

Math 701 Fall 2026, Lecture 9: The Pieri rules

website: <https://www.cip.ifi.lmu.de/~grinberg/t/26fs/>

(GPT-6 was used to create examples and to polish and detail some of the arguments below. **The current version contains unverified GPT output.**)

End of the proof of Lemma 1.7.8. As we recall, $\mathcal{X}$ denotes the set of all SSYT $T \in \text{SSYT}(\lambda/\mu, N)$ that are not ν -Yamanouchi. We have defined a map

$$f : \mathcal{X} \rightarrow \mathcal{X}$$

as follows: Let $T \in \mathcal{X}$. We let j be the largest violator of T (that is, the largest $j \geq 1$ such that $\nu + \text{cont}(\text{col}_{\geq j} T)$ is **not** an N -partition). Define the N -tuple

$$b := \nu + \text{cont}(\text{col}_{\geq j} T) \quad (\text{not an } N\text{-partition}).$$

We let k be the smallest misstep of T (that is, the smallest $k \in [N-1]$ such that $b_k < b_{k+1}$). Now we define the tableau $f(T)$ by keeping columns $j, j+1, j+2, \dots$ of T unchanged while applying the Bender–Knuth involution β_k to the remaining columns $1, 2, \dots, j-1$ (that is, replacing $\text{col}_{< j} T$ by $\beta_k(\text{col}_{< j} T)$). We recall the five claims that stood between us and the proof of Lemma 1.7.8:

1. The resulting tableau $f(T)$ is still semistandard.
2. This tableau $f(T)$ again belongs to $\mathcal{X}$.
3. The resulting map f is an involution, i.e., we have $f(f(T)) = T$.
4. We have $a_{\nu + \text{cont} T + \rho} + a_{\nu + \text{cont}(f(T)) + \rho} = 0$.
5. If $f(T) = T$, then $a_{\nu + \text{cont} T + \rho} = 0$.

Claims 1 and 2 were already proved in Lecture 8. Now we prove the rest:

Proof of Claim 3. Fix $T \in \mathcal{X}$. We did not change anything in columns $j, j+1, j+2, \dots$ of T . So j is still a violator of $f(T)$, while none of $j+1, j+2, j+3, \dots$ is. So j is still the largest violator of $f(T)$. Furthermore, k is still the smallest misstep of $f(T)$, because the tuple $\nu + \text{cont}(\text{col}_{\geq j}(f(T)))$ is precisely the tuple $\nu + \text{cont}(\text{col}_{\geq j} T)$ (again since we did not change anything in columns $j, j+1, j+2, \dots$ of T). Hence, when we apply f again to $f(T)$, the course is set precisely in the same way as it was when we applied f to T : we again leave columns $j, j+1, j+2, \dots$ unchanged and apply β_k to the first $j-1$ columns. Thus, in total, the first $j-1$ columns of T end up undergoing β_k twice, which brings them back to their original state (since β_k is an involution); meanwhile, all the remaining columns of T never change at all. Altogether, we thus conclude that $f(f(T)) = T$. This proves Claim 3. $\square$

It remains to prove Claims 4 and 5. In these proofs, we will use the following notation: For any N -tuple α and any $i \in [N]$, we let α_i denote the i -th entry of α . If $\alpha \in \mathbb{N}^N$ is an N -tuple and $\sigma \in S_N$ is a permutation, then $\alpha \circ \sigma$ denotes the N -tuple $(\alpha_{\sigma(1)}, \alpha_{\sigma(2)}, \dots, \alpha_{\sigma(N)}) \in \mathbb{N}^N$. This gives a right action of the group S_N on $\mathbb{N}^N$; it satisfies

$$(\alpha + \beta) \circ \sigma = \alpha \circ \sigma + \beta \circ \sigma \quad \text{for all } \alpha, \beta \in \mathbb{N}^N \text{ and } \sigma \in S_N.$$

We shall use this equality without further mention. Note also that if $\alpha \in \mathbb{N}^N$ and $k \in [N-1]$, then the N -tuple $\alpha \circ s_k$ is obtained from α by swapping the k -th and $(k+1)$ -st entries. In particular, if $\alpha_k = \alpha_{k+1}$, then $\alpha \circ s_k = \alpha$.

Proof of Claim 4. We must prove that $a_{\nu + \text{cont } T + \rho} + a_{\nu + \text{cont}(f(T)) + \rho} = 0$.

