Drexel University, Department of Mathematics

Math 332: Undergraduate Abstract Algebra II, Winter 2025: Midterm 1

Please solve **at most 3 of the 6 problems!** **No collaboration** is allowed on the midterm.

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1 Exercise 1

1.1 Problem

(a) Is the map

$$\mathbb{Z}^{2\times 2} \to \mathbb{Z}^{2\times 2},$$
$$A \mapsto A^T$$

(which sends each 2×2-matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ to its transpose $A^T = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$) a ring morphism?

(b) Is the map

$$\mathbb{Z}^{2\times 2} \to \mathbb{Z}^{2\times 2},$$
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} d & c \\ b & a \end{pmatrix}$$

a ring morphism?

Keep in mind that claims should be proved.

1.2 Solution

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2 Exercise 2

2.1 Problem

Let R be a ring. Let a and b be two units of R such that $a+b$ is a unit as well.

(a) Prove that $a^{-1}+b^{-1}$, too, is a unit, and its inverse is

$$\left(a^{-1}+b^{-1}\right)^{-1} = a \cdot (a+b)^{-1} \cdot b = b \cdot (a+b)^{-1} \cdot a.$$

(b) Show on an example that $\left(a^{-1}+b^{-1}\right)^{-1}$ can differ from $ab \cdot (a+b)^{-1}$.

2.2 Solution

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3 Exercise 3

3.1 Problem

Let R be a ring. If A, B, C, D are four subsets of R, then the notation $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ shall denote the set of all 2×2-matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in R^{2\times 2}$ with $a \in A$, $b \in B$, $c \in C$ and $d \in D$. (For instance, $\begin{pmatrix} \mathbb{N} & 2\mathbb{Z} \\ 2\mathbb{Z} & \mathbb{N} \end{pmatrix}$ is the set of all 2×2-matrices whose diagonal entries are nonnegative integers and whose off-diagonal entries are even integers.)

(a) Let I be a subset of R. Prove that I is an ideal of R if and only if $\begin{pmatrix} R & I \\ \{0\} & R \end{pmatrix}$ is a subring of $R^{2\times 2}$.

(b) Does the same claim hold for $\begin{pmatrix} R & I \\ I & R \end{pmatrix}$ instead of $\begin{pmatrix} R & I \\ \{0\} & R \end{pmatrix}$?

(c) Does the same claim hold for $\begin{pmatrix} R & I \\ R & R \end{pmatrix}$ instead of $\begin{pmatrix} R & I \\ \{0\} & R \end{pmatrix}$?

(d) Does the same claim hold for $\begin{pmatrix} R & R \\ R & I \end{pmatrix}$ instead of $\begin{pmatrix} R & I \\ \{0\} & R \end{pmatrix}$?

For the sake of brevity, you need not give proofs for parts (b), (c) and (d).

3.2 Solution

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4 Exercise 4

4.1 Problem

Let R be a finite ring. Assume that its size $|R|$ is either a prime number p or a product pq of two (not necessarily distinct!) prime numbers p and q. Our goal is to show that R is commutative.

Consider the abelian group $(R, +, 0)$. If $u_1, u_2, \ldots, u_k$ are any elements of R, then $\langle u_1, u_2, \ldots, u_k \rangle$ shall denote the subgroup of this abelian group $(R, +, 0)$ generated by the elements $u_1, u_2, \ldots, u_k$. (Explicitly, this subgroup consists of all sums of the form $a_1 u_1 + a_2 u_2 + \cdots + a_k u_k$ with $a_1, a_2, \ldots, a_k \in \mathbb{Z}$.)

Let $x, y \in R$. Consider the following chain of subgroups of $(R, +, 0)$:

$$0 \leq \langle 1 \rangle \leq \langle x, 1 \rangle \leq R.$$

(The symbol $\leq$ means "subgroup of".)

(a) Prove that at least one of the three "$\leq$" signs in this chain must be an "$=$" sign.

(b) Prove that $xy = yx$ if the first "$\leq$" sign is a "$=$" sign.

(c) Prove that $xy = yx$ if the second "$\leq$" sign is a "$=$" sign.

(d) Prove that $xy = yx$ if the third "$\leq$" sign is a "$=$" sign.

(e) Conclude that R is commutative.

4.2 Hint

In part **(a)**, recall Lagrange's theorem about subgroups, and observe that a number m of the form p or pq cannot have a nontrivial chain of three divisors $1 \mid d \mid e \mid m$. Parts **(b)**, **(c)** and **(d)** are easy in their own ways.

4.3 Solution

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5 Exercise 5

5.1 Problem

Consider the ring $\mathbb{R}^{2\times 2}$ of all 2×2-matrices with real entries.

Define two subsets $\mathcal{P}$ and $\mathcal{M}$ of $\mathbb{R}^{2\times 2}$ by

$$\mathcal{P} := \left\{ \begin{pmatrix} a & b \\ b & a \end{pmatrix} \mid a, b \in \mathbb{R} \right\} \qquad \text{and}$$
$$\mathcal{M} := \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \mid a, b \in \mathbb{R} \right\}.$$

(a) Show that $\mathcal{P}$ and $\mathcal{M}$ are commutative subrings of $\mathbb{R}^{2\times 2}$.

(b) Prove that $\mathcal{P}$ is not an integral domain.

(c) Prove that $\mathcal{M}$ is a field.

5.2 Solution

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6 Exercise 6

6.1 Problem

Let R be a ring. For any two subsets A and B of R, we define a subset $A + B$ of R by

$$A + B := \{a + b \mid a \in A \text{ and } b \in B\}.$$

Which of the following three claims are true? (Prove the true ones and give counterexamples to the false ones.)

(a) If I and J are two ideals of R, then $I + J$ is again an ideal of R.

(b) If A and B are two subrings of R, then $A + B$ is again a subring of R.

(c) If I is an ideal of R, and if S is a subring of R, then $S + I$ is a subring of R.

6.2 Solution

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References

[23wa] Darij Grinberg, *Math 332 Winter 2023 notes: An introduction to the algebra of rings and fields*, 17 January 2025. `https://www.cip.ifi.lmu.de/~grinberg/t/23wa/23wa.pdf`