phi \approx 1.618\ldots$ and $\phi^2 = \phi + 1$.

We recall some basic and well-known facts about the Fibonacci sequence:

Lemma 2.1. Let S be a finite lacunar subset of $\mathbb{N}_2$. Then, $\sum\limits_{t\in S} f_t < f_{\max S+1}$.

Proof. WLOG assume that the set S is nonempty (else, the lemma follows from our convention that $\max \varnothing = 0$). Write the set S in the form $\{s_1, s_2, \ldots, s_k\}$ with $s_1 <$

[1] Here and in the following, $\max \varnothing$ is understood to be 0.

$s_2 < \cdots < s_k$. Every $i \in \{1, 2, \ldots, k-1\}$ satisfies $s_i + 1 \leq s_{i+1} - 1$ (because the set S is lacunar, so s_{i+1} cannot be $s_i + 1$, whence $s_{i+1} > s_i + 1$ and thus $s_{i+1} - 1 \geq s_i + 1$), so that

$$f_{s_i+1} \leq f_{s_{i+1}-1} \tag{1}$$

(since the Fibonacci sequence $(f_1, f_2, f_3, \ldots)$ is weakly increasing). Thus,

$$\begin{aligned}
\sum_{t\in S} f_t &= \sum_{i=1}^{k} \underbrace{f_{s_i}}_{\substack{=f_{s_i+1}-f_{s_i-1} \\ (\text{since } f_{s_i+1}=f_{s_i}+f_{s_i-1})}} = \sum_{i=1}^{k} \left(f_{s_i+1} - f_{s_i-1}\right) = \underbrace{\sum_{i=1}^{k} f_{s_i+1}}_{=\sum_{i=1}^{k-1} f_{s_i+1}+f_{s_k+1}} - \underbrace{\sum_{i=1}^{k} f_{s_i-1}}_{=f_{s_1-1}+\sum_{i=2}^{k} f_{s_i-1}} \\
&= \left(\sum_{i=1}^{k-1} \underbrace{f_{s_i+1}}_{\substack{\leq f_{s_{i+1}-1} \\ (\text{by (1)})}} + f_{s_k+1} \right) - \left(f_{s_1-1} + \sum_{i=2}^{k} f_{s_i-1} \right) \\
&\leq \left(\sum_{i=1}^{k-1} f_{s_{i+1}-1} + f_{s_k+1} \right) - \left(f_{s_1-1} + \sum_{i=2}^{k} f_{s_i-1} \right) \\
&= \left(\sum_{i=2}^{k} f_{s_i-1} + f_{s_k+1} \right) - \left(f_{s_1-1} + \sum_{i=2}^{k} f_{s_i-1} \right) \\
&\qquad (\text{here, we substituted } i \text{ for } i+1 \text{ in the first sum}) \\
&= f_{s_k+1} - f_{s_1-1} < f_{s_k+1} \qquad (\text{since } f_{s_1-1} > 0) \\
&= f_{\max S+1}
\end{aligned}$$

(since $s_k = \max S$), which proves Lemma 2.1. (An alternative proof proceeds by strong induction over $\max S$; it uses $f_{\max S+1} = f_{\max S} + f_{\max S-1}$ in the induction step.) □

Lemma 2.2 (existence part of the Zeckendorf theorem)**.** Let $n \in \mathbb{N}$. Then, there exists a finite lacunar subset T of $\mathbb{N}_2$ such that $n = \sum_{t\in T} f_t$.

Proof. Strong induction over n. The case $n = 0$ needs to be treated separately. In the induction step for $n > 0$, the main idea is to let t_1 be the maximal $\tau \in \mathbb{N}_2$ satisfying $f_\tau \leq n$ (this exists because $f_2 = 1 \leq n$ and because the Fibonacci sequence is increasing and unbounded from above), and to apply Lemma 2.2 to $n - f_{t_1}$ instead of n. (This yields a finite lacunar subset T' of $\mathbb{N}_2$ satisfying $n - f_{t_1} = \sum_{t\in T'} f_t$; now, it remains to be shown that the set $T' \cup \{t_1\}$ is still lacunar. To check this, observe that $n < f_{t_1+1}$, so that $n - f_{t_1} < f_{t_1+1} - f_{t_1} = f_{t_1-1}$, which shows that no addend f_t of the sum $\sum_{t\in T'} f_t$ can be f_{t_1-1} or larger.) The details are left to the reader (and can be found in [Grinbe11]). □

Lemma 2.3 (uniqueness part of the Zeckendorf theorem)**.** Let $n \in \mathbb{N}$, and let T and T' be two finite lacunar subsets of $\mathbb{N}_2$ such that $n = \sum_{t \in T} f_t$ and $n = \sum_{t \in T'} f_t$. Then, $T = T'$.