We shall show that the N -tuple $\nu + \text{cont}(f(T)) + \rho$ is obtained from $\nu + \text{cont } T + \rho$ by swapping the k -th and $(k+1)$ -st entries. Thus, we shall show that

$$\nu + \text{cont}(f(T)) + \rho = (\nu + \text{cont } T + \rho) \circ s_k. \quad (1)$$

By Lemma 1.7.9 (b), this will entail $a_{\nu + \text{cont}(f(T)) + \rho} = -a_{\nu + \text{cont } T + \rho}$, thus $a_{\nu + \text{cont } T + \rho} + a_{\nu + \text{cont}(f(T)) + \rho} = 0$, which will prove Claim 4.

By the definition of $f(T)$, we have

$$\text{col}_{\geq j}(f(T)) = \text{col}_{\geq j} T$$

(since only the first $j-1$ columns change from T to $f(T)$). Moreover, the first $j-1$ columns of $f(T)$ are obtained from the corresponding columns of T by the Bender–Knuth involution β_k ; thus, $\text{col}_{< j}(f(T)) = \beta_k(\text{col}_{< j} T)$, so that

$$\text{cont}(\text{col}_{< j}(f(T))) = \text{cont}(\beta_k(\text{col}_{< j} T)) = \text{cont}(\text{col}_{< j} T) \circ s_k$$

(since the Bender–Knuth involution β_k swaps the #s of k 's and $(k+1)$'s in a tableau, while leaving all other #s unchanged (by Observation 4 in the proof of Theorem 1.6.14); thus, it acts on the content by swapping the k -th and $(k+1)$ -st entries). Finally,

$$\text{cont } T = \text{cont}(\text{col}_{\geq j} T) + \text{cont}(\text{col}_{< j} T) \quad (2)$$

(since each entry of T belongs either to $\text{col}_{\geq j} T$ or to $\text{col}_{< j} T$).

Now, recall that $b = \nu + \text{cont}(\text{col}_{\geq j} T)$. Define also $a := \text{cont}(\text{col}_{< j} T)$. So

$$\begin{aligned} \nu + \underbrace{\text{cont } T}_{= \text{cont}(\text{col}_{\geq j} T) + \text{cont}(\text{col}_{< j} T)} &= \underbrace{\nu + \text{cont}(\text{col}_{\geq j} T)}_{=b} + \underbrace{\text{cont}(\text{col}_{< j} T)}_{=a} = b + a. \end{aligned}$$

Meanwhile, (2) (applied to $f(T)$ instead of T) yields

$$\begin{aligned} \text{cont}(f(T)) &= \text{cont}\left(\underbrace{\text{col}_{\geq j}(f(T))}_{=\text{col}_{\geq j} T}\right) + \underbrace{\text{cont}(\text{col}_{< j}(f(T)))}_{=\text{cont}(\text{col}_{< j} T) \circ s_k} \\ &= \text{cont}(\text{col}_{\geq j} T) + \underbrace{\text{cont}(\text{col}_{< j} T)}_{=a} \circ s_k = \text{cont}(\text{col}_{\geq j} T) + a \circ s_k. \end{aligned}$$

Adding v to both sides, we thus obtain

$$v + \text{cont}(f(T)) = \underbrace{v + \text{cont}(\text{col}_{\geq j} T)}_{=b} + a \circ s_k = b + a \circ s_k. \quad (3)$$

But $\rho = (N-1, N-2, \dots, 1, 0)$ yields $\rho_{k+1} = \rho_k - 1$. Recall also that $b_{k+1} = b_k + 1$, as we saw previously in the construction of $f(T)$. Thus,

$$(b + \rho)_{k+1} = \underbrace{b_{k+1}}_{=b_k+1} + \underbrace{\rho_{k+1}}_{=\rho_k-1} = (b_k + 1) + (\rho_k - 1) = b_k + \rho_k = (b + \rho)_k$$

and therefore

$$(b + \rho) \circ s_k = b + \rho$$

(since an N -tuple $\alpha \in \mathbb{N}^N$ that satisfies $\alpha_k = \alpha_{k+1}$ always satisfies $\alpha \circ s_k = \alpha$).

