

Math 701 Fall 2026, Lecture 8: Proof of the Littlewood–Richardson rule

website: <https://www.cip.ifi.lmu.de/~grinberg/t/26fs/>

(GPT-6 was used to create examples and to polish and detail some of the arguments below.)

In order to prove the Littlewood–Richardson rule, we will first show **Stembridge’s lemma**:

Lemma 1.7.8 (Stembridge’s lemma). Let λ/μ be a skew partition and let ν be an N -partition. Then,

$$a_{\nu+\rho} \cdot s_{\lambda/\mu} = \sum_{\substack{T \in \text{SSYT}(\lambda/\mu, N); \\ T \text{ is } \nu\text{-Yamanouchi}}} a_{\nu+\text{cont } T+\rho}.$$

This lemma, due to Stembridge [Stembr02], will easily yield both the Littlewood–Richardson rule (Theorem 1.7.6 in Lecture 7) and the alternant formula for Schur polynomials (Theorem 1.5.22 (b) in Lecture 5). Let me explain how. The proof of Lemma 1.7.8 will be given afterwards.

First, we recover the alternant formula:

Proof of Theorem 1.5.22 (b) using Lemma 1.7.8. Let λ be a partition of length $\leq N$. We identify λ with the N -partition $(\lambda_1, \lambda_2, \dots, \lambda_N)$ formed by its first N entries (since all its other entries are 0). Set $\mu = \emptyset$ and $\nu = \emptyset = (0, 0, \dots, 0)$ in Lemma 1.7.8. We obtain

$$a_{\rho} \cdot s_{\lambda} = \sum_{\substack{T \in \text{SSYT}(\lambda, N); \\ T \text{ is } \emptyset\text{-Yamanouchi}}} a_{\text{cont } T+\rho} \quad (1)$$

(since $s_{\lambda/\emptyset} = s_{\lambda}$ and $\text{SSYT}(\lambda/\emptyset, N) = \text{SSYT}(\lambda, N)$). But what are the $\emptyset$ -Yamanouchi tableaux $T \in \text{SSYT}(\lambda, N)$?

Let T_0 be the tableau of shape λ whose entries $T_0(i, j)$ are simply given by

$$T_0(i, j) = i \quad \text{for all } (i, j) \in Y(\lambda) \quad (2)$$

(that is, all entries of the i -th row of T_0 equal i for each i). For example, if $\lambda = (5, 3, 2)$, then

$$T_0 = \begin{array}{|c|c|c|c|c|} \hline 1 & 1 & 1 & 1 & 1 \\ \hline 2 & 2 & 2 & & \\ \hline 3 & 3 & & & \\ \hline \end{array}.$$

It is clear that the tableau T_0 is semistandard, and that its entries belong to $[N]$ (since $\ell(\lambda) \leq N$); hence, $T_0 \in \text{SSYT}(\lambda, N)$. We call this tableau T_0 the

minimalistic tableau of shape λ , because its entry in each cell is the smallest that it can be (indeed, it is easy to see that any SSYT T of shape λ must satisfy

$$T(i, j) \geq i \quad \text{for all } (i, j) \in Y(\lambda) \quad (3)$$

¹, and of course the tableau T_0 achieves equality in this inequality). Others call it the **highest-weight tableau**, since its content is lexicographically larger than the content of any other SSYT of shape λ with entries in $[N]$ (we will prove this later on).

The tableau T_0 has content $\text{cont } T_0 = \lambda$ (since the number of i 's in T_0 is the number of cells in the i -th row of $Y(\lambda)$, but the latter number is λ_i). It is furthermore easy to see that this tableau T_0 is $\emptyset$ -Yamanouchi (indeed, for each $j \geq 1$, the tableau $\text{col}_{\geq j}(T_0)$ has content $\text{cont}(\text{col}_{\geq j}(T_0)) = (\lambda_{1,j}, \lambda_{2,j}, \dots, \lambda_{N,j})$, where $\lambda_{i,j} := \max\{\lambda_i - j + 1, 0\}$; and this N -tuple is easily seen to be weakly decreasing because λ is). Now I claim:

Claim 1: The **only** $\emptyset$ -Yamanouchi tableau $T \in \text{SSYT}(\lambda, N)$ is T_0 .

Proof of Claim 1. Let T be any $\emptyset$ -Yamanouchi tableau in $\text{SSYT}(\lambda, N)$. We must show that $T = T_0$.

