Math 701 Fall 2026, Lecture 2: The classical families: e, h, p, m

website: `https://www.cip.ifi.lmu.de/~grinberg/t/26fs/`

1. Symmetric polynomials

So we have seen that symmetric polynomials can be useful. What can we learn about them?

1.1. Definitions and examples

We begin by defining them.

Convention 1.1.1. We fix a commutative ring K and a nonnegative integer $N \in \mathbb{N}$.

Throughout this chapter, K and N will be fixed.

We let S_N denote the N-th symmetric group, i.e., the group of all permutations of $[N] = \{1, 2, \ldots, N\}$. (The notation $[n]$ stands for the set $\{1, 2, \ldots, n\}$ whenever n is an integer; in particular, $[n] = \varnothing$ whenever $n \leq 0$.)

Definition 1.1.2.

(a) Let $\mathcal{P}$ be the polynomial ring $K[x_1, x_2, \ldots, x_N]$. This is a commutative K-algebra.

(b) The symmetric group S_N acts on $\mathcal{P}$ by the formula

$$\sigma \cdot f = f\left(x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(N)}\right) \qquad \text{for any } \sigma \in S_N \text{ and } f \in \mathcal{P}.$$

In other words, $\sigma \cdot f$ is obtained from the polynomial f by substituting $x_{\sigma(i)}$ for each x_i (simultaneously). For example, if $N = 3$ and $f = x_1^2 + x_2 - x_3$, then

$$\underbrace{\mathrm{cyc}_{1,2,3}}_{\text{the cycle } 1 \to 2 \to 3 \to 1} \cdot f = f(x_2, x_3, x_1) = x_2^2 + x_3 - x_1.$$

So the group S_N acts on $\mathcal{P}$ by permuting the variables. (See [21s, Proposition 7.1.4 and Proposition 7.1.5] for a proof that this is an actual group action by algebra automorphisms, i.e., we have $\sigma \cdot (fg) = (\sigma \cdot f)(\sigma \cdot g)$ for all $\sigma \in S_N$ and $f, g \in \mathcal{P}$.)

(c) A polynomial $f \in \mathcal{P}$ is said to be **symmetric** if it satisfies

$$\sigma \cdot f = f \qquad \text{for all } \sigma \in S_N.$$

(d) We let $\mathcal{S}$ be the set of all symmetric polynomials $f \in \mathcal{P}$.

Example 1.1.3. Let $N = 3$ and $K = \mathbb{Q}$, and let us rename the variables x_1, x_2, x_3 as x, y, z for brevity (I will often do this for $N = 3$).

(a) We have $0, 1 \in \mathcal{S}$ and $x + y + z \in \mathcal{S}$ and $(x+1)(y+1)(z+1) \in \mathcal{S}$ and

$$x^2y + y^2z + z^2x + y^2x + z^2y + x^2z \in \mathcal{S}.$$

(b) We have $x + y \notin \mathcal{S}$ (since the transposition $t_{1,3} \in S_3$ sends $x + y$ to $z + y \neq x + y$).

(c) We have $(x-y)(y-z)(z-x) \notin \mathcal{S}$. This polynomial $(x-y)(y-z)(z-x)$ is an example of an **antisymmetric** polynomial, meaning a polynomial $f \in \mathcal{P}$ that satisfies $\sigma \cdot f = (-1)^\sigma f$ for each $\sigma \in S_N$, where $(-1)^\sigma$ denotes the sign of a permutation σ.

(d) We have $\dfrac{1}{(1-x)(1-y)(1-z)} \notin \mathcal{S}$; this is symmetric but not a polynomial. It is a symmetric power series.

Theorem 1.1.4. The subset $\mathcal{S}$ is a K-subalgebra of $\mathcal{P}$.

Proof. This is just saying that $\mathcal{S}$ is closed under addition, scaling and multiplication and contains 0 and 1. All of this is clear because S_N acts on $\mathcal{P}$ by algebra automorphisms. $\square$

Definition 1.1.5. The K-subalgebra $\mathcal{S}$ of $\mathcal{P}$ is called the **ring of symmetric polynomials in N variables over K.**

We now shall define some specific symmetric polynomials. First, some notations:

Definition 1.1.6.

(a) A **monomial** means an expression of the form $x_1^{a_1} x_2^{a_2} \cdots x_N^{a_N}$ with $a_1, a_2, \ldots, a_N \in \mathbb{N}$. (Recall that $0 \in \mathbb{N}$.)

(b) The **degree** $\deg \mathfrak{m}$ of a monomial $\mathfrak{m} = x_1^{a_1} x_2^{a_2} \cdots x_N^{a_N}$ is defined to be $a_1 + a_2 + \cdots + a_N$.