Altogether,

$$\begin{aligned} \left(\underbrace{v + \text{cont} T}_{=b+a} + \rho\right) \circ s_k &= \underbrace{(b + a + \rho)}_{=(b+\rho)+a} \circ s_k = ((b + \rho) + a) \circ s_k \\ &= \underbrace{(b + \rho) \circ s_k}_{=b+\rho} + a \circ s_k = b + \rho + a \circ s_k \\ &= \underbrace{b + a \circ s_k}_{=v+\text{cont}(f(T))}_{\text{(by (3))}} + \rho = v + \text{cont}(f(T)) + \rho, \end{aligned}$$

which proves (1). As we explained above, this completes the proof of Claim 4. $\square$

Proof of Claim 5. Assume that $f(T) = T$. But Claim 4 tells us that

$$a_{v+\text{cont} T+\rho} + a_{v+\text{cont}(f(T))+\rho} = 0.$$

In view of $f(T) = T$, this simplifies to $a_{v+\text{cont} T+\rho} + a_{v+\text{cont} T+\rho} = 0$. That is, $2a_{v+\text{cont} T+\rho} = 0$. If $K = \mathbb{Z}$, then this immediately implies that $a_{v+\text{cont} T+\rho} = 0$, since we can cancel the factor 2 in a polynomial ring over $\mathbb{Z}$. So at least

we have proved the equality $a_{\nu+\text{cont } T+\rho} = 0$ for $K = \mathbb{Z}$. Now, applying the coefficientwise ring morphism $\mathcal{P}_{\mathbb{Z}} \rightarrow \mathcal{P}_K$ from Proposition 1.4.1 (b) proves the same equality over any K (since this morphism sends the α -alternant a_α defined over $\mathbb{Z}$ to the analogous α -alternant a_α defined over K for any $\alpha \in \mathbb{N}^N$); thus, Claim 5 is proved. Note that this argument uses functoriality (in the form of the morphism $\mathcal{P}_{\mathbb{Z}} \rightarrow \mathcal{P}_K$) to avoid the need for cancelling 2 in $\mathcal{P}_K$; roughly speaking, it transfers this cancellation to the polynomial ring $\mathcal{P}_{\mathbb{Z}}$, where it can be easily justified.

Alternatively, we can argue as follows: In the proof of Claim 4, we proved the equality (1). Since $f(T) = T$, this equality rewrites as

$$\nu + \text{cont } T + \rho = (\nu + \text{cont } T + \rho) \circ s_k.$$

This means that the tuple $\nu + \text{cont } T + \rho$ has equal entries in positions k and $k + 1$. Therefore, Lemma 1.7.9 (a) gives $a_{\nu+\text{cont } T+\rho} = 0$. So Claim 5 is proved again. This argument works directly over any K . $\square$

Example 1.7.12. Fixed points of f really can occur. Let $N = 2$, $\nu = (0, 0)$, $\lambda = (2, 1)$ and $\mu = \emptyset$, and consider the tableau

$$T = \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 2 & \\ \hline \end{array}.$$

Its largest violator is $j = 2$, since the last column has content $(0, 1)$, whereas every column restriction starting at 3 or later is empty. Thus, $b = (0, 1)$ and $k = 1$. In the first column, the 1 and 2 are matched, so β_1 leaves that column unchanged. Consequently, $f(T) = T$. Just as Claim 5 predicts, the corresponding summand $a_{\nu+\text{cont } T+\rho}$ is zero: since $\text{cont } T = (1, 2)$ and $\rho = (1, 0)$, we have

$$a_{\nu+\text{cont } T+\rho} = a_{(2,2)} = \det \begin{pmatrix} x_1^2 & x_1^2 \\ x_2^2 & x_2^2 \end{pmatrix} = 0.$$

We can now complete the cancellation. The set $\mathcal{X}$ is finite, since its tableaux have a fixed finite shape and entries in $[N]$. By Claim 3, it splits into one-element and two-element orbits of f . A one-element orbit contributes 0 to the sum $\sum_{T \in \mathcal{X}} a_{\nu+\text{cont } T+\rho}$ by Claim 5, and a two-element orbit contributes 0 by Claim 4. Hence,

$$\sum_{T \in \mathcal{X}} a_{\nu+\text{cont } T+\rho} = 0.$$

But we proved in Lecture 8 that

$$a_{\nu+\rho} \cdot s_{\lambda/\mu} = \sum_{T \in \text{SSYT}(\lambda/\mu, N)} a_{\nu+\text{cont } T+\rho}.$$

