Five questions on symmetric group algebras

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slides:
http://www.cip.ifi.lmu.de/~grinberg/t/24s/japan25.pdf

The symmetric group algebra

- Fix an $n \in \mathbb{N}$ and a commutative ring $\mathbf{k}$.
- The symmetric group S_n (aka $\mathfrak{S}_n$) consists of the permutations of $[n] := \{1, 2, \ldots, n\}$.
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- **Example:** For $n = 3$, we have

$$(1 + s_1)(1 - s_1) = 1 + s_1 - s_1 - s_1^2 = 1 + s_1 - s_1 - 1 = 0;$$

$$(1 + s_2)(1 + s_1 + s_1 s_2) = 1 + s_2 + s_1 + s_2 s_1 + s_1 s_2 + s_2 s_1 s_2 = \sum_{w \in S_3} w.$$

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  Here, s_i is the simple transposition swapping i with $i+1$.
- $\mathcal{A}$ has been studied since the late 1890s (Alfred Young's "substitutional analysis"). Today I will show that many questions are still open.

- Some of the nicest elements of $\mathcal{A}$ are the **Young–Jucys–Murphy elements** (short: **YJM elements**)

$$\mathbf{m}_k := t_{1,k} + t_{2,k} + \cdots + t_{k-1,k} \qquad \text{for all } 1 \leq k \leq n,$$

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- **Theorem (easy exercise).** The YJM elements $\mathbf{m}_1, \mathbf{m}_2, \ldots, \mathbf{m}_n$ commute.

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- The subalgebra of $\mathcal{A}$ they generate is called the **Gelfand–Tsetlin subalgebra** GZ_n.
- **Theorem (Murphy, ca. 1980?).** If $\mathbf{k}$ is a field of characteristic 0, then GZ_n (as a $\mathbf{k}$-vector space) has dimension equal to

$$(\# \text{ of involutions in } S_n) = \sum_{\lambda \vdash n} (\# \text{ of standard Young tableaux of shape } \lambda)$$

  (OEIS sequence A000085).

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- For $\mathbf{k} = \mathbb{Q}$, it has a basis $\left(\mathbf{e}_{T,T}\right)_{\lambda \vdash n;\ T \text{ is a standard tableau of shape } \lambda}$ coming from the seminormal basis of $\mathbf{k}\left[S_n\right]$.

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- Questions 1, 1', 1" are true for $n \leq 6$.
- https://mathoverflow.net/questions/497831/

The Young–Jucys–Murphy elements, 3: Some results

- **Fact:** The $2^n n!$ products $\mathbf{m}_1^{a_1} \mathbf{m}_2^{a_2} \cdots \mathbf{m}_n^{a_n}$ with $a_i \leq 2(i-1)$ span GZ_n.

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- **Fact (Jucys–Murphy theorem):** The **symmetric** polynomials in $\mathbf{m}_1, \mathbf{m}_2, \ldots, \mathbf{m}_n$ form the center $Z(\mathbf{k}[S_n])$ of $\mathbf{k}[S_n]$. This center has basis

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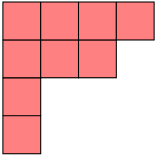
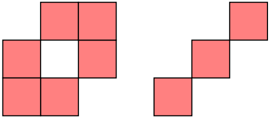
- **Fact (Olshanskii? Okounkov/Vershik?):** You don't need the YJM elements to define GZ_n. Two other characterizations:
  - GZ_n is the $\mathbf{k}$-subalgebra of $\mathbf{k}[S_n]$ generated by the centers of $\mathbf{k}[S_k]$ for all $k \in \{0, 1, \ldots, n\}$.
  - GZ_n is the $\mathbf{k}$-subalgebra of $\mathbf{k}[S_n]$ generated by the centralizers of $\mathbf{k}[S_{k-1}]$ in $\mathbf{k}[S_k]$ for all $k \in \{1, 2, \ldots, n\}$.

Specht modules: a quick introduction

- Let D be a diagram with n cells. For instance, for $n = 9$, we can have

$D =$  or $D =$  .

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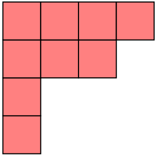
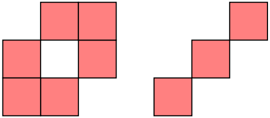
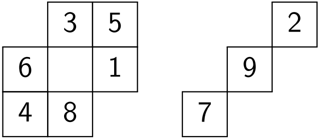
$D =$  or $D =$  .

