

Math 701 Fall 2026, Lecture 10: The Murnaghan–Nakayama rule

website: <https://www.cip.ifi.lmu.de/~grinberg/t/26fs/>

(GPT-6 was used to create examples and to polish and detail some of the arguments below. **The current version contains unverified GPT output.**)

1.10. The Murnaghan–Nakayama rule

So we can multiply Schur polynomials by e_n and by h_n . What about multiplying them by p_n ?

Again, we recall $s_\mu = a_{\mu+\rho}/a_\rho$ for an N -partition μ , and we try to compute $p_n a_{\mu+\rho}$, or, in general, $p_n a_\alpha$. Here and throughout this lecture, n is a positive integer. Recall that

$$a_\alpha = \det \left(\left(x_i^{\alpha_j} \right)_{i,j \in [N]} \right) \quad \text{and} \quad p_n = \sum_{k=1}^N x_k^n.$$

We will use the following lemma about determinants:

Lemma 1.10.1 (Moving column lemma). Let $(b_{i,j})_{i,j \in [N]}$ be an $N \times N$ -matrix over some commutative ring R , and let $c_1, c_2, \dots, c_N$ be N elements of R . Then,

$$\left(\sum_{k=1}^N c_k \right) \det (b_{i,j})_{i,j \in [N]} = \sum_{k=1}^N \det (b_{i,j} c_i^{[j=k]})_{i,j \in [N]}.$$

Here, we are using the Iverson bracket notation: If $\mathcal{A}$ is a statement, then $[\mathcal{A}]$ means the truth value of $\mathcal{A}$:

$$[\mathcal{A}] := \begin{cases} 1, & \text{if } \mathcal{A} \text{ is true;} \\ 0, & \text{if } \mathcal{A} \text{ is false.} \end{cases}$$

So the factor $c_i^{[j=k]}$ is $c_i^0 = 1$ if $j \neq k$ and is $c_i^1 = c_i$ if $j = k$. Thus, the matrix $(b_{i,j} c_i^{[j=k]})_{i,j \in [N]}$ is the original matrix $(b_{i,j})_{i,j \in [N]}$ with its k -th column entrywise multiplied by the vector $(c_1, c_2, \dots, c_N)^T$.

Example 1.10.2. Let $N = 3$. Here the claim of Lemma 1.10.1 is

$$\begin{aligned}
 & (c_1 + c_2 + c_3) \det \begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{pmatrix} \\
 &= \det \begin{pmatrix} c_1 a_{1,1} & a_{1,2} & a_{1,3} \\ c_2 a_{2,1} & a_{2,2} & a_{2,3} \\ c_3 a_{3,1} & a_{3,2} & a_{3,3} \end{pmatrix} + \det \begin{pmatrix} a_{1,1} & c_1 a_{1,2} & a_{1,3} \\ a_{2,1} & c_2 a_{2,2} & a_{2,3} \\ a_{3,1} & c_3 a_{3,2} & a_{3,3} \end{pmatrix} \\
 &+ \det \begin{pmatrix} a_{1,1} & a_{1,2} & c_1 a_{1,3} \\ a_{2,1} & a_{2,2} & c_2 a_{2,3} \\ a_{3,1} & a_{3,2} & c_3 a_{3,3} \end{pmatrix}.
 \end{aligned}$$

Each determinant on the right changes just one column; the three terms account for the three possible choices of that column.