Splitting this sum into two parts, according to whether T is ν -Yamanouchi or not, we find

$$a_{\nu+\rho} \cdot s_{\lambda/\mu} = \sum_{\substack{T \in \text{SSYT}(\lambda/\mu, N); \\ T \text{ is } \nu\text{-Yamanouchi}}} a_{\nu+\text{cont } T+\rho} + \sum_{\substack{T \in \text{SSYT}(\lambda/\mu, N); \\ T \text{ is not } \nu\text{-Yamanouchi}}} a_{\nu+\text{cont } T+\rho}.$$

Since

$$\begin{aligned} \sum_{\substack{T \in \text{SSYT}(\lambda/\mu, N); \\ T \text{ is not } \nu\text{-Yamanouchi}}} a_{\nu+\text{cont } T+\rho} &= \sum_{T \in \mathcal{X}} a_{\nu+\text{cont } T+\rho} \quad (\text{by the definition of } \mathcal{X}) \\ &= \sum_{\substack{T \in \mathcal{X}; \\ f(T)=T}} \underbrace{a_{\nu+\text{cont } T+\rho}}_{=0 \text{ (by Claim 5)}} + \sum_{\substack{T \in \mathcal{X}; \\ f(T) \neq T}} a_{\nu+\text{cont } T+\rho} \\ &= \sum_{\substack{T \in \mathcal{X}; \\ f(T) \neq T}} a_{\nu+\text{cont } T+\rho} = 0 \end{aligned}$$

(because the tableaux indexing this sum can be split into unordered pairs $\{T, f(T)\}$, and each pair contributes two addends $a_{\nu+\text{cont } T+\rho}$ and $a_{\nu+\text{cont}(f(T))+\rho}$ that sum to 0 by Claim 4), this rewrites as

$$a_{\nu+\rho} \cdot s_{\lambda/\mu} = \sum_{\substack{T \in \text{SSYT}(\lambda/\mu, N); \\ T \text{ is } \nu\text{-Yamanouchi}}} a_{\nu+\text{cont } T+\rho} + 0 = \sum_{\substack{T \in \text{SSYT}(\lambda/\mu, N); \\ T \text{ is } \nu\text{-Yamanouchi}}} a_{\nu+\text{cont } T+\rho}.$$

This proves Lemma 1.7.8 (and thus completes the proofs of Theorem 1.7.6 and Theorem 1.5.22 (b) that relied on it). $\square$

1.8. Intermezzo: Antisymmetric polynomials

Antisymmetric polynomials are an analogue of symmetric polynomials, much as antisymmetric matrices are analogous to symmetric matrices. Let us define them:

Definition 1.8.1. A polynomial $f \in \mathcal{P}$ is said to be **antisymmetric** if it satisfies

$$\sigma \cdot f = (-1)^\sigma \cdot f \quad \text{for all } \sigma \in S_N.$$

Here, $(-1)^\sigma$ means the sign of the permutation σ .

Again, this condition is equivalent to requiring that $s_k \cdot f = -f$ for all $k \in [N-1]$ (for a similar reason as Lemma 1.2.9 in Lecture 3).

A standard example of an antisymmetric polynomial is the Vandermonde determinant $a_\rho = \prod_{i < j} (x_i - x_j)$. But beware: Over a field of characteristic 2, antisymmetric polynomials are the same as symmetric ones, since $-1 = 1$. Our

definition of antisymmetric polynomials is not well-adapted to “characteristic-2” behavior. This can be fixed by switching to a more careful definition (see Remark 1.8.5 below), but we shall instead fix it by stating our main result (Theorem 1.8.4) under the condition that we are not in characteristic 2 (or, more precisely, that 2 is a regular element of our base ring K). We will mostly need the case $K = \mathbb{Z}$ anyway.

Definition 1.8.2. Let $\mathcal{A}$ be the K -module of all antisymmetric polynomials $f \in \mathcal{P}$. (In general, $\mathcal{A}$ is not a subring of $\mathcal{P}$: the product of two antisymmetric polynomials is symmetric, since the two signs cancel. But it is obviously a K -submodule.)

Every alternant a_α belongs to $\mathcal{A}$: Indeed, permuting the variables $x_1, x_2, \dots, x_N$ permutes the rows of the determinant $\det \left(\left(x_i^{\alpha_j} \right)_{i,j \in [N]} \right)$ defining a_α , and hence multiplies it by the sign of the permutation.

Example 1.8.3. Let $N = 2$, and write x, y for x_1, x_2 .