We can argue column-by-column (by induction, starting with the rightmost column) that all entries in each column must be $1, 2, \dots, h$, where h is the length of the column. Here are the details. Suppose that all columns strictly to the right of a given column j have already been shown to have this form. We must show that column j has this form as well.²

Let r be the length of column $j + 1$ (or 0 if column j is the rightmost column), and let h be the length of column j . Since the shape of T is straight, its columns get shorter as we move east; thus, $r \leq h$. For each $i \leq r$, we have $T(i, j) \geq i$ by (3), while weak increase along row i gives $T(i, j) \leq T(i, j + 1) = i$ (by induction). Thus, the first r entries in column j are $1, 2, \dots, r$.

¹*Proof.* Let T be an SSYT of shape λ . Let $(i, j) \in Y(\lambda)$. Since $Y(\lambda)$ is a straight Young diagram, all the i cells $(1, j), (2, j), \dots, (i, j)$ must belong to $Y(\lambda)$ as well. The entries in these cells are positive integers and satisfy $T(1, j) < T(2, j) < \dots < T(i, j)$ (since the entries of T strictly increase down each column). Thus, the entry $T(i, j)$ must have $i - 1$ distinct smaller entries above it in its same column; consequently, it must itself be $\geq i$. That is, $T(i, j) \geq i$, qed.

²Here is an example of our situation (column j is highlit in gray):

?	?	?	1	1	1
?	?	?	2	2	
?	?	?			
?	?				

It remains to prove³ that the remaining entries in column j must be $r+1, r+2, \dots, h$. Assume the contrary. Since these remaining entries of column j are all $\geq r+1$ (because of (3)) and are strictly increasing down this column (since T is semistandard), we conclude that they must skip one of the integers $r+1, r+2, \dots, h$; that is, there exists some $k > r$ such that k is not an entry of column j but $k+1$ is. Now, this k cannot appear in any of the columns $j+1, j+2, j+3, \dots$ either (since all entries in these columns are $\leq r$ by induction); hence, k does not appear in any of the columns $j, j+1, j+2, \dots$. In other words, the k -th entry of $\text{cont}(\text{col}_{\geq j} T)$ is 0. But the $(k+1)$ -st entry of $\text{cont}(\text{col}_{\geq j} T)$ is positive (since $k+1$ appears at least in column j). Therefore, the N -tuple $\text{cont}(\text{col}_{\geq j} T)$ cannot be weakly decreasing (since its k -th entry is 0, but its $(k+1)$ -st entry is positive). This contradicts the fact that $\text{cont}(\text{col}_{\geq j} T)$ is weakly decreasing (since T is $\emptyset$ -Yamanouchi).

This contradiction shows that our assumption was wrong. So the entries in column j must be $1, 2, \dots, h$. This proves the claim by induction.

So the entries of T are given by $T(i, j) = i$. In other words, $T = T_0$. This proves Claim 1. $\square$

Claim 1 yields

$$\sum_{\substack{T \in \text{SSYT}(\lambda, N); \\ T \text{ is } \emptyset\text{-Yamanouchi}}} a_{\text{cont } T + \rho} = a_{\lambda + \rho},$$

and thus our above equality (1) rewrites as $a_\rho \cdot s_\lambda = a_{\lambda + \rho}$. This proves Theorem 1.5.22 (b), assuming that Lemma 1.7.8 holds. $\square$

Proof of Theorem 1.7.6 using Lemma 1.7.8. We have assumed that Lemma 1.7.8 holds, i.e., we have

$$a_{\nu + \rho} \cdot s_{\lambda/\mu} = \sum_{\substack{T \in \text{SSYT}(\lambda/\mu, N); \\ T \text{ is } \nu\text{-Yamanouchi}}} a_{\nu + \text{cont } T + \rho}. \quad (4)$$

By Theorem 1.5.22 (b), we have $a_{\nu + \rho} = a_\rho \cdot s_\nu$ and $a_{\nu + \text{cont } T + \rho} = a_\rho \cdot s_{\nu + \text{cont } T}$ for any ν -Yamanouchi tableau $T \in \text{SSYT}(\lambda/\mu, N)$ (the latter because $\nu + \text{cont } T$ is an N -partition whenever T is ν -Yamanouchi). Thus, the equality (4) rewrites as

$$a_\rho \cdot s_\nu \cdot s_{\lambda/\mu} = \sum_{\substack{T \in \text{SSYT}(\lambda/\mu, N); \\ T \text{ is } \nu\text{-Yamanouchi}}} a_\rho \cdot s_{\nu + \text{cont } T}. \quad (5)$$

³Here is an example of our situation (column j is highlit in gray):

?	?	1	1	1	1
?	?	2	2	2	
?	?	?			
?	?				