- Let T be an n-**tableau of shape** D, that is, a filling of D with the numbers $1, 2, \ldots, n$. (Not necessarily standard, but bijective!)
  For example, if D is the second diagram above, we can have

$T =$ 

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- The **Specht module** $\mathcal{S}^D$ is the left ideal of $\mathcal{A}$ generated by

$$\left(\sum_{\substack{w \in S_n \text{ preserves} \\ \text{the columns of } T}} (-1)^w \, w \right) \left(\sum_{\substack{w \in S_n \text{ preserves} \\ \text{the rows of } T}} w \right).$$

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- **Question 2 (Rota, Buchsbaum, 1980s??):** Does $\mathcal{S}^D$ have a basis for arbitrary $\mathbf{k}$?
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- **Equivalent definitions of Specht modules #1:**
  We define a *n*-**tabloid** of shape D to be an equivalence class of n-tableaux under row equivalence (i.e., permuting numbers within rows of D). For instance,

$$n\text{-tableau } \begin{array}{|c|c|c|c|}\hline 3 & 2 & 7 & 5 \\ \hline 4 & 6 & 1 \\ \cline{1-3} 9 \\ \cline{1-1} 2 \\ \cline{1-1}\end{array} \quad \mapsto \quad n\text{-tabloid } \begin{array}{cccc}\hline 2 & 3 & 5 & 7 \\ \hline 1 & 4 & 6 \\ \cline{1-3} 9 \\ \cline{1-1} 2 \\ \cline{1-1}\end{array} .$$

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  We take the free $\mathbf{k}$-module with basis the n-tabloids of shape D. Inside it, we define the vector

$$\mathbf{e}_T := \sum_{\substack{w \in S_n \text{ preserves} \\ \text{the columns of } T}} (-1)^w \, \overline{wT}$$

  for each n-tableau T (where $\overline{wT}$ means the n-tabloid that is the equivalence class of wT). Then, the span of these $\mathbf{e}_T$'s is $\cong \mathcal{S}^D$ as S_n-representation.

Specht modules: alternative constructions, 2

- **Equivalent definitions of Specht modules #2:**
  For each n-tableau T of shape D with $\leq k$ columns (labelled $1, 2, \ldots, k$), we define the **Specht polynomial**

$$\mathrm{sp}_T := \prod_{c=1}^{k} \det \left(\begin{array}{c} x_i^j \text{ where } i \text{ runs over the entries} \\ \text{in the } c\text{-th column of } T \\ \text{while } j \text{ runs over the indexes} \\ \text{of the rows they occupy} \end{array} \right).$$

  For instance,

$$T = \begin{array}{|c|c|c|} \hline 2 & & \\ \hline 6 & & 4 \\ \hline 3 & 5 & 1 \\ \hline & 7 & \\ \hline \end{array} \quad \mapsto \quad \mathrm{sp}_T = \begin{vmatrix} x_2^0 & x_6^0 & x_3^0 \\ x_2^1 & x_6^1 & x_3^1 \\ x_2^2 & x_6^2 & x_3^2 \end{vmatrix} \cdot \begin{vmatrix} x_5^2 & x_7^2 \\ x_5^3 & x_7^3 \end{vmatrix} \cdot \begin{vmatrix} x_4^1 & x_1^1 \\ x_4^2 & x_1^2 \end{vmatrix}.$$

  Then, $\mathcal{S}^D \cong \mathrm{span}_{\mathbf{k}} \{ \mathrm{sp}_T \mid T \text{ is an } n\text{-tableau of shape } D \}$.

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- The general question of bases is wide open.
- The next question would be to decompose $\mathcal{S}^D$ into irreducibles ("Specht filtration").
- Same questions can be posed for Schur and Weyl modules (over GL_n), but not sure if still equivalent.

- Let $\mathbf{z}_n$ be the sum of all n-cycles in S_n. This is also the product of all nonzero Young–Jucys–Murphy elements:

$$\mathbf{z}_n = \mathbf{m}_2 \mathbf{m}_3 \cdots \mathbf{m}_n.$$

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- **Conjecture (Nathan Lindzey,** MathOverflow #334899**):** For $\mathbf{k} = \mathbb{Z}$, the quotient $\mathbb{Z}$-module $\mathcal{A}/\mathbf{z}_n\mathcal{A}$ is a direct sum of $\mathbb{Z}$'s and $\mathbb{Z}/2^k$'s.
Equivalently: All nonzero invariant factors of the "multiply by $\mathbf{z}_n$" operator $L(\mathbf{z}_n) = R(\mathbf{z}_n) : \mathcal{A} \to \mathcal{A}$ are powers of 2.