Proof. Strong induction over n. In the induction step for $n > 0$, use Lemma 2.1 to show that $\max T < \max T' + 1$ and $\max T' < \max T + 1$; these together result in $\max T = \max T'$. Hence, the sets T and T' have an element in common, and we can reduce the situation to one with a smaller n by removing this common element from both sets. □

Lemmata 2.2 and 2.3 together yield the following theorem:

Theorem 2.4 (Zeckendorf theorem)**.** Let $n \in \mathbb{N}$. Then, there exists one and only one finite lacunar subset T of $\mathbb{N}_2$ such that $n = \sum_{t \in T} f_t$.

Theorem 2.4 is a classical result that can be found in various places (e.g., [Hender16]). Hoggatt proved a generalization of Theorem 2.4 in [Hoggat72].

Definition 2.5. Let $n \in \mathbb{N}$. Theorem 2.4 shows that there exists one and only one finite lacunar subset T of $\mathbb{N}_2$ such that $n = \sum_{t \in T} f_t$. We will denote this set T by Z_n. Thus, $n = \sum_{t \in Z_n} f_t$.

3. Inequalities for the golden ratio

Next, we state a completely straightforward (and well-known, cf. [BenQui03, Chapter 9, Corollary 34]) theorem, which shows that the Fibonacci sequence grows roughly exponentially (with the exponent being the golden ratio ϕ):

Theorem 3.1. For every positive integer n, we have $|f_{n+1} - \phi f_n| = \frac{1}{\sqrt{5}} (\phi - 1)^n$.

Proof. Binet's formula (see, e.g., [BenQui03, Identity 240] or [Vorobi02, (1.20)]) yields $f_n = \frac{\phi^n - \phi^{-n}}{\sqrt{5}}$ and $f_{n+1} = \frac{\phi^{n+1} - \phi^{-(n+1)}}{\sqrt{5}}$; the rest is computation. □

Let us show yet another lemma for later use:

Lemma 3.2. Let S be a finite lacunar subset of $\mathbb{N}_2$. Then, $\sum_{s \in S} (\phi - 1)^s \leq \phi - 1$.

Proof of Lemma 3.2. Since S is a lacunar subset of $\mathbb{N}_2$, the smallest element of S is at least 2, the second smallest element of S is at least 4 (since it is larger than the smallest element by at least 2), the third smallest element of S is at least 6 (since it is larger than the second smallest element by at least 2), and so on. Since $\mathbb{N} \to \mathbb{R}$, $s \mapsto (\phi - 1)^s$ is a weakly decreasing function (as $0 \leq \phi - 1 \leq 1$), we thus have

$$\sum_{s \in S} (\phi - 1)^s \leq \sum_{s \in \{2,4,6,\ldots\}} (\phi - 1)^s = \sum_{t \in \{1,2,3,\ldots\}} (\phi - 1)^{2t} = \phi - 1$$

(by the formula for the sum of the geometric series, along with some computations). This proves Lemma 3.2. □

4. Proving Theorem 1.4

Let us now come to the proof of Theorem 1.4. First, we formulate the existence part of this theorem:

Theorem 4.1 (existence part of the generalized Zeckendorf family identities)**.** Let T be a finite set, and let a_t be an integer for every $t \in T$.

Then, there exists a finite lacunar subset S of $\mathbb{Z}$ such that

$$\left(\begin{array}{c} \sum\limits_{t \in T} f_{n+a_t} = \sum\limits_{s \in S} f_{n+s} \text{ for every } n \in \mathbb{Z} \text{ which} \\ \text{satisfies } n > \max\left(\{-a_t \mid t \in T\} \cup \{-s \mid s \in S\}\right) \end{array} \right).$$

Before we start proving this, let us introduce a notation:

Definition 4.2. Let K be a subset of $\mathbb{Z}$, and let $a \in \mathbb{Z}$. Then, $K + a$ will denote the subset $\{k + a \mid k \in K\}$ of $\mathbb{Z}$.