(c) A monomial $\mathfrak{m} = x_1^{a_1} x_2^{a_2} \cdots x_N^{a_N}$ is said to be **squarefree** if $a_1, a_2, \ldots, a_N \in \{0, 1\}$.

(d) A monomial $\mathfrak{m} = x_1^{a_1} x_2^{a_2} \cdots x_N^{a_N}$ is said to be **primal** if there is at most one $i \in [N]$ satisfying $a_i > 0$ (that is, if $\mathfrak{m}$ is 1 or some power of some x_i).

Definition 1.1.7.

(a) For each $n \in \mathbb{Z}$, define the n**-th elementary symmetric polynomial** $e_n \in \mathcal{S}$ by

$$\begin{aligned} e_n &= \sum_{\substack{(i_1,i_2,\ldots,i_n)\in[N]^n;\\ i_1<i_2<\cdots<i_n}} x_{i_1}x_{i_2}\cdots x_{i_n} \\ &= \sum_{\substack{L\subseteq[N];\\ |L|=n}} \prod_{i\in L} x_i \\ &= (\text{sum of all squarefree monomials of degree } n)\,. \end{aligned}$$

(b) For each $n \in \mathbb{Z}$, define the n**-th complete homogeneous symmetric polynomial** $h_n \in \mathcal{S}$ by

$$\begin{aligned} h_n &= \sum_{\substack{(i_1,i_2,\ldots,i_n)\in[N]^n;\\ i_1\leq i_2\leq\cdots\leq i_n}} x_{i_1}x_{i_2}\cdots x_{i_n} \\ &= \sum_{\substack{L \text{ is a multisubset of } [N];\\ |L|=n \text{ (counting multiplicities)}}} \prod_{\substack{i\in L\\ \text{(with multiplicities)}}} x_i \\ &\qquad \left(\begin{array}{c}\text{here we are using the language of multisets;}\\ \text{we shall mostly avoid this language}\end{array}\right) \\ &= (\text{sum of all monomials of degree } n)\,. \end{aligned}$$

(c) For each $n \in \mathbb{Z}$, define the n**-th power-sum symmetric polynomial** (aka n**-th power sum**) $p_n \in \mathcal{S}$ by

$$\begin{aligned} p_n &= \begin{cases} x_1^n + x_2^n + \cdots + x_N^n, & \text{if } n > 0;\\ 1, & \text{if } n = 0;\\ 0, & \text{if } n < 0 \end{cases} \\ &= (\text{sum of all primal monomials of degree } n)\,. \end{aligned}$$

Example 1.1.8.

(a) The 2-nd elementary symmetric polynomial is

$$\begin{aligned} e_2 = \sum_{\substack{(i_1,i_2)\in[N]^2;\\ i_1<i_2}} x_{i_1}x_{i_2} &= \sum_{\substack{(i,j)\in[N]^2;\\ i<j}} x_i x_j \\ = x_1x_2 + x_1x_3 + \cdots &+ x_1x_N \\ + x_2x_3 + \cdots &+ x_2x_N \\ \ddots \quad & \quad \vdots \\ &+ x_{N-1}x_N. \end{aligned}$$

(b) The 2-nd complete homogeneous symmetric polynomial is

$$\begin{aligned} h_2 &= \sum_{\substack{(i_1,i_2)\in[N]^2;\\ i_1\le i_2}} x_{i_1}x_{i_2} = \sum_{\substack{(i,j)\in[N]^2;\\ i\le j}} x_i x_j \\ &= x_1^2 + x_1x_2 + x_1x_3 + \cdots + x_1x_{N-1} + x_1x_N \\ &\quad + x_2^2 + x_2x_3 + \cdots + x_2x_{N-1} + x_2x_N \\ &\qquad\qquad \ddots \qquad \vdots \qquad\qquad \vdots \\ &\qquad\qquad + x_{N-1}^2 + x_{N-1}x_N \\ &\qquad\qquad\qquad + x_N^2. \end{aligned}$$

(c) The 2-nd power sum is

$$p_2 = x_1^2 + x_2^2 + \cdots + x_N^2.$$

Note that

$$h_2 = e_2 + p_2.$$

(d) We have

$$e_1 = h_1 = p_1 = x_1 + x_2 + \cdots + x_N.$$

(e) We have

$$e_0 = h_0 = p_0 = 1,$$

since the only monomial of degree 0 is 1.

(f) If $n < 0$, then $e_n = h_n = p_n = 0$, since there are no monomials of negative degree.

(g) If $N = 0$ (so $\mathcal{P}$ is a polynomial ring in 0 variables), then

$$e_n = h_n = p_n = 0 \qquad \text{for all } n > 0,$$

but $e_0 = h_0 = p_0 = 1$. (Note that polynomials in 0 variables are just constants in a trenchcoat.)