Proof of Lemma 1.10.1. For each $k \in [N]$, we have

$$\begin{aligned}
 & \det \left(b_{i,j} c_i^{[j=k]} \right)_{i,j \in [N]} \\
 &= \sum_{\sigma \in S_N} (-1)^\sigma \prod_{i=1}^N b_{i,\sigma(i)} c_i^{[\sigma(i)=k]} \\
 &= \sum_{\sigma \in S_N} (-1)^\sigma \prod_{i=1}^N b_{i,\sigma(i)} \underbrace{\prod_{i=1}^N c_i^{[\sigma(i)=k]}}_{=c_{\sigma^{-1}(k)}} \\
 &= \sum_{\sigma \in S_N} (-1)^\sigma c_{\sigma^{-1}(k)} \prod_{i=1}^N b_{i,\sigma(i)}.
 \end{aligned}$$

Summing this over all $k \in [N]$, we find

$$\begin{aligned}
 & \sum_{k=1}^N \det \left(b_{i,j} c_i^{[j=k]} \right)_{i,j \in [N]} \\
 &= \sum_{k=1}^N \sum_{\sigma \in S_N} (-1)^\sigma c_{\sigma^{-1}(k)} \prod_{i=1}^N b_{i,\sigma(i)} \\
 &= \sum_{\sigma \in S_N} (-1)^\sigma \underbrace{\left(\sum_{k=1}^N c_{\sigma^{-1}(k)} \right)}_{=c_1+c_2+\dots+c_N} \prod_{i=1}^N b_{i,\sigma(i)} \\
 & \quad \text{(since } \sigma \text{ is a permutation)} \\
 &= \sum_{\sigma \in S_N} (-1)^\sigma (c_1 + c_2 + \dots + c_N) \prod_{i=1}^N b_{i,\sigma(i)}
 \end{aligned}$$

$$\begin{aligned}
&= (c_1 + c_2 + \cdots + c_N) \underbrace{\left(\sum_{\sigma \in S_N} (-1)^\sigma \prod_{i=1}^N b_{i, \sigma(i)} \right)}_{=\det(b_{i,j})_{i,j \in [N]}} \\
&= (c_1 + c_2 + \cdots + c_N) \det(b_{i,j})_{i,j \in [N]},
\end{aligned}$$

as desired. The same identity holds when $N = 0$: both sides are zero. $\square$

So let us apply this to compute $p_n a_\alpha$, where n is a positive integer and $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_N) \in \mathbb{N}^N$ is an N -tuple. We have

$$\begin{aligned}
p_n a_\alpha &= (x_1^n + x_2^n + \cdots + x_N^n) \det(x_i^{\alpha_j})_{i,j \in [N]} \\
&= \sum_{k=1}^N \det(x_i^{\alpha_j} (x_i^n)^{[j=k]})_{i,j \in [N]} \quad (\text{by Lemma 1.10.1}) \\
&= \sum_{k=1}^N \det(x_i^{\alpha_j + n[j=k]})_{i,j \in [N]}.
\end{aligned}$$

Let $\vec{e}_1, \vec{e}_2, \dots, \vec{e}_N \in \mathbb{N}^N$ be the standard basis vectors of the $\mathbb{Z}$ -module $\mathbb{Z}^N$; that is, $\vec{e}_i$ has a 1 in its i -th position and zeroes everywhere else. Then, the exponent $\alpha_j + n[j=k]$ can be rewritten as $(\alpha + n\vec{e}_k)_j$ (since both are α_j when $j \neq k$, and are $\alpha_j + n$ when $j = k$). So the above equality rewrites as

$$p_n a_\alpha = \sum_{k=1}^N \underbrace{\det(x_i^{(\alpha + n\vec{e}_k)_j})_{i,j \in [N]}}_{=a_{\alpha + n\vec{e}_k}} = \sum_{k=1}^N a_{\alpha + n\vec{e}_k}. \quad (1)$$

Setting $\alpha = \mu + \rho$ (where μ is an N -partition) and dividing this by a_ρ should give us a formula for $p_n s_\mu$ as a sum of Schur polynomials. As in Lecture 8, division by a_ρ here means cancelling this regular element from a polynomial identity; no assumption that K is a field is needed.