Cancelling a_ρ would give us precisely the claim of Theorem 1.7.6.

Why can we cancel a_ρ ? Well, if K is an integral domain, then the polynomial ring $\mathcal{P} = K[x_1, x_2, \dots, x_N]$ is an integral domain as well, and the polynomial $a_\rho = \prod_{1 \leq i < j \leq N} (x_i - x_j)$ (the Vandermonde determinant) is nonzero, so it can be cancelled. In the general case, we can argue as follows:⁴

- Either argue that Theorem 1.7.6 is true for $K = \mathbb{Z}$ (by what we just said, since $\mathbb{Z}$ is an integral domain) and therefore for arbitrary K (since we know from Proposition 1.4.1 (b) that there is a functorial ring morphism $\mathcal{P}_{\mathbb{Z}} \rightarrow \mathcal{P}_K$ that sends skew Schur polynomials in $\mathcal{P}_{\mathbb{Z}}$ to skew Schur polynomials in $\mathcal{P}_K$; applying this morphism to the claim of Theorem 1.7.6 for $K = \mathbb{Z}$ yields the general case of Theorem 1.7.6).
- Or argue that the element $a_\rho = \prod_{1 \leq i < j \leq N} (x_i - x_j) \in \mathcal{P}$ is regular (i.e., multiplication by this element is injective as a map from $\mathcal{P}$ to $\mathcal{P}$) for any K . Indeed, each factor $x_i - x_j$ is monic as a polynomial in x_i over the ring of polynomials in the other variables. Multiplication by a monic polynomial is injective over any commutative ring (the leading nonzero coefficient of a nonzero polynomial stays nonzero in the product). Multiplication by the product of these factors $x_i - x_j$ is therefore injective as well. That is, multiplication by a_ρ is injective. Hence, we can cancel a_ρ from equalities such as (5).

Either way, Theorem 1.7.6 is proved provided that we can prove Lemma 1.7.8. $\square$

So it remains to prove Lemma 1.7.8. We prepare with a simple observation about alternants:

Lemma 1.7.9 (Alternants are alternating in α). Let $\alpha \in \mathbb{N}^N$ be an N -tuple.

- (a) If α has two equal entries, then $a_\alpha = 0$.
- (b) If β is an N -tuple obtained from α by swapping two entries, then $a_\beta = -a_\alpha$.
- (c) Let $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_N) \in \mathbb{N}^N$ be any N -tuple. Let $\sigma \in S_N$ be a permutation. As before, let $\alpha \circ \sigma$ denote the N -tuple $(\alpha_{\sigma(1)}, \alpha_{\sigma(2)}, \dots, \alpha_{\sigma(N)})$. Then, $a_{\alpha \circ \sigma} = (-1)^\sigma a_\alpha$, where $(-1)^\sigma$ denotes the sign of a permutation σ .

⁴See [21s, Lemma 7.3.38] for details.

Proof. Recall that $a_\alpha = \det \left(\left(x_i^{\alpha_j} \right)_{i,j \in [N]} \right)$, and use the fact that the determinant of a matrix is alternating in its columns. (See [21s, Lemma 7.3.39] for detailed proofs of parts **(a)** and **(b)**. Part **(c)** follows from [21s, second equality in Corollary 6.4.15].) $\square$

Beginning of the proof of Lemma 1.7.8.

This proof is due to Stembridge [Stembr02], although Gasharov gave a less elementary proof in 1998 based on some of the same ideas [Gashar98].

We will use the notation β_i for the i -th entry of any tuple β .