Clearly, $(K + a) + b = K + (a + b)$ for any two integers a and b. Also, $K + 0 = K$. Finally, if K is a lacunar subset of $\mathbb{Z}$, and if $a \in \mathbb{Z}$, then $K + a$ is lacunar as well.

Proof of Theorem 4.1. Choose a high enough integer N. Here, "high enough" means that N should satisfy $N \in \mathbb{N}_2$ and $N > \max\{-a_t \mid t \in T\}$ and

$$(\phi - 1)^N \sum_{t \in T} (\phi - 1)^{a_t} + (\phi - 1) < 1. \qquad (2)$$

(Such an N can indeed be found[2].)

[2] *Proof.* We have $(\phi - 1)^N \to 0$ for $N \to \infty$ (since $0 < \phi - 1 < 1$). Therefore, the left hand side of (2) tends to $\phi - 1$ as $N \to \infty$. Thus, for all sufficiently high N, the left hand side of (2) will be < 1, because $\phi - 1 < 1$. So, if we take N sufficiently high, then (2) will hold. Of course, our other two requirements on N (namely, $N \in \mathbb{N}_2$ and $N > \max\{-a_t \mid t \in T\}$) can also be achieved by taking N sufficiently high.

Let $\nu = \sum_{t \in T} f_{N+a_t}$. Then, Z_ν is a finite lacunar subset of $\mathbb{N}_2$ satisfying $\nu = \sum_{t \in Z_\nu} f_t$. Hence, Lemma 3.2 yields

$$\sum_{s \in Z_\nu} (\phi - 1)^s \leq \phi - 1. \tag{3}$$

Define a subset S of $\mathbb{Z}$ by $S = Z_\nu + (-N)$. Then, S is a finite lacunar subset of $\mathbb{Z}$ (since Z_ν is a finite lacunar subset of $\mathbb{Z}$). Furthermore, from $S = Z_\nu + (-N)$, we obtain $Z_\nu = S + N$. Thus, the map $S \to Z_\nu$, $s \mapsto N + s$ is a bijection. This allows us to substitute $N + s$ for t in sums over all $t \in Z_\nu$; we thus obtain

$$\sum_{t \in Z_\nu} f_t = \sum_{s \in S} f_{N+s} \qquad \text{and}$$

$$\sum_{t \in Z_\nu} (\phi - 1)^t = \sum_{s \in S} (\phi - 1)^{N+s}. \tag{4}$$

Hence,

$$\sum_{t \in T} f_{N+a_t} = \nu = \sum_{t \in Z_\nu} f_t = \sum_{s \in S} f_{N+s}, \tag{5}$$

while the equality (4) yields

$$\sum_{s \in S} (\phi - 1)^{N+s} = \sum_{t \in Z_\nu} (\phi - 1)^t = \sum_{s \in Z_\nu} (\phi - 1)^s \leq \phi - 1 \tag{6}$$

(by (6)).

So, we have chosen a high N and found a finite lacunar subset S of $\mathbb{Z}$ which satisfies $\sum_{t \in T} f_{N+a_t} = \sum_{s \in S} f_{N+s}$. But Theorem 4.1 is not proven yet: Theorem 4.1 requires us to prove that there exists *one* finite lacunar subset S of $\mathbb{Z}$ which works for *every* N, while at the moment we cannot be sure yet whether different N's wouldn't produce *different* sets S. And, in fact, different N's *can* produce different sets S, but (fortunately!) only if the N's are too small. As we have taken N high enough, the set S that we obtained turns out to be *universal*, i.e., it satisfies

$$\left(\begin{array}{c} \sum_{t \in T} f_{n+a_t} = \sum_{s \in S} f_{n+s} \text{ for every } n \in \mathbb{Z} \text{ which} \\ \text{satisfies } n > \max\left(\{-a_t \mid t \in T\} \cup \{-s \mid s \in S\}\right) \end{array} \right). \tag{7}$$

We are now going to prove this.