- (a) Over $\mathbb{Z}$, the polynomial $x - y$ is antisymmetric, but $(x - y)^2$ is not (swapping x and y leaves this nonzero square unchanged, rather than multiplying it by -1).
- (b) Over $K = \mathbb{Z}/4\mathbb{Z}$, the nonzero constant polynomial 2 is antisymmetric, because $2 = -2$ in K . Yet it is not a K -linear combination of alternants a_α ; nor is it divisible by $a_\rho = x - y$. This explains why we require the regularity of 2 in Theorem 1.8.4 below (at least in parts (a), (b) and (c); part (d) does not actually need this requirement).

We can now state the antisymmetric analogue of Theorem 1.3.10:

Theorem 1.8.4. Assume that 2 is regular in K (that is, any $u \in K$ satisfying $2u = 0$ must itself be 0). Then:

- (a) The family $(a_\alpha)_{\alpha \in \mathbb{N}^N}$ is strictly decreasing is a basis of the K -module $\mathcal{A}$.
- (b) Each antisymmetric polynomial $f \in \mathcal{A}$ satisfies

$$f = \sum_{\substack{\alpha = (\alpha_1, \alpha_2, \dots, \alpha_N) \in \mathbb{N}^N \\ \text{is strictly decreasing}}} \underbrace{([x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_N^{\alpha_N}] f)}_{\text{the } x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_N^{\alpha_N}\text{-coefficient of } f} a_\alpha.$$

Or, to put this more compactly:

$$f = \sum_{\substack{\alpha \in \mathbb{N}^N \\ \text{is strictly decreasing}}} ([x^\alpha] f) a_\alpha. \quad (4)$$

(c) The map

$$\begin{aligned} \mathcal{S} &\rightarrow \mathcal{A}, \\ f &\mapsto a_\rho f \end{aligned}$$

is a K -module isomorphism. (In particular, $\mathcal{A} = a_\rho \mathcal{S}$.)

(d) The family $(s_\lambda)_\lambda$ is an N -partition is a basis of the K -module $\mathcal{S}$.

Remark 1.8.5. Part (d) of Theorem 1.8.4 holds even if we don't assume anything about K (and we will prove this later). For parts (a), (b) and (c), if we want K to be arbitrary, we need to adapt the definition of “antisymmetric” to additionally require that f becomes 0 whenever two of the N variables are set equal. Antisymmetric polynomials f that satisfy this additional requirement are called *alternating*; if we replace $\mathcal{A}$ by the K -module of alternating polynomials, then parts (a), (b) and (c) of Theorem 1.8.4 still hold (see [LLPT, (SYM.6.1), (SYM.6.6) and (SYM.6.7)]). We will not need these subtleties, since the case most important to us is the case $K = \mathbb{Z}$; almost all combinatorially defined symmetric polynomials are defined over $\mathbb{Z}$ and can be mapped into $\mathcal{P}_K$ for any K by functoriality.

Parts (a) and (b) of Theorem 1.8.4 are analogues of respective parts of Theorem 1.3.10 in Lecture 4; their proofs are mostly similar, but somewhat complicated by signs and by monomials with equal exponents. Due to these complications, we shall give all the details. First, recall Definition 1.3.5 in Lecture 4. Note that if $a \in \mathbb{N}^N$ is an N -tuple whose entries are distinct, then $\text{sort } a$ is a **strictly** decreasing N -tuple. The following is an analogue of Lemma 1.3.6 (b) in Lecture 4:

Lemma 1.8.6. Assume that 2 is regular in K (that is, any $u \in K$ satisfying $2u = 0$ must itself be 0). Let $f \in \mathcal{P}$ be a polynomial. Then:

(a) The polynomial f is antisymmetric (i.e., belongs to $\mathcal{A}$) if and only if all $a \in \mathbb{N}^N$ satisfy

$$[x^a] f = (-1)^\sigma \cdot [x^{\text{sort } a}] f,$$

where $\sigma \in S_N$ is such that $\text{sort } a = a \circ \sigma$. (Here, $a \circ \sigma$ denotes the N -tuple $(a_{\sigma(1)}, a_{\sigma(2)}, \dots, a_{\sigma(N)})$, where $a = (a_1, a_2, \dots, a_N)$.)