Recall that any $\sigma \in S_N$ and any $f, g \in \mathcal{P}$ satisfy

$$\sigma \cdot (fg) = (\sigma \cdot f) \cdot (\sigma \cdot g). \quad (6)$$

For any $\beta \in \mathbb{N}^N$, we have

$$\begin{aligned} a_\beta &= \det \left(\left(x_i^{\beta_j} \right)_{i,j \in [N]} \right) = \det \left(\left(x_j^{\beta_i} \right)_{i,j \in [N]} \right) \quad \left(\begin{array}{l} \text{since } \det(A^T) = \det A \\ \text{for any matrix } A \end{array} \right) \\ &= \sum_{\sigma \in S_N} \underbrace{(-1)^\sigma}_{\text{the sign of } \sigma} \underbrace{x_{\sigma(1)}^{\beta_1} x_{\sigma(2)}^{\beta_2} \cdots x_{\sigma(N)}^{\beta_N}}_{\substack{= \sigma \cdot (x_1^{\beta_1} x_2^{\beta_2} \cdots x_N^{\beta_N}) \\ = \sigma \cdot x^\beta}} \quad \left(\begin{array}{l} \text{by the definition of} \\ \text{a determinant} \end{array} \right) \\ &= \sum_{\sigma \in S_N} (-1)^\sigma \sigma \cdot x^\beta. \end{aligned} \quad (7)$$

Applying this to $\beta = \nu + \rho$, we find

$$a_{\nu+\rho} = \sum_{\sigma \in S_N} (-1)^\sigma \sigma \cdot x^{\nu+\rho}.$$

Multiplying this by $s_{\lambda/\mu}$, we find

$$\begin{aligned} a_{\nu+\rho} \cdot s_{\lambda/\mu} &= \sum_{\sigma \in S_N} (-1)^\sigma (\sigma \cdot x^{\nu+\rho}) \cdot \underbrace{s_{\lambda/\mu}}_{\substack{= \sigma \cdot s_{\lambda/\mu} \\ \text{(since } s_{\lambda/\mu} \text{ is symmetric)}}} \\ &= \sum_{\sigma \in S_N} (-1)^\sigma \underbrace{(\sigma \cdot x^{\nu+\rho}) \cdot (\sigma \cdot s_{\lambda/\mu})}_{\substack{= \sigma \cdot (x^{\nu+\rho} s_{\lambda/\mu}) \\ \text{(by (6))}}} \\ &= \sum_{\sigma \in S_N} (-1)^\sigma \sigma \cdot (x^{\nu+\rho} s_{\lambda/\mu}). \end{aligned}$$

Since

$$\begin{aligned} x^{\nu+\rho} s_{\lambda/\mu} &= x^{\nu+\rho} \sum_{T \in \text{SSYT}(\lambda/\mu, N)} x^{\text{cont } T} = \sum_{T \in \text{SSYT}(\lambda/\mu, N)} \underbrace{x^{\nu+\rho} x^{\text{cont } T}}_{= x^{\nu+\text{cont } T+\rho}} \\ &= \sum_{T \in \text{SSYT}(\lambda/\mu, N)} x^{\nu+\text{cont } T+\rho}, \end{aligned}$$

we can rewrite this as

$$\begin{aligned}
 a_{\nu+\rho} \cdot s_{\lambda/\mu} &= \sum_{\sigma \in S_N} (-1)^\sigma \sigma \cdot \sum_{T \in \text{SSYT}(\lambda/\mu, N)} x^{\nu+\text{cont } T+\rho} \\
 &= \sum_{T \in \text{SSYT}(\lambda/\mu, N)} \underbrace{\sum_{\sigma \in S_N} (-1)^\sigma \sigma \cdot x^{\nu+\text{cont } T+\rho}}_{\substack{= a_{\nu+\text{cont } T+\rho} \\ \text{(by (7), applied} \\ \text{backwards to } \beta=\nu+\text{cont } T+\rho)}} \\
 &= \sum_{T \in \text{SSYT}(\lambda/\mu, N)} a_{\nu+\text{cont } T+\rho}. \tag{8}
 \end{aligned}$$

This is **almost** the sum we want; we just want the right-hand side to be

$$\sum_{\substack{T \in \text{SSYT}(\lambda/\mu, N); \\ T \text{ is } \nu\text{-Yamanouchi}}} a_{\nu+\text{cont } T+\rho} \tag{9}$$

instead. All that remains is to prove that these two sums are equal, i.e., that the non- ν -Yamanouchi tableaux in the sum in (8) cancel out. To be precise, we must show that

$$\sum_{\substack{T \in \text{SSYT}(\lambda/\mu, N); \\ T \text{ is not } \nu\text{-Yamanouchi}}} a_{\nu+\text{cont } T+\rho} = 0. \tag{10}$$

But we will achieve this by cancelling addends that are equal-up-to-sign and throwing out addends that are zero.