Example 1.10.3. Let us compute $p_2 s_{(2,2,1)}$ for $N = 3$. We have

$$s_{(2,2,1)} = a_{(2,2,1)+\rho} / a_\rho = a_{(4,3,1)} / a_{(2,1,0)},$$

so we must find $p_2 a_{(4,3,1)}$. The above formula gives us

$$\begin{aligned}
p_2 a_{(4,3,1)} &= a_{(6,3,1)} + \underbrace{a_{(4,5,1)}}_{=-a_{(5,4,1)} \text{ (since alternants are alternating)}} + \underbrace{a_{(4,3,3)}}_{=0 \text{ (since alternants are alternating)}} \\
&= a_{(6,3,1)} - a_{(5,4,1)}.
\end{aligned}$$

Dividing back by a_ρ gives us

$$p_2 s_{(2,2,1)} = s_{(6,3,1)-\rho} - s_{(5,4,1)-\rho} = s_{(4,2,1)} - s_{(3,3,1)}.$$

So we get an expansion, but the coefficients need not be nonnegative, and we have to sort our $\alpha + n \vec{e}_k$'s to make them strictly decreasing again (and remove the ones that have equal entries!). The general case uses Lemma 1.7.9 from Lecture 8. If sorting requires a permutation σ , then

$$a_{\alpha+n \vec{e}_k} = (-1)^\sigma a_{(\alpha+n \vec{e}_k) \circ \sigma}.$$

Thus, we keep track of the sign of the sorting permutation. Tuples with repeated entries contribute zero by part (a) of that lemma.

Back to the combinatorics: can we actually describe the partitions we get in the expansion? For example, for $N \geq 7$, we get

$$p2^s(2,2,1) = s(4,2,1) - s(3,3,1) - s(2,2,1,1,1);$$

$$p3^s(3,2,1) = s(3,2,1,1,1,1) - s(3,2,2,2) - s(3,3,3) - s(4,4,1) + s(6,2,1).$$

(this still holds when $N < 7$, but then some of these Schurs will be 0). What is the pattern here? Again, $p_n s_\mu = \sum \pm s_\lambda$ over certain partitions λ for which λ/μ is a skew partition (that is, $\mu \subseteq \lambda$). But which skew partitions appear in the sum, and with which signs?

The answer can be described visually in terms of so-called **ribbons** aka **rim hooks** aka **border strips**. These are skew partitions “of width 1”. There are many equivalent formal ways to describe them. We will introduce them by giving all these descriptions and showing that they are equivalent.

Notations:

- We view the set $\mathbb{Z}^2$ as a directed graph with arcs $(i, j) \rightarrow (i+1, j)$ (“south-steps”) and $(i, j) \rightarrow (i, j-1)$ (“west-steps”). This digraph will be called $\mathbb{Z}_{\text{sw}}^2$. We allow a directed path to consist of just one vertex (and no arcs).
- Forgetting the directions of the arcs turns this digraph into an undirected graph, which we just call $\mathbb{Z}^2$.
- For any $k \in \mathbb{Z}$, the k -th *diagonal* means the set of all $(i, j) \in \mathbb{Z}^2$ with $j - i = k$.
- The *overlap* of two rows R and S of a diagram means the set of all $j \in \mathbb{Z}$ such that both R and S have a cell in the j -th column.
- If λ is a partition, then

$$R(\lambda) := \{(i, j) \in Y(\lambda) \mid (i+1, j+1) \notin Y(\lambda)\}.$$

We call this set the **rim** of λ . It consists of the cells of $Y(\lambda)$ that are the southeasternmost cells on their diagonal to belong to $Y(\lambda)$.

Moreover, for each $c \in Y(\lambda)$, we set

$$R_c(\lambda) := \{(i, j) \in R(\lambda) \mid (i, j) \geq c\},$$

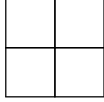
where the $\geq$ relation refers to the entrywise order on $\mathbb{Z}^2$ (that is, “ (i, j) lies weakly southeast of c ”).