(b) If f is antisymmetric, and if $a \in \mathbb{N}^N$ is an N -tuple with two equal entries, then

$$[x^a] f = 0.$$

Proof. TODO. □

Proof of Theorem 1.8.4. (a), (b) This is similar to proving that $(m_\lambda)_\lambda$ is an N -partition is a basis of $\mathcal{S}$ (see Theorem 1.3.10 (a) in Lecture 4). The only complication now is that signs appear, and we must rule out monomials with equal exponents.

Linear independence of $(a_\alpha)_{\alpha \in \mathbb{N}^N}$ is strictly decreasing is easy: Each

$$a_\alpha = \sum_{\sigma \in S_N} (-1)^\sigma \sigma \cdot x^\alpha$$

contains the monomial $x^\alpha = x_1^{\alpha_1} \cdots x_N^{\alpha_N}$ with coefficient 1 (nothing can cancel it because α is strictly decreasing, so every nonidentity permutation of its entries changes the tuple), and contains none of the x^β for other strictly decreasing N -tuples β . So any linear combination $\sum c_\alpha a_\alpha$ will “remember” the coefficient c_α as its x^α -coefficient; in particular, if it is 0, then all c_α ’s must be 0.

It remains to show that this family $(a_\alpha)_{\alpha \in \mathbb{N}^N}$ is strictly decreasing spans $\mathcal{A}$. So we fix an antisymmetric polynomial $f \in \mathcal{A}$, and we try to decompose it as a linear combination of $(a_\alpha)_{\alpha \in \mathbb{N}^N}$ is strictly decreasing. Namely, we shall show that

$$f = \sum_{\alpha \in \mathbb{N}^N \text{ is strictly decreasing}} ([x^\alpha] f) a_\alpha.$$

Only finitely many coefficients in this sum are nonzero, since f is a polynomial. Both sides are antisymmetric, because every a_α is antisymmetric. They agree in all coefficients of the strictly decreasing monomials (that is, monomials whose exponent tuples are strictly decreasing), by the coefficient calculation just used for linear independence.

Here is why this also gives equality of the coefficients of all monomials with distinct exponents. Each such monomial is $m = \sigma \cdot x^\alpha$ for some strictly decreasing α and some $\sigma \in S_N$. For any antisymmetric polynomial g , we have

$$[x^\alpha]g = [m](\sigma \cdot g) = (-1)^\sigma [m]g, \quad \text{so} \quad [m]g = (-1)^\sigma [x^\alpha]g.$$

Applying this to both sides proves that their m -coefficients agree.

It remains to handle those monomials whose exponent tuples have repeated entries, such as $x^2 y z^2$. The RHS of the above equality does not contain any such monomials, since none of these a_α ’s contains them. We must show that the LHS does not contain them either. In other words, we must show that if $\alpha \in \mathbb{N}^N$ has two equal entries – say, $\alpha_i = \alpha_j$ for $i < j$ – then $[x^\alpha] f = 0$.

But f is antisymmetric, so the transposition $t_{i,j} \in S_N$ that swaps i with j satisfies $t_{i,j} \cdot f = (-1)^{t_{i,j}} \cdot f = -f$. But this transposition $t_{i,j}$ does not change the tuple α (since $\alpha_i = \alpha_j$), and thus does not change the monomial x^α . So

$$[x^\alpha] f = [x^\alpha] \underbrace{(t_{i,j} \cdot f)}_{=-f} = [x^\alpha] (-f) = -[x^\alpha] f.$$

Therefore, $2[x^\alpha] f = 0$. Since 2 is regular in K , this entails $[x^\alpha] f = 0$. Thus all coefficients agree, which proves the desired expansion and completes part (a) and (b).

(c) Well-definedness: If $f \in \mathcal{S}$, then $a_\rho f \in \mathcal{A}$, since

$$\sigma \cdot (a_\rho f) = (\sigma \cdot a_\rho) (\sigma \cdot f) = (-1)^\sigma a_\rho f$$

for every $\sigma \in S_N$.

K -linearity is even more obvious.

Injectivity follows from the regularity of a_ρ , proved in Lecture 8 while deducing Theorem 1.7.6 from Stembridge's lemma. Recall the argument: each factor $x_i - x_j$ is monic in x_i , so multiplication by it is injective; hence multiplication by their product a_ρ is injective as well.

Surjectivity: Every $f \in \mathcal{A}$ is a linear combination of $(a_\alpha)_{\alpha \in \mathbb{N}^N}$ is strictly decreasing by part (a). But each strictly decreasing N -tuple α can be written uniquely as $\lambda + \rho$ for an N -partition λ (by Lemma 1.5.7), and furthermore we then have $a_\alpha = a_{\lambda+\rho} = a_\rho s_\lambda$ (by Theorem 1.5.22 (b)). So each a_α lies in the image of the map; therefore the map is surjective.