The cancellation, as so often, is produced by a sign-reversing involution. Let $\mathcal{X}$ be the set of all $T \in \text{SSYT}(\lambda/\mu, N)$ that are not ν -Yamanouchi. We shall find an involution

$$f : \mathcal{X} \rightarrow \mathcal{X}$$

with the properties that

1. if a tableau $T \in \mathcal{X}$ satisfies $f(T) = T$, then $a_{\nu+\text{cont } T+\rho} = 0$;
2. for all $T \in \mathcal{X}$, we have $a_{\nu+\text{cont } T+\rho} + a_{\nu+\text{cont}(f(T))+\rho} = 0$.

If such an involution f is found, then the sum (10) will break up into addends that are simply 0 (the ones corresponding to T that satisfy $f(T) = T$) and pairs of addends that cancel one another (the ones corresponding to all the other T 's and their respective $f(T)$'s). As a consequence, the whole sum (10) will therefore be 0. (This argument is spelt out in more detail in [21s, end of proof of Lemma 7.3.34].)

To define our involution f , let us analyze the non- ν -Yamanouchi tableaux $T \in \mathcal{X}$ more closely. Let $T \in \mathcal{X}$ be such a tableau. Then, there exists at least one $j \geq 1$ such that $\nu + \text{cont}(\text{col}_{\geq j} T)$ is **not** an N -partition (since T is not ν -Yamanouchi). Any such j will be called a **violation** of T . Note that T has only

finitely many violators (since $\text{col}_{\geq j} T$ is empty for all sufficiently large j , so its content is the zero N -tuple and therefore $\nu + \text{cont}(\text{col}_{\geq j} T) = \nu$, which is an N -partition).

Let j be the **largest** violator of T . Define the two N -tuples

$$\begin{aligned} b &:= \nu + \text{cont}(\text{col}_{\geq j} T) && \text{(not an } N\text{-partition)} && \text{and} \\ c &:= \nu + \text{cont}(\text{col}_{\geq j+1} T) && \text{(an } N\text{-partition, since } j \text{ is largest).} \end{aligned}$$

Since b is not an N -partition, there exists a $k \in [N-1]$ such that $b_k < b_{k+1}$. Call such a k a **misstep** of T . Let k be the **smallest** misstep of T . Thus,

$$\begin{aligned} b_1 &\geq b_2 \geq \cdots \geq b_k < b_{k+1} && \text{whereas} \\ c_1 &\geq c_2 \geq \cdots \geq c_k \geq c_{k+1} \geq \cdots \geq c_N && \text{(since } c \text{ is an } N\text{-partition).} \end{aligned}$$

Example 1.7.10. Let $N = 4$, $\nu = (1, 1, 0, 0)$, $\lambda = (4, 4)$ and $\mu = \emptyset$. Consider the SSYT

$$T = \begin{array}{|c|c|c|c|} \hline 1 & 2 & 2 & 3 \\ \hline 2 & 3 & 4 & 4 \\ \hline \end{array}.$$

For each positive integer r , write

$$C_r := \nu + \text{cont}(\text{col}_{\geq r} T).$$

Starting with ν and adding the columns of T from right to left, we get

$$\begin{array}{c|cccc} r & \geq 5 & 4 & 3 & 2 & 1 \\ \hline C_r & (1, 1, 0, 0) & (1, 1, 1, 1) & \mathbf{(1, 2, 1, 2)} & \mathbf{(1, 3, 2, 2)} & \mathbf{(2, 4, 2, 2)} \end{array}$$

(the boldfaced tuples are not weakly decreasing). Thus, the violators of T are 1, 2, 3, and we choose the largest one: $j = 3$. This gives

$$b = C_3 = (1, 2, 1, 2), \quad c = C_4 = (1, 1, 1, 1).$$

Now $b_1 < b_2$ and $b_3 < b_4$, whereas $b_2 \geq b_3$. Thus, the missteps are 1 and 3, and we choose the smallest one: $k = 1$. Notice the distinction: j is a column number in T , whereas k is an index of an inequality between consecutive entries of b .

