

Math 332: Undergraduate Abstract Algebra II, Winter 2025: Homework 3

Please solve **at most 3 of the 6 problems!**

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1 EXERCISE 1

1.1 PROBLEM

Let R and S be two rings. Let $f : R \rightarrow S$ be a ring morphism.

Which of the following two claims are true? (Prove the true ones and give counterexamples to the false ones.)

- (a) If I is an ideal of R , then $f(I) := \{f(i) \mid i \in I\}$ is an ideal of S .
- (b) If K is an ideal of S , then $f^{-1}(K) := \{i \in I \mid f(i) \in K\}$ is an ideal of R .

1.2 SOLUTION

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2 EXERCISE 2

2.1 PROBLEM

Let R be any ring. Recall that $R^{n \leq n}$ denotes the ring of all upper-triangular $n \times n$ -matrices with entries in R . In particular,

$$R^{2 \leq 2} = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \mid a, b, d \in R \right\}.$$

Define four subsets I, J, K, L of $R^{2 \leq 2}$ by

$$\begin{aligned} I &:= \left\{ \begin{pmatrix} 0 & b \\ 0 & d \end{pmatrix} \mid b, d \in R \right\}; \\ J &:= \left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \mid a, b \in R \right\}; \\ K &:= \left\{ \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} \mid b \in R \right\}; \\ L &:= \left\{ \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \mid a, d \in R \right\}. \end{aligned}$$

Prove that I, J and K are ideals of $R^{2 \leq 2}$, but L is not (unless R is trivial).

Feel free to omit the proofs of the “contains zero” and “is closed under addition” ideal axioms. (These axioms are satisfied for all four of I, J, K, L for pretty obvious reasons.)

2.2 SOLUTION

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3 EXERCISE 3

3.1 PROBLEM

Let R be a ring. An element $a \in R$ is said to be *nilpotent* if there exists an $n \in \mathbb{N}$ such that $a^n = 0$. (For example, the residue class $\bar{6}$ in $\mathbb{Z}/8$ is nilpotent, since its 3-rd power is $\bar{0}$.)

- (a) If a and b are two nilpotent elements of R satisfying $ab = ba$, then prove that $a + b$ is nilpotent as well.
- (b) Find a counterexample to part (a) if we don't assume $ab = ba$.
- (c) Assume that the ring R is commutative. Let N be the set of all nilpotent elements of R . Prove that N is an ideal of R .

3.2 REMARK

The ideal N defined in part (c) is called the *nilradical* of R .

3.3 SOLUTION

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4 EXERCISE 4

4.1 PROBLEM

Let n be a positive integer with prime factorization $n = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$, where $p_1, p_2, \dots, p_k$ are distinct primes and $a_1, a_2, \dots, a_k$ are positive integers. Let R be the ring $\mathbb{Z}/n$. Prove that the nilradical¹ of R is the principal ideal $\overline{p_1 p_2 \cdots p_k} R$.

4.2 SOLUTION

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5 EXERCISE 5

5.1 PROBLEM

Let R be the ring of all real numbers of the form $a + b\sqrt{5}$ with $a, b \in \mathbb{Z}$. (You can take for granted that this is indeed a ring; the proof is analogous to the proof for $\mathbb{S}$ in [23wa, §2.1.2].)

- (a) Is the quotient ring $R/(2R)$ a field?
- (b) Is the quotient ring $R/(3R)$ a field?
- (c) Prove that the quotient ring $R/(5R)$ is not a field, and in fact the residue class $\overline{\sqrt{5}}$ in this quotient is nilpotent. (See Exercise 3 above for the definition of “nilpotent”.)

5.2 SOLUTION

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6 EXERCISE 6

6.1 PROBLEM

Let R be the ring $\mathbb{Z}[i]$ of Gaussian integers. Let $z = 1 + 2i \in R$.

¹See the remark after Exercise 3 for the definition of this notion.

- (a) Prove that the quotient ring $R/(zR)$ has exactly 5 elements, namely $\bar{0}$, $\bar{1}$, $\bar{2}$, $\bar{3}$, $\bar{4}$.
(This requires, in particular, showing that these 5 elements are distinct.)
- (b) Which of these five elements is $\overline{12 + 7i}$?

6.2 HINT

Proceed as in Winter 2023 Lecture 8. In particular, first show that $5 \in zR$.

6.3 SOLUTION

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REFERENCES

- [23wa] Darij Grinberg, *Math 332 Winter 2023 notes: An introduction to the algebra of rings and fields*, 17 January 2025. <https://www.cip.ifi.lmu.de/~grinberg/t/23wa/23wa.pdf>