- For an integer N -tuple $\alpha \in \mathbb{Z}^N$, we write $\text{sort}(\alpha)$ for its entries rearranged in weakly decreasing order. This is an integer tuple, not necessarily a partition: its entries may be negative. Repeated entries are not removed.
- For integers u, v , we write $[u, v] := \{i \in \mathbb{Z} \mid u \leq i \leq v\}$. In particular, this interval is empty when $u > v$.
- As usual, ρ means the N -partition $(N-1, N-2, \dots, 1, 0)$.

Theorem 1.10.4 (Ribbon criteria). Let λ and μ be two N -partitions with $\mu \subseteq \lambda$ and $n := |Y(\lambda/\mu)| > 0$. Then, the following conditions are equivalent:

$\mathcal{R}_{\text{path}}$: The diagram $Y(\lambda/\mu)$ is the set of vertices of a (directed!) path of the digraph $\mathbb{Z}_{\text{sw}}^2$.

$\mathcal{R}_{\text{diag}}$: The diagram $Y(\lambda/\mu)$ has exactly one cell in each of n consecutive diagonals (and these exhaust all cells of $Y(\lambda/\mu)$).

$\mathcal{R}_{2 \times 2}$: The diagram $Y(\lambda/\mu)$ is connected as a subset of the undirected graph $\mathbb{Z}^2$ and contains no 2×2 -square  as a subset.

$\mathcal{R}_{\text{ovl}}$: The nonempty rows of the diagram $Y(\lambda/\mu)$ form an interval (i.e., they are the u -th, $(u+1)$ -st, $\dots$, v -th row for some $1 \leq u \leq v \leq N$), and any two consecutive nonempty rows have an overlap of size 1.

$\mathcal{R}_{+1}$: There exist $1 \leq u \leq v \leq N$ such that

$$\begin{aligned} \lambda_i &= \mu_{i-1} + 1 && \text{for all } i \in [u+1, v], && \text{and} \\ \lambda_i &= \mu_i && \text{for all } i \in [N] \setminus [u, v]. \end{aligned}$$

$\mathcal{R}_{\rho+}$: We have $\lambda + \rho = \text{sort}(\mu + \rho + n \vec{e}_k)$ for some $k \in [N]$, where $\vec{e}_k \in \mathbb{N}^N$ denotes the vector with a 1 in its k -th position and zeroes everywhere else.

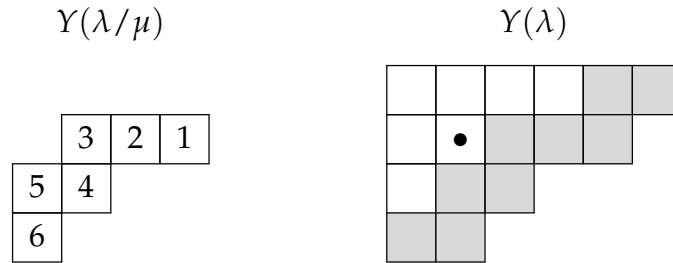
$\mathcal{R}_{\rho-}$: We have $\mu + \rho = \text{sort}(\lambda + \rho - n \vec{e}_k)$ for some $k \in [N]$, where $\vec{e}_k \in \mathbb{N}^N$ denotes the vector with a 1 in its k -th position and zeroes everywhere else.

$\mathcal{R}_R$: We have $Y(\lambda/\mu) = R_c(\lambda)$ for some $c \in Y(\lambda)$.

Moreover, the u and v in condition $\mathcal{R}_{\text{ovl}}$ are precisely the u and v in condition $\mathcal{R}_{+1}$.

A skew partition satisfying these equivalent conditions is called a **ribbon**, a **rim hook**, or a **border strip**. In particular, a ribbon is nonempty.