We want to define $f(T)$ to be again a non- ν -Yamanouchi SSYT of shape λ/μ with the same largest violator j and the same smallest misstep k , but we want it to satisfy $a_{\nu + \text{cont } T + \rho} + a_{\nu + \text{cont}(f(T)) + \rho} = 0$. The simplest way to achieve the latter equality is to have $\nu + \text{cont } T + \rho$ and $\nu + \text{cont}(f(T)) + \rho$ differ by a switch of two entries – let's even make it two consecutive entries, in order for the respective entries of ρ to differ only by 1.

It stands to reason that $f(T)$ should agree with T in all columns $j, j+1, j+2, \dots$. Indeed, if this is the case, then it is automatic that $f(T)$ still has largest

violator j and smallest misstep k (since $\nu + \text{cont}(\text{col}_{\geq i} T)$ remains unchanged from T to $f(T)$ for all $i \geq j$, and therefore j remains the largest violator; moreover, k remains the smallest misstep, since $\nu + \text{cont}(\text{col}_{\geq j} T)$ remains unchanged). So let us try to define the tableau $f(T)$ in such a way that it differs from T only in the first $j-1$ columns.

Let $\text{col}_{< j} T$ be the tableau obtained from T by removing all the columns $j, j+1, j+2, \dots$ (and leaving all remaining columns in their places). This is again an SSYT of skew shape (namely, of shape $\bar{\lambda}/\bar{\mu}$, where $\bar{\lambda}_i = \min\{j-1, \lambda_i\}$ and $\bar{\mu}_i = \min\{j-1, \mu_i\}$ for each i). (If $j=1$, this is the empty tableau.)

Before we define $f(T)$, let us make some observations. The N -tuples $c = \nu + \text{cont}(\text{col}_{\geq j+1} T)$ and $b = \nu + \text{cont}(\text{col}_{\geq j} T)$ differ only in the fact that the j -th column of T is counted in b but not in c . Hence, for each $i \in [N]$, we have

$$c_i \leq b_i \leq c_i + 1$$

(since the j -th column of T contains at most 1 copy of i). So, in particular,

$$c_k \leq b_k \leq c_k + 1 \quad \text{and} \quad c_{k+1} \leq b_{k+1} \leq c_{k+1} + 1.$$

But as we have seen above,

$$c_k \geq c_{k+1} \quad \text{and} \quad b_k < b_{k+1}.$$

These inequalities give the chain

$$c_k \leq b_k < b_{k+1} \leq c_{k+1} + 1 \leq c_k + 1 \quad (\text{since } c_k \geq c_{k+1}).$$

Since all entries here are integers, this entails that all $\leq$ signs must be $=$ signs (because there are no integers lying strictly between c_k and $c_k + 1$), and therefore

$$c_k = c_{k+1}, \quad b_k = c_k, \quad \text{and} \quad b_{k+1} = c_{k+1} + 1.$$

Hence, the j -th column of T has no k (since $b_k = c_k$) and one $k+1$ (since $b_{k+1} = c_{k+1} + 1$). Moreover,

$$b_{k+1} = \underbrace{c_{k+1}}_{=c_k=b_k} + 1 = b_k + 1.$$

Now, we define the tableau $f(T)$ by keeping columns $j, j+1, j+2, \dots$ of T unchanged while applying the Bender–Knuth involution β_k (from the proof of Theorem 1.6.14 in Lecture 7) to the remaining columns $1, 2, \dots, j-1$ (that is, replacing $\text{col}_{< j} T$ by $\beta_k(\text{col}_{< j} T)$). We must now prove five claims:

1. The resulting tableau $f(T)$ is still semistandard.
 2. This tableau $f(T)$ again belongs to $\mathcal{X}$.
 3. The resulting map f is an involution, i.e., we have $f(f(T)) = T$.
 4. We have $a_{\nu + \text{cont } T + \rho} + a_{\nu + \text{cont}(f(T)) + \rho} = 0$.
 5. If $f(T) = T$, then $a_{\nu + \text{cont } T + \rho} = 0$.
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Example 1.7.11. Continue Example 1.7.10, where $j = 3$ and $k = 1$. We apply β_1 only to the first two columns:

$$\text{col}_{<3} T = \begin{array}{|c|c|} \hline 1 & \mathbf{2} \\ \hline 2 & 3 \\ \hline \end{array} \mapsto \beta_1(\text{col}_{<3} T) = \begin{array}{|c|c|} \hline 1 & \mathbf{1} \\ \hline 2 & 3 \\ \hline \end{array}.$$