(d) The isomorphism from part (c) sends s_λ to $a_{\lambda+\rho}$. By Lemma 1.5.7, as λ runs through all N -partitions, $\lambda + \rho$ runs through all strictly decreasing N -tuples in $\mathbb{N}^N$, without repetitions. Thus, the s_λ are exactly the preimages of the basis in part (a), and hence form a basis of $\mathcal{S}$. $\square$

In particular, any symmetric polynomial defined over $\mathbb{Z}$ (for example, any product of known symmetric polynomials such as $h_5 p_7 e_3 s_{(3,2)/(1)}$) can be expanded in the Schur basis $(s_\lambda)_{\lambda \text{ is an } N\text{-partition}}$ – first over the ring $\mathbb{Z}$, then (by functoriality) over any K . This is called the **Schur expansion** of that polynomial. We will now try to find Schur expansions of many common symmetric polynomials.

Example 1.8.7. Let $N = 2$, $K = \mathbb{Z}$, and write x, y for x_1, x_2 . The antisymmetric polynomial

$$F = 2x^3 - x^2y + xy^2 - 2y^3$$

has the alternant expansion

$$F = 2a_{(3,0)} - a_{(2,1)},$$

since $a_{(3,0)} = x^3 - y^3$ and $a_{(2,1)} = x^2y - xy^2$. Notice that the coefficients 2 and -1 can be read off as $[x^3]F$ and $[x^2y]F$. Now $\rho = (1,0)$, so dividing by $a_\rho = x - y$ gives

$$2x^2 + xy + 2y^2 = 2s_{(2)} - s_{(1,1)}.$$

Indeed, $s_{(2)} = h_2 = x^2 + xy + y^2$ and $s_{(1,1)} = e_2 = xy$. Thus, a symmetric polynomial can have nonnegative monomial coefficients but a negative coefficient in its Schur expansion.

The easiest way to do this is by finding rules for multiplying a Schur polynomial by another polynomial – mainly by h_n , e_n or p_n .

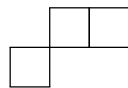
1.9. The Pieri rules

Definition 1.9.1. Let λ/μ be a skew partition, and let $n \in \mathbb{N}$.

- (a) We say that λ/μ is a **horizontal strip** if $Y(\lambda/\mu)$ has no two cells in the same column.
- (b) We say that λ/μ is a **vertical strip** if $Y(\lambda/\mu)$ has no two cells in the same row.
- (c) We say that λ/μ is a **horizontal n -strip** if λ/μ is a horizontal strip of size n (that is, $|Y(\lambda/\mu)| = n$).
- (d) We say that λ/μ is a **vertical n -strip** if λ/μ is a vertical strip of size n (that is, $|Y(\lambda/\mu)| = n$).

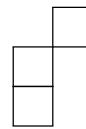
Example 1.9.2. All four skew partitions below have inner partition $\mu = (2, 1)$ and size 3:

$(4, 2)/(2, 1)$



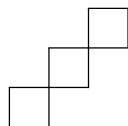
horizontal only

$(2, 2, 1, 1)/(2, 1)$



vertical only

$(3, 2, 1)/(2, 1)$



both horizontal and vertical

$(3, 3)/(2, 1)$



neither

In the first diagram, the occupied columns are 2, 3, 4, but two cells are in row 1. In the second, the occupied rows are 2, 3, 4, but two cells are in column 1. In the third, no row or column contains two cells. In the fourth, row 2 and column 3 each contain two cells.

In particular, a horizontal strip need not lie in a single row, and a vertical strip need not lie in a single column. The empty skew diagram is both a horizontal 0-strip and a vertical 0-strip.

Proposition 1.9.3. Let λ and μ be two N -partitions with $\mu \subseteq \lambda$. Then:

(a) The skew partition λ/μ is a horizontal strip if and only if

$$\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \cdots \geq \lambda_N \geq \mu_N.$$

(b) The skew partition λ/μ is a vertical strip if and only if

$$\mu_i \leq \lambda_i \leq \mu_i + 1 \quad \text{for each } i \in [N].$$

Proof. (a) Since $\mu \subseteq \lambda$, the displayed inequalities amount to $\lambda_{i+1} \leq \mu_i$ for every $i \in [N-1]$.

