

Math 332: Undergraduate Abstract Algebra II, Winter 2025: Homework 2

Please solve **at most 3 of the 6 problems!***

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1 EXERCISE 1

1.1 PROBLEM

Let R be a commutative ring. Prove the following:

- (a) If $2 \cdot 1_R = 0_R$, then the set of all idempotent elements $a \in R$ is a subring of R .
- (b) If $8 \cdot 1_R = 0_R$, then the set of all elements $a \in R$ satisfying $(1 - 2a)^2 = 1$ is a subring of R .
- (c) **(Optional, for extra points if you obtain something interesting:)** Are there similar statements for higher powers of 2? For instance, could $16 \cdot 1_R = 0_R$ entail that the set of all $a \in R$ satisfying $(1 - 4a)^2 = 1$ is a subring of R ?

*I recommend solving as many problems as you can and wish, but I will only grade and score 3 solutions per submission (and if you submit more, I get to pick which ones I grade).

Results stated in class, and claims of previous problems (even if you did not solve these previous problems), can be used without proof. For example, in solving Problem 5, you can use the result of Problem 2 without proof.

I expect approximately the level of detail that I give in class. Purely straightforward arguments (like checking the ring axioms for a direct product of rings) need not be spelled out; I only expect a note of their necessity.

1.2 REMARK

Examples of commutative rings R satisfying $2 \cdot 1_R = 0_R$ are $\mathbb{Z}/2$ or polynomial rings over $\mathbb{Z}/2$.

Examples of commutative rings R satisfying $8 \cdot 1_R = 0_R$ are $\mathbb{Z}/2$, $\mathbb{Z}/4$, $\mathbb{Z}/8$ and polynomial rings over any of these.

1.3 SOLUTION

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2 EXERCISE 2

2.1 PROBLEM

Let R be a ring. Let $a, b \in R$ be such that $a + b$ is central¹. Prove that $ab = ba$.

2.2 SOLUTION

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3 EXERCISE 3

3.1 PROBLEM

Let R be a ring. Let $a, b \in R$ be such that ab is central. Prove that $(ab)^n = a^n b^n$ for all $n \in \mathbb{N}$.

3.2 SOLUTION

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4 EXERCISE 4

4.1 PROBLEM

Let R be the matrix ring $\mathbb{R}^{2 \times 2}$. In this ring R , consider the two matrices

$$A := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad B := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

¹An element $c \in R$ is said to be *central* if it satisfies $cx = xc$ for all $x \in R$. See Subsection 2.3.5 of the notes for more.

- (a) Describe the centralizer² $Z_R(\{A, B\})$.
- (b) Describe the centralizer $Z_R(\{A + B\})$.
- (c) Describe the centralizer $Z_R(\{A - B\})$.

You **do not need to give proofs** in this exercise if your claims are correct.

4.2 SOLUTION

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5 EXERCISE 5

5.1 PROBLEM

In Exercise 2.3.6 in the notes, a commutative subring $\mathcal{F}$ of the matrix ring $\mathbb{Z}^{2 \times 2}$ was constructed. This ring $\mathcal{F}$ consists of all matrices of the form

$$\begin{pmatrix} b & a \\ a & a+b \end{pmatrix} \quad \text{with } a, b \in \mathbb{Z}.$$

(You can take all claims of Exercise 2.3.6 for granted; you can also find its solution on my website³.)

- (a) Prove that this ring $\mathcal{F}$ is an integral domain.
- (b) Let $\mathcal{F}_{\mathbb{Q}}$ be the ring defined just like $\mathcal{F}$, but using $\mathbb{Q}$ instead of $\mathbb{Z}$ (that is, by replacing “ $a, b \in \mathbb{Z}$ ” with “ $a, b \in \mathbb{Q}$ ” in the definition of $\mathcal{F}$). Is $\mathcal{F}_{\mathbb{Q}}$ an integral domain?
- (c) Let $\mathcal{F}_{\mathbb{R}}$ be the ring defined just like $\mathcal{F}$, but using $\mathbb{R}$ instead of $\mathbb{Z}$ (that is, by replacing “ $a, b \in \mathbb{Z}$ ” with “ $a, b \in \mathbb{R}$ ” in the definition of $\mathcal{F}$). Is $\mathcal{F}_{\mathbb{R}}$ an integral domain?

5.2 HINT

For part (a), argue that the determinant

$$\det \begin{pmatrix} b & a \\ a & a+b \end{pmatrix} = -a^2 + ab + b^2 = \frac{5b^2 - (2a - b)^2}{4}$$

of any nonzero matrix in $\mathcal{F}$ is nonzero, since $\sqrt{5}$ is irrational. Now recall that $\det(AB) = \det A \cdot \det B$ for any $A, B \in \mathbb{R}^{2 \times 2}$.

²See Subsection 2.3.5 in the notes for a definition.

³It's the solution to Exercise 5 in my Spring 2019 Math 4281 class, available at <https://www.cip.ifi.lmu.de/~grinberg/t/19s/mt2s.pdf>.

5.3 SOLUTION

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6 EXERCISE 6

6.1 PROBLEM

Let p be a prime such that $p > 3$. Prove that $2^{p-2} + 3^{p-2} + 6^{p-2} \equiv 1 \pmod{p}$.

6.2 HINT

First, show that $u^{p-2} = \frac{1}{u}$ for every nonzero $u \in \mathbb{Z}/p$. Then, recall the formula $\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}$.

6.3 SOLUTION

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