The 1 and 2 in the first column are matched and stay unchanged. The only free 1 or 2 in $\text{col}_{<j} T$ is the 2 in cell $(1, 2)$, so it becomes a 1. The entry 3 is unaffected. Restoring the last two columns, we obtain

$$T = \begin{array}{|c|c|c|c|} \hline 1 & \mathbf{2} & 2 & 3 \\ \hline 2 & 3 & 4 & 4 \\ \hline \end{array} \mapsto f(T) = \begin{array}{|c|c|c|c|} \hline 1 & \mathbf{1} & 2 & 3 \\ \hline 2 & 3 & 4 & 4 \\ \hline \end{array}.$$

The shaded columns are left unchanged; the boldfaced entry is the only entry that changes. Both tableaux are semistandard. Since columns 3 and 4 agree, the largest violator is still $j = 3$ and the smallest misstep is still $k = 1$. Applying the procedure again changes the free 1 back to 2, so $f(f(T)) = T$ in this example.

The other violators need not stay the same: for $f(T)$, the tallies at $r = 2$ and $r = 1$ are $(2, 2, 2, 2)$ and $(3, 3, 2, 2)$, respectively. Thus, $f(T)$ has only one violator, namely 3.

Finally, let us check the cancellation of the corresponding alternants. The contents are

$$\text{cont } T = (1, 3, 2, 2), \quad \text{cont}(f(T)) = (2, 2, 2, 2).$$

Since $\rho = (3, 2, 1, 0)$ and $\nu = (1, 1, 0, 0)$, we obtain

$$\begin{aligned} \nu + \text{cont } T + \rho &= (5, 6, 3, 2), \\ \nu + \text{cont}(f(T)) + \rho &= (6, 5, 3, 2). \end{aligned}$$

These tuples differ by swapping their first two entries. Hence, Lemma 1.7.9 (b) gives $a_{(5,6,3,2)} = -a_{(6,5,3,2)}$, and finally we have

$$a_{\nu + \text{cont } T + \rho} + a_{\nu + \text{cont}(f(T)) + \rho} = a_{(5,6,3,2)} + a_{(6,5,3,2)} = 0.$$

Proof of Claim 1. Assume the contrary. Thus, T is semistandard but $f(T)$ is not. If $j = 1$, then no entries change from T to $f(T)$, so this is clearly impossible. Thus, we must have $j > 1$. Since $f(T)$ is not semistandard, we must have $(f(T))(i, j-1) > (f(T))(i, j)$ for some i for which both cells $(i, j-1)$ and (i, j) belong to $Y(\lambda/\mu)$ (because all the other semistandardness inequalities for $f(T)$ are guaranteed either by the semistandardness of T or by the semistandardness of $\beta_k(\text{col}_{<j} T)$). But $T(i, j-1) \leq T(i, j)$ (since T is semistandard).

So this inequality $T(i, j-1) \leq T(i, j)$ must have flipped to $(f(T))(i, j-1) > (f(T))(i, j)$.

Since $(f(T))(i, j) = T(i, j)$ (because $f(T)$ only differs from T in the first $j-1$ columns), this can only happen if the $(i, j-1)$ -th entry has increased from T to $f(T)$. Since this entry is affected only by β_k applied to $\text{col}_{<j} T$, this means that it would have increased from k to $k+1$. That is, we must have $T(i, j-1) = k$ and $(f(T))(i, j-1) = k+1$. Combining the original row inequality with the assumed failure of the new one, we obtain

$$k = T(i, j-1) \leq T(i, j) = (f(T))(i, j) < (f(T))(i, j-1) = k+1.$$

Hence, $T(i, j) = k$, since this entry is an integer.

But the j -th column of T has no k (as we saw above), so $T(i, j) \neq k$. This contradiction shows that our assumption was false, and so Claim 1 is proved.

Proof of Claim 2. As we have said above, $f(T)$ agrees with T in all columns $j, j+1, j+2, \dots$. Thus, $f(T)$ still has largest violator j , hence in particular has a violator; that is, $f(T)$ is not ν -Yamanouchi. Therefore, $f(T) \in \mathcal{X}$ (since Claim 1 already shows that $f(T)$ is semistandard). This proves Claim 2.

We will finish the other three claims next time.

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