In order to prove (7), we need two assertions:

> *Assertion 1:* If some $n \in \mathbb{Z}$ satisfies $n \geq N$ and $\sum_{t \in T} f_{n+a_t} = \sum_{s \in S} f_{n+s}$, then $\sum_{t \in T} f_{(n+1)+a_t} = \sum_{s \in S} f_{(n+1)+s}$.
>
> *Assertion 2:* If some $n \in \mathbb{Z}$ satisfies $\sum_{t \in T} f_{n+a_t} = \sum_{s \in S} f_{n+s}$ and $\sum_{t \in T} f_{(n+1)+a_t} = \sum_{s \in S} f_{(n+1)+s}$, then $\sum_{t \in T} f_{(n-1)+a_t} = \sum_{s \in S} f_{(n-1)+s}$ (if $n - 1 > \max\left(\{-a_t \mid t \in T\} \cup \{-s \mid s \in S\}\right)$).

Obviously, Assertion 1 yields (by induction) the equality $\sum_{t\in T} f_{n+a_t} = \sum_{s\in S} f_{n+s}$ for every $n \geq N$ (the induction base follows from (5)), and Assertion 2 then proves it for the remaining values of n (by backwards induction, or, to be more precise, by an induction step from $n+1$ and n to $n-1$). Thus, once both Assertions 1 and 2 are proven, (7) will follow and thus Theorem 4.1 will be proven.

Assertion 2 follows from comparing the equalities

$$\sum_{t\in T} \underbrace{f_{(n-1)+a_t}}_{\substack{=f_{n+a_t-1} \\ =f_{n+a_t+1}-f_{n+a_t}}} = \sum_{t\in T} f_{n+a_t+1} - \sum_{t\in T} f_{n+a_t} = \sum_{t\in T} f_{(n+1)+a_t} - \sum_{t\in T} f_{n+a_t}$$

and

$$\sum_{s\in S} \underbrace{f_{(n-1)+s}}_{\substack{=f_{n+s-1} \\ =f_{n+s+1}-f_{n+s}}} = \sum_{s\in S} f_{n+s+1} - \sum_{s\in S} f_{n+s} = \sum_{s\in S} f_{(n+1)+s} - \sum_{s\in S} f_{n+s}$$

(whose right hand sides are equal by the assumptions of Assertion 2); thus, it only remains to prove Assertion 1.

So let us prove Assertion 1. Here we are going to use that N is high enough (because otherwise, Assertion 1 wouldn't hold). We have $\sum_{t\in T} f_{n+a_t} = \sum_{s\in S} f_{n+s}$ by assumption, so that $\sum_{t\in T} f_{n+a_t} - \sum_{s\in S} f_{n+s} = 0$. Thus,

$$\begin{aligned}
\sum_{t\in T} f_{(n+1)+a_t} - \sum_{s\in S} f_{(n+1)+s} &= \sum_{t\in T} f_{(n+1)+a_t} - \sum_{s\in S} f_{(n+1)+s} - \phi \left(\sum_{t\in T} f_{n+a_t} - \sum_{s\in S} f_{n+s} \right) \\
&= \sum_{t\in T} \left(f_{(n+1)+a_t} - \phi f_{n+a_t} \right) - \sum_{s\in S} \left(f_{(n+1)+s} - \phi f_{n+s} \right) \\
&= \sum_{t\in T} \left(f_{n+a_t+1} - \phi f_{n+a_t} \right) - \sum_{s\in S} \left(f_{n+s+1} - \phi f_{n+s} \right),
\end{aligned}$$

