**Math 331, Fall 2026: homework set 1**
due date: Monday 5 October 2026 at 3:30 PM on Canvas.
Please solve **5** of the 10 exercises!

**Exercise 1.** For the following five binary operations $*$ on the set $\mathbb{Z}$, determine whether $(\mathbb{Z}, *)$ is a semigroup, a monoid, a group, or none of these:

**(a)** the operation $*$ given by $a * b = \max\{a, b\}$;

**(b)** the operation $*$ given by $a * b = a + b + 1$;

**(c)** the operation $*$ given by $a * b = |ab|$;

**(d)** the operation $*$ given by $a * b = ab + a + b + 1$;

**(e)** the operation $*$ given by $a * b = ab - a - b + 2$.

Proofs are not required in this exercise.

If $s$ is an element of a monoid $(S, *)$, and if $n$ is a nonnegative integer, then $s^n$ denotes the $n$-fold product $\underbrace{s * s * \cdots * s}_{n \text{ times } s}$. For $n = 0$, this is simply the neutral element of $(S, *)$. If $s$ is furthermore invertible, then $s^n$ is defined for negative integers $n$ as well, namely by setting $s^n = \left(s^{-1}\right)^{-n}$ when $n$ is negative.

**Exercise 2.** Let $(S, *)$ be a monoid with neutral element $e$. Let $s \in S$.

**(a)** Prove that if $s^3 = e$ and $s^5 = e$, then $s = e$.

**(b)** Prove that if $s^6 = e$ and $s^{10} = e$, then $s^2 = e$.

**(c)** Is it true that if $s^6 = e$ and $s^{10} = e$, then $s = e$ ?

**Exercise 3.** Let $(S, *)$ be a semigroup. Let $a, b \in S$ be such that $ab$ is central in $S$. Prove that

$$(ab)^n = a^n b^n \qquad \text{for all positive integers } n.$$

**Exercise 4.** Let $(S, *)$ be a monoid, and let $a \in S$ be an invertible element. Let $b \in S$.

**(a)** Prove that $\left(aba^{-1}\right)^k = ab^k a^{-1}$ for each nonnegative integer $k$.

**(b)** Now assume that $b$, too, is invertible. Prove that $\left(aba^{-1}\right)^k = ab^k a^{-1}$ holds for all integers $k$, including the negative ones.

**Exercise 5.** Consider a group $G = \{e, a, b, c, d\}$ with five elements, of which you only know that $ee = e$ and $aa = b$ and $ab = c$ and $ac = d$. Reconstruct the entire multiplication table of this group, i.e., compute all products $xy$ with $x, y \in G$.

| $xy$ | $y = e$ | $y = a$ | $y = b$ | $y = c$ | $y = d$ |
|---|---|---|---|---|---|
| $x = e$ | $e$ | | | | |
| $x = a$ | | $b$ | $c$ | $d$ | |
| $x = b$ | | | | | |
| $x = c$ | | | | | |
| $x = d$ | | | | | |

Just fill in this table; don't show your proof.

[You should not assume that $e$ is the neutral element just because I called it $e$. But it follows from $ee = e$ – do you see how?]

**Exercise 6.** The following is the multiplication table of a group $\{e, p, q, r, u, v\}$ with 6 elements:

| $xy$ | $y = e$ | $y = p$ | $y = q$ | $y = r$ | $y = u$ | $y = v$ |
|---|---|---|---|---|---|---|
| $x = e$ | $e$ | | | | | |
| $x = p$ | | $e$ | | | $q$ | |
| $x = q$ | | | $e$ | | | |
| $x = r$ | | | | | | |
| $x = u$ | | $r$ | | | $v$ | $e$ |
| $x = v$ | | | | | | |

Fill in the missing values. (Again, don't write down your argument.)

**Exercise 7.** Let $(S, *)$ be a nonempty semigroup. Assume that for each $a \in S$, the maps

$$\begin{aligned} L_a : S &\to S, \\ x &\mapsto ax \end{aligned}$$

and

$$\begin{aligned} R_a : S &\to S, \\ x &\mapsto xa \end{aligned}$$

are bijective. Prove that $(S, *)$ is a group.

[**Hint:** How would you construct a neutral element?]

A **submonoid** of a monoid $(S, *)$ is defined to be a subset $T$ of $S$ that contains the neutral element of $S$ and is closed under the operation $*$ (that is, satisfies $a * b \in T$ for all $a, b \in T$). For example, $\mathbb{N}$ is a submonoid of $(\mathbb{Z}, +)$. Note that a submonoid $T$ of a monoid $(S, *)$ always becomes a monoid itself if we restrict the operation $*$ to it (more precisely: to $T \times T$).

**Exercise 8.** Let $\mathbb{Z}_{\text{ev}}$ be the set of all even integers, and let $\mathbb{Z}_{\text{od}}$ be the set of all odd integers. Consider the two monoids $(\mathbb{Z}, +)$ and $(\mathbb{Z}, \cdot)$.

**(a)** Is $\mathbb{Z}_{\text{ev}}$ a submonoid of $(\mathbb{Z}, +)$ ?

**(b)** Is $\mathbb{Z}_{\text{od}}$ a submonoid of $(\mathbb{Z}, +)$ ?

**(c)** Is $\mathbb{Z}_{\text{ev}}$ a submonoid of $(\mathbb{Z}, \cdot)$ ?

**(d)** Is $\mathbb{Z}_{\text{od}}$ a submonoid of $(\mathbb{Z}, \cdot)$ ?

**(e)** Is $\{1\} \cup \mathbb{Z}_{\text{ev}}$ a submonoid of $(\mathbb{Z}, \cdot)$ ?

**Exercise 9.** Let $(S, *)$ be a monoid with neutral element $e$. Let $a \in S$ be any element. The **centralizer** of $a$ is defined to be the set $C_a$ of all $s \in S$ that commute with $a$ (that is, that satisfy $as = sa$).

**(a)** Prove that $C_a$ is a submonoid of $S$ (that is: $e \in C_a$ and $st \in C_a$ for all $s, t \in C_a$).

**(b)** Assume that $S$ is a group. Prove that $C_a$ is a subgroup of $S$ (that is: in addition to what was shown in part **(a)**, we also have $s^{-1} \in C_a$ for each $s \in C_a$).

**(c)** What is $C_e$?

Recall that a semigroup $(S, *)$ is said to be **abelian** if $ab = ba$ for all $a, b \in S$.

**Exercise 10.** Let $(S, *)$ be a monoid with neutral element $e$. Let $n$ be a positive integer. Define the set

$$T_n := \{s \in S \mid s^n = e\}.$$

**(a)** Prove that if $(S, *)$ is abelian, then $T_n$ is a submonoid of $S$.

**(b)** Prove that this does not necessarily hold when $(S, *)$ is not abelian.

[**Hint:** Try the group from Exercise 6 and $n = 2$.]