If $\lambda_{i+1} > \mu_i$ for some $i \in [N-1]$, then both cells $(i, \mu_i + 1)$ and $(i+1, \mu_i + 1)$ belong to $Y(\lambda/\mu)$: we use $\lambda_i \geq \lambda_{i+1} > \mu_i$ and $\mu_{i+1} \leq \mu_i$. Thus, the diagram has two cells in the same column, so it is not a horizontal strip.

Conversely, assume that $\lambda_{i+1} \leq \mu_i$ for all $i \in [N-1]$. If (r, t) and (s, t) were two cells of $Y(\lambda/\mu)$ in the same column, with $r < s$, then

$$\mu_r < t \leq \lambda_s \leq \lambda_{r+1} \leq \mu_r,$$

a contradiction. Hence the diagram is a horizontal strip.

(b) Row i of $Y(\lambda/\mu)$ has $\lambda_i - \mu_i$ cells. Thus, every row has at most one cell if and only if $0 \leq \lambda_i - \mu_i \leq 1$ for each $i \in [N]$, which is precisely the stated condition. $\square$

Theorem 1.9.4 (Pieri rules). Let $n \in \mathbb{N}$. Let μ be an N -partition. Then:

(a) We have

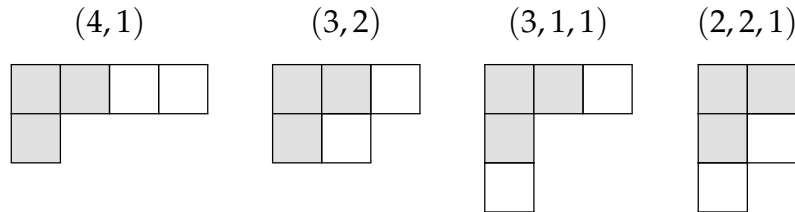
$$h_n s_\mu = \sum_{\substack{\lambda \text{ is an } N\text{-partition;} \\ \lambda/\mu \text{ is a horizontal } n\text{-strip}}} s_\lambda.$$

(b) We have

$$e_n s_\mu = \sum_{\substack{\lambda \text{ is an } N\text{-partition;} \\ \lambda/\mu \text{ is a vertical } n\text{-strip}}} s_\lambda.$$

Example 1.9.5. Let $N \geq 4$ and $\mu = (2, 1)$. We illustrate the Pieri rules for $n = 2$. In the diagrams below, the shaded cells belong to $Y(\mu)$ and the unshaded cells are the two added cells.

(a) The horizontal 2-strips give the four outer partitions

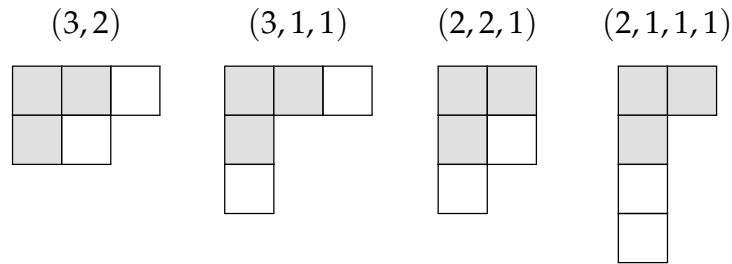


Thus, the first Pieri rule gives

$$h_2 s_{(2,1)} = s_{(4,1)} + s_{(3,2)} + s_{(3,1,1)} + s_{(2,2,1)}.$$

For instance, $(4,1)/(2,1)$ is allowed because its two new cells lie in columns 3 and 4, even though they lie in the same row.

(b) The vertical 2-strips give



Hence, the second Pieri rule gives

$$e_2 s_{(2,1)} = s_{(3,2)} + s_{(3,1,1)} + s_{(2,2,1)} + s_{(2,1,1,1)}.$$

Here, $(2,1,1,1)/(2,1)$ is allowed because its two new cells lie in distinct rows, even though both lie in column 1.

For $N = 3$, the horizontal expansion is unchanged, while the term $s_{(2,1,1,1)}$ disappears from the vertical expansion by Proposition 1.7.7.

References

[LLPT] Dan Laksov, Alain Lascoux, Piotr Pragacz and Anders Thorup, *The LLPT Notes*, edited by A. Thorup, 1995; version of 28 March 2018, with corrections and additions suggested by D. Grinberg.