so that

$$\begin{aligned}
&\left|\sum_{t\in T} f_{(n+1)+a_t} - \sum_{s\in S} f_{(n+1)+s}\right| = \left|\sum_{t\in T}\left(f_{n+a_t+1} - \phi f_{n+a_t}\right) - \sum_{s\in S}\left(f_{n+s+1} - \phi f_{n+s}\right)\right| \\
&\leq \sum_{t\in T}\left|f_{n+a_t+1} - \phi f_{n+a_t}\right| + \sum_{s\in S}\left|f_{n+s+1} - \phi f_{n+s}\right| \qquad \text{(by the triangle inequality)} \\
&= \sum_{t\in T} \frac{1}{\sqrt{5}} (\phi - 1)^{n+a_t} + \sum_{s\in S} \frac{1}{\sqrt{5}} (\phi - 1)^{n+s} \qquad \text{(by Theorem 3.1)} \\
&< \sum_{t\in T} \underbrace{(\phi - 1)^{n+a_t}}_{\substack{\leq (\phi-1)^{N+a_t} \\ \text{(since } n\geq N \text{ and } 0<\phi-1<1)}} + \sum_{s\in S} \underbrace{(\phi - 1)^{n+s}}_{\substack{\leq (\phi-1)^{N+s} \\ \text{(since } n\geq N \text{ and } 0<\phi-1<1)}} \qquad \left(\text{since } \frac{1}{\sqrt{5}} < 1\right) \\
&\leq \underbrace{\sum_{t\in T} (\phi - 1)^{N+a_t}}_{=(\phi-1)^N \sum_{t\in T} (\phi-1)^{a_t}} + \underbrace{\sum_{s\in S} (\phi - 1)^{N+s}}_{\substack{\leq \phi - 1 \\ \text{(by (6))}}} \leq (\phi - 1)^N \sum_{t\in T} (\phi - 1)^{a_t} + (\phi - 1) < 1
\end{aligned}$$

(by (2)). This leads to $\left|\sum_{t\in T} f_{(n+1)+a_t} - \sum_{s\in S} f_{(n+1)+s}\right| = 0$ (since $\left|\sum_{t\in T} f_{(n+1)+a_t} - \sum_{s\in S} f_{(n+1)+s}\right|$ is a nonnegative integer). In other words, $\sum_{t\in T} f_{(n+1)+a_t} = \sum_{s\in S} f_{(n+1)+s}$. This completes the proof of Assertion 1, and, with it, the proof of Theorem 4.1. □

All that remains now is the (rather trivial) uniqueness part of Theorem 1.4:

Lemma 4.3 (uniqueness part of the generalized Zeckendorf family identities)**.** Let T be a finite set, and let a_t be an integer for every $t \in T$.

Let S be a finite lacunar subset of $\mathbb{Z}$ such that

$$\left(\begin{array}{c} \sum_{t\in T} f_{n+a_t} = \sum_{s\in S} f_{n+s} \text{ for every } n \in \mathbb{Z} \text{ which} \\ \text{satisfies } n > \max\left(\{-a_t \mid t \in T\} \cup \{-s \mid s \in S\}\right) \end{array}\right). \tag{8}$$

Let S' be a finite lacunar subset of $\mathbb{Z}$ such that

$$\left(\begin{array}{c} \sum_{t\in T} f_{n+a_t} = \sum_{s\in S'} f_{n+s} \text{ for every } n \in \mathbb{Z} \text{ which} \\ \text{satisfies } n > \max\left(\{-a_t \mid t \in T\} \cup \{-s \mid s \in S'\}\right) \end{array}\right). \tag{9}$$

Then, $S = S'$.

Proof of Lemma 4.3. Let

$$n = \max\left(\{-a_t \mid t \in T\} \cup \{-s \mid s \in S\} \cup \{-s \mid s \in S'\}\right) + 2. \tag{10}$$

Then, n satisfies $n > \max\left(\{-a_t \mid t \in T\} \cup \{-s \mid s \in S\}\right)$. Thus, (8) yields

$$\sum_{t\in T} f_{n+a_t} = \sum_{s\in S} f_{n+s} = \sum_{t\in S+n} f_t$$

(here, we substituted t for $n+s$, since the map $S \to S+n$, $s \mapsto n+s$ is a bijection). Similarly, $\sum\limits_{t\in T} f_{n+a_t} = \sum\limits_{t\in S'+n} f_t$. Since the sets $S+n$ and $S'+n$ are both lacunar (since so are S and S') and finite (since so are S and S'), and are subsets of $\mathbb{N}_2$ (by (10)), we can now conclude from Lemma 2.3 (applied to $\sum\limits_{t\in T} f_{n+a_t}$, $S+n$ and $S'+n$ instead of n, S and S') that $S+n = S'+n$, so that $S = S'$. This proves Lemma 4.3. □

Proof of Theorem 1.4. Now, Theorem 1.4 is clear, since the existence follows from Theorem 4.1 and the uniqueness from Lemma 4.3. □

References

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