Proposition 1.1.9. For each integer $n > N$, we have $e_n = 0$.

Proof. We defined e_n as a sum over all n-tuples $(i_1, i_2, \ldots, i_n) \in [N]^n$ with $i_1 < i_2 < \cdots < i_n$. But there are no such n-tuples when $n > N$ (since any such n-tuple must have n distinct entries, but no n distinct entries exist in $[N]$). So the sum is empty and equals 0. $\square$

Thus, there are only N "interesting" elementary symmetric polynomials: $e_1, e_2, \ldots, e_N$. In contrast, there are infinitely many "interesting" complete homogeneous polynomials and power sums (when $N > 0$): For example, if $N = 2$, then

$$h_5 = x_1^5 + x_1^4 x_2 + x_1^3 x_2^2 + x_1^2 x_2^3 + x_1 x_2^4 + x_2^5 \qquad \text{and}$$
$$p_5 = x_1^5 + x_2^5.$$

1.2. First identities and generating functions

So we have three sequences of symmetric polynomials:

$$\underbrace{(e_0, e_1, e_2, \ldots)}_{\substack{\text{elementary}\\ \text{symmetric}\\ \text{polynomials}}}, \qquad \underbrace{(h_0, h_1, h_2, \ldots)}_{\substack{\text{complete}\\ \text{homogeneous}\\ \text{symmetric}\\ \text{polynomials}}}, \qquad \underbrace{(p_0, p_1, p_2, \ldots)}_{\text{power sums}}.$$

They are so fundamental and widely known that they are colloquially just called "the e's", "the h's" and "the p's" nowadays (with the caveat that sometimes these names are also used for the products of several entries of each sequence). In algebraic combinatorics, the notations e_n, h_n, p_n are completely standards, but outside of the field you will often see people refer to them by completely different (and oftentimes random) names.[1]

Our first nontrivial result are three formulas that connect these sequences recursively (the so-called **Newton–Girard identities**):

Theorem 1.2.1 (Newton–Girard identities)**.** For any positive integer n, we have

$$\sum_{j=0}^{n} (-1)^j e_j h_{n-j} = 0; \tag{1}$$

$$\sum_{j=1}^{n} (-1)^{j-1} e_{n-j} p_j = n e_n; \tag{2}$$

$$\sum_{j=1}^{n} h_{n-j} p_j = n h_n. \tag{3}$$

[1] Particularly the authors of algebra texts have a bad habit of denoting the e's by whatever yet-unused letter comes to their mind first.

Expanded, these are saying that

$$\underbrace{e_0}_{=1} h_n - e_1 h_{n-1} + e_2 h_{n-2} - e_3 h_{n-3} \pm \cdots + (-1)^n e_n \underbrace{h_0}_{=1} = 0;$$

$$e_{n-1} p_1 - e_{n-2} p_2 + e_{n-3} p_3 - e_{n-4} p_4 \pm \cdots + (-1)^{n-1} \underbrace{e_0}_{=1} p_n = n e_n;$$

$$h_{n-1} p_1 + h_{n-2} p_2 + h_{n-3} p_3 + h_{n-4} p_4 + \cdots + \underbrace{h_0}_{=1} p_n = n h_n.$$

Example 1.2.2. For $n = 2$, the formula (2) says

$$e_1 p_1 - \underbrace{e_0}_{=1} p_2 = 2 e_2,$$

so that

$$p_2 = e_1 \underbrace{p_1}_{=e_1} - 2e_2 = e_1^2 - 2e_2$$
$$= (x_1 + x_2 + \cdots + x_N)^2 - 2 \sum_{i<j} x_i x_j.$$

This is precisely the $e_1 \left(\lambda_1^2, \lambda_2^2, \ldots, \lambda_N^2\right)$ identity that we saw at the end of Lecture 1. Sadly, $e_2 \left(\lambda_1^2, \lambda_2^2, \ldots, \lambda_N^2\right)$ cannot be computed this easily.

How do we prove Theorem 1.2.1? I will only show the first identity, i.e., (1), leaving the other two to homework. To prove it, I will use power series in additional variables t, u, v. The crucial fact here is the following:

Proposition 1.2.3. In addition to the indeterminates x_i already in $\mathcal{P}$, we introduce three further indeterminates t, u, v. Then:

(a) In the polynomial ring $\mathcal{P}[t]$, we have

$$\prod_{i=1}^{N} (1 - t x_i) = \sum_{n \in \mathbb{N}} (-1)^n t^n e_n.$$

(b) In the polynomial ring $\mathcal{P}[u, v]$, we have

$$\prod_{i=1}^{N} (u - v x_i) = \sum_{n=0}^{N} (-1)^n u^{N-n} v^n e_n.$$

(c) In the formal power series ring $\mathcal{P}[[t]]$, we have

$$\prod_{i=1}^{N} \frac{1}{1 - t x_i} = \sum_{n \in \mathbb{N}} t^n h_n.$$

Proof. **(c)** For each $i \in [N]$, the geometric series formula $\dfrac{1}{1-y} = 1 + y + y^2 + y^3 + \cdots$ yields

$$\frac{1}{1-tx_i} = 1 + tx_i + (tx_i)^2 + (tx_i)^3 + \cdots = \sum_{a \in \mathbb{N}} (tx_i)^a .$$

Multiplying these equalities over all the i's yields

$$\begin{aligned}
\prod_{i=1}^{N} \frac{1}{1-tx_i} &= \prod_{i=1}^{N} \sum_{a \in \mathbb{N}} (tx_i)^a \\
&= \sum_{(a_1,a_2,\ldots,a_N) \in \mathbb{N}^N} \underbrace{(tx_1)^{a_1} (tx_2)^{a_2} \cdots (tx_N)^{a_N}}_{= t^{a_1+a_2+\cdots+a_N} x_1^{a_1} x_2^{a_2} \cdots x_N^{a_N}} \\
&\qquad \text{(by the product rule, aka generalized distributivity)} \\
&= \sum_{(a_1,a_2,\ldots,a_N) \in \mathbb{N}^N} t^{a_1+a_2+\cdots+a_N} x_1^{a_1} x_2^{a_2} \cdots x_N^{a_N} \\
&= \sum_{n \in \mathbb{N}} \sum_{\substack{(a_1,a_2,\ldots,a_N) \in \mathbb{N}^N; \\ a_1+a_2+\cdots+a_N = n}} \underbrace{t^{a_1+a_2+\cdots+a_N}}_{= t^n} x_1^{a_1} x_2^{a_2} \cdots x_N^{a_N} \\
&= \sum_{n \in \mathbb{N}} t^n \underbrace{\sum_{\substack{(a_1,a_2,\ldots,a_N) \in \mathbb{N}^N; \\ a_1+a_2+\cdots+a_N = n}} x_1^{a_1} x_2^{a_2} \cdots x_N^{a_N}}_{\substack{= (\text{sum of all monomials of degree } n) \\ = h_n}} \\
&= \sum_{n \in \mathbb{N}} t^n h_n,
\end{aligned}$$

which proves part **(c)**.

(a) Upon substituting $-t$ for t, the claim of part **(a)** takes the form

$$\prod_{i=1}^{N} (1 + tx_i) = \sum_{n \in \mathbb{N}} t^n e_n.$$

This can be shown similarly to part **(c)**, where instead of the geometric series formula we use the trivial formula $1 + y = \sum_{a \in \{0,1\}} y^a$. Thus, $1 + tx_i =$

$\sum_{a\in\{0,1\}} (tx_i)^a$ for each $i \in [N]$. Multiplying this over all the i's, we obtain

$$\begin{aligned}
\prod_{i=1}^{N} (1 + tx_i) &= \prod_{i=1}^{N} \sum_{a\in\{0,1\}} (tx_i)^a \\
&= \sum_{(a_1,a_2,\ldots,a_N)\in\{0,1\}^N} (tx_1)^{a_1} (tx_2)^{a_2} \cdots (tx_N)^{a_N} \\
&= \sum_{(a_1,a_2,\ldots,a_N)\in\{0,1\}^N} t^{a_1+a_2+\cdots+a_N} x_1^{a_1} x_2^{a_2} \cdots x_N^{a_N} \\
&= \sum_{n\in\mathbb{N}} \sum_{\substack{(a_1,a_2,\ldots,a_N)\in\{0,1\}^N;\\ a_1+a_2+\cdots+a_N=n}} \underbrace{t^{a_1+a_2+\cdots+a_N}}_{=t^n} x_1^{a_1} x_2^{a_2} \cdots x_N^{a_N} \\
&= \sum_{n\in\mathbb{N}} t^n \underbrace{\sum_{\substack{(a_1,a_2,\ldots,a_N)\in\{0,1\}^N;\\ a_1+a_2+\cdots+a_N=n}} x_1^{a_1} x_2^{a_2} \cdots x_N^{a_N}}_{\substack{=(\text{sum of all squarefree monomials of degree } n)\\ =e_n}} \\
&= \sum_{n\in\mathbb{N}} t^n e_n,
\end{aligned}$$

just as we wanted to prove.

(b) is similar to **(a)** (homework). □

Next time we will derive the first Newton–Girard formula from this.

References

[21s] Darij Grinberg, *An Introduction to Algebraic Combinatorics (Math 531, Winter 2024 lecture notes)*, 22 July 2026.
`http://www.cip.ifi.lmu.de/~grinberg/t/21s/lecs.pdf`