Example 1.10.5. Let $N = 4$, $\lambda = (6, 5, 3, 2)$ and $\mu = (6, 2, 1, 1)$. Thus, $n = 6$. The left-hand picture shows the skew diagram, with its cells numbered in southwest path order. The right-hand picture shows $Y(\lambda)$, with the rim $R(\lambda)$ shaded and the cell $c = (2, 2)$ marked by a bullet:



The path in the left-hand picture is

$$(2, 5) \longrightarrow (2, 4) \longrightarrow (2, 3) \longrightarrow (3, 3) \longrightarrow (3, 2) \longrightarrow (4, 2).$$

Its six cells lie on diagonals $3, 2, 1, 0, -1, -2$, respectively. The diagram is connected and contains no 2×2 -square. Its nonempty rows are rows $2, 3, 4$, with consecutive overlaps $\{3\}$ and $\{2\}$. Thus, $u = 2$ and $v = 4$, and the equalities in $\mathcal{R}_{+1}$ are

$$\lambda_3 = 3 = \mu_2 + 1, \quad \lambda_4 = 2 = \mu_3 + 1, \quad \lambda_1 = \mu_1 = 6.$$

Also, $Y(\lambda/\mu) = R_{(2,2)}(\lambda)$: we keep precisely the shaded cells weakly south-east of the bullet. Notice that c itself need not belong to the rim or to the skew diagram.

Finally, $\rho = (3, 2, 1, 0)$ gives

$$\mu + \rho = (9, 4, 2, 1), \quad \lambda + \rho = (9, 7, 4, 2).$$

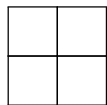
Thus, the two sorting conditions hold as well:

$$\begin{aligned} \text{sort}(\mu + \rho + 6\vec{e}_4) &= \text{sort}(9, 4, 2, 7) = (9, 7, 4, 2) = \lambda + \rho, \\ \text{sort}(\lambda + \rho - 6\vec{e}_2) &= \text{sort}(9, 1, 4, 2) = (9, 4, 2, 1) = \mu + \rho. \end{aligned}$$

The chosen positions in these two conditions need not be the same: here they are 4 and 2. In the first sorting, the 7 moves left past two entries, so $a_{(9,4,2,7)} = a_{(9,7,4,2)}$ with a plus sign.

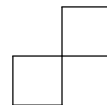
Example 1.10.6. The following two diagrams are not ribbons:

$Y((2,2))$



connected, but contains
a 2×2 -square

$Y((2,1)/(1))$



no 2×2 -square,
but disconnected

The two cells in the second diagram touch only at a corner, which does not make them adjacent in $\mathbb{Z}^2$. They lie on diagonals 1 and -1 , with diagonal 0 missing. Thus, both clauses in $\mathcal{R}_{2 \times 2}$ matter. A diagram with just one cell, on the other hand, is a ribbon; the empty diagram is not.

Beginning of the proof of Theorem 1.10.4.

We prove the equivalence of the first three conditions in this lecture.

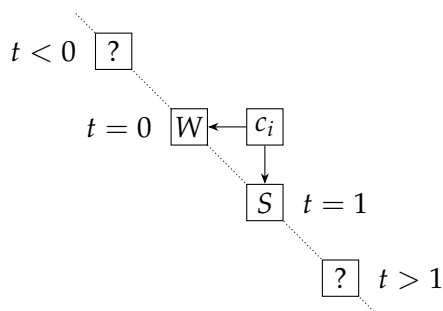
Proof of $\mathcal{R}_{\text{path}} \implies \mathcal{R}_{\text{diag}}$: Assume $\mathcal{R}_{\text{path}}$. Then, the n cells of $Y(\lambda/\mu)$ are the vertices of a path of $\mathbb{Z}_{\text{sw}}^2$. Each arc $c \rightarrow d$ of $\mathbb{Z}_{\text{sw}}^2$ has the property that if c lies on the k -th diagonal, then d lies on the $(k-1)$ -th diagonal. So the vertices of a path must lie on consecutive diagonals. In particular, the n cells of $Y(\lambda/\mu)$ must therefore lie on n consecutive diagonals. That is, $\mathcal{R}_{\text{diag}}$ holds.

Proof of $\mathcal{R}_{\text{diag}} \implies \mathcal{R}_{\text{path}}$: Assume $\mathcal{R}_{\text{diag}}$. List the cells as $c_1, \dots, c_n$ so that c_i lies on diagonal $d+i$, for some integer d . We shall show that there is an arc from c_i to c_{i-1} for each $i \in \{2, \dots, n\}$. Then $c_n, c_{n-1}, \dots, c_1$ is the required path in $\mathbb{Z}_{\text{sw}}^2$.

Fix $i \in \{2, \dots, n\}$ and write $c_i = (a, b)$. Since c_{i-1} lies on the preceding diagonal, it has the form

$$c_{i-1} = (a+t, b+t-1) \quad \text{for some } t \in \mathbb{Z}.$$

Let $W = (a, b-1)$ and $S = (a+1, b)$ be the western and southern neighbors of c_i . The possibilities $t = 0$ and $t = 1$ give $c_{i-1} = W$ and $c_{i-1} = S$, respectively, and therefore give the desired arc. All other positions lie farther northwest or southeast on the same diagonal:



The two question marks illustrate positions with $t < 0$ and $t > 1$; such positions can be arbitrarily far along the dotted diagonal.

If $t < 0$, then $c_{i-1} \leq W \leq c_i$ in the northwest order. By the convexity of skew diagrams (Lemma 1.6.7), the cell W must also belong to $Y(\lambda/\mu)$. But W and c_{i-1} are distinct cells on the same diagonal, contradicting $\mathcal{R}_{\text{diag}}$.

If $t > 1$, then $c_i \leq S \leq c_{i-1}$. Convexity now puts S in the diagram, again giving two distinct cells, S and c_{i-1} , on the same diagonal. This is the same contradiction. Therefore $t \in \{0, 1\}$, as required. For $n = 1$, there are no arcs to check and the one cell is already a path.

Thus, $\mathcal{R}_{\text{path}} \iff \mathcal{R}_{\text{diag}}$.

Proof of $\mathcal{R}_{\text{diag}} \implies \mathcal{R}_{2 \times 2}$: Assume $\mathcal{R}_{\text{diag}}$ holds. Then, there are no 2×2 -squares in $Y(\lambda/\mu)$ because a 2×2 -square would contain two cells on the same diagonal. Moreover, $\mathcal{R}_{\text{diag}} \implies \mathcal{R}_{\text{path}}$ shows that we have $\mathcal{R}_{\text{path}}$, and therefore $Y(\lambda/\mu)$ is connected (since a path is connected). So $\mathcal{R}_{2 \times 2}$ holds.

Proof of $\mathcal{R}_{2 \times 2} \implies \mathcal{R}_{\text{diag}}$: Assume $\mathcal{R}_{2 \times 2}$. Adjacent cells of $\mathbb{Z}^2$ lie on adjacent diagonals. Since $Y(\lambda/\mu)$ is connected, any two of its cells can be joined by a path within the diagram. Along such a path the diagonal number changes by 1 or -1 at each step, so it takes every integer value between its starting and ending values. Consequently, the diagonals met by $Y(\lambda/\mu)$ form an interval of integers.

Moreover, no two cells of $Y(\lambda/\mu)$ can lie on the same diagonal. Indeed, two distinct cells on the same diagonal have the form (a, b) and $(a + t, b + t)$ for some integer $t \geq 1$, after interchanging them if needed. Every cell of the square

$$\{(a, b), (a, b + 1), (a + 1, b), (a + 1, b + 1)\}$$

lies between these two cells in the northwest order. Convexity would therefore put this entire 2×2 -square in $Y(\lambda/\mu)$, contrary to our assumption. Thus, there is exactly one cell on each occupied diagonal; since there are n cells, there are exactly n consecutive occupied diagonals. This proves $\mathcal{R}_{\text{diag}}$.

Please try to think about the other equivalences. We will finish this proof next time and then state the Murnaghan–Nakayama rule for $p_n s_\mu$ in terms of ribbons.