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- Verified for all $n \leq 7$.
- The rank of $\mathbf{z}_n$ (or, rather, of the operator $L(\mathbf{z}_n)$) is

$$\sum_{\lambda \vdash n \text{ is a hook}} \left(f^\lambda\right)^2 = \binom{2(n-1)}{n-1}.$$

- The next two questions are related to my new preprint *Rook sums in the symmetric group algebra* (arXiv:2507.22386).

- **Definition.** For any two subsets A and B of $[n]$, we define the elements

$$\nabla_{B,A} := \sum_{\substack{w \in S_n; \\ w(A)=B}} w \qquad \text{and} \qquad \widetilde{\nabla}_{B,A} := \sum_{\substack{w \in S_n; \\ w(A)\subseteq B}} w$$

of $\mathbf{k}[S_n] = \mathcal{A}$. We shall call these elements the **rectangular rook sums**.

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- **Examples.**

$$\nabla_{\varnothing,\varnothing} = \nabla_{[n],[n]} = (\text{sum of all } w \in S_n);$$
$$\nabla_{\{2\},\{1\}} = (\text{sum of all } w \in S_n \text{ sending } 1 \text{ to } 2);$$
$$\widetilde{\nabla}_{\{2,3\},\{1\}} = (\text{sum of all } w \in S_n \text{ sending } 1 \text{ to } 2 \text{ or } 3).$$

- **Proposition.** Let A and B be two subsets of $[n]$. Then:
  (a) We have $\nabla_{B,A} = 0$ if $|A| \neq |B|$.
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  Next, let $S : \mathbf{k}[S_n] \to \mathbf{k}[S_n]$ be the **antipode** of $\mathbf{k}[S_n]$; this is the $\mathbf{k}$-linear map sending each permutation $w \in S_n$ to w^{-1}. Then:
  (f) We have $S(\nabla_{B,A}) = \nabla_{A,B}$.
  (g) We have $S\left(\widetilde{\nabla}_{B,A}\right) = \widetilde{\nabla}_{[n] \setminus A,\ [n] \setminus B}$.

- The simplest rectangular rook sum is

$$\nabla_{\varnothing,\varnothing} = (\text{sum of all } w \in S_n).$$

Easily, $\nabla_{\varnothing,\varnothing}^2 = n!\nabla_{\varnothing,\varnothing}$, so that

$$P(\nabla_{\varnothing,\varnothing}) = 0 \qquad \text{for the polynomial } P(x) = x(x - n!).$$

Minimal polynomials: a question

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- **Question:** What polynomials P satisfy $P(\nabla_{B,A}) = 0$ or $P\left(\widetilde{\nabla}_{B,A}\right) = 0$ for arbitrary A, B ?
  In particular, what is the minimal polynomial of $\widetilde{\nabla}_{B,A}$? (The only interesting $\nabla_{B,A}$'s are those for $|A| = |B|$, and they agree with $\widetilde{\nabla}_{B,A}$, so that we need not study them separately.)

- **Example.** The minimal polynomial of $\widetilde{\nabla}_{\{2,4,5,6\},\ \{1,2\}}$ for $n = 6$ is $(x - 288)x(x + 12)(x + 36)$.

Minimal polynomials: experimental data

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- **Theorem (G. 2025).** The minimal polynomial of $\widetilde{\nabla}_{B,A}$ always splits over $\mathbb{Z}$ (i.e., factors into linear factors)!
- Also true for arbitrary linear combinations

$$\nabla_{B,\alpha} := \sum_{\substack{A \subseteq [n];\\ |A|=|B|}} \alpha_A \nabla_{B,A}.$$

A product rule

- A crucial step in the proof is a product rule for ∇s:
- **Theorem (product rule).** Let A, B, C, D be four subsets of $[n]$ such that $|A| = |B|$ and $|C| = |D|$. Then,

$$\nabla_{D,C} \nabla_{B,A} = \omega_{B,C} \sum_{\substack{U \subseteq D, \\ V \subseteq A; \\ |U| = |V|}} (-1)^{|U| - |B \cap C|} \binom{|U|}{|B \cap C|} \nabla_{U,V}.$$

Here, for any two subsets B and C of $[n]$, we set

$$\omega_{B,C} := |B \cap C|! \cdot |B \setminus C|! \cdot |C \setminus B|! \cdot |[n] \setminus (B \cup C)|! \in \mathbb{Z}.$$

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- **Proof.** Nice exercise in enumeration! First step is to show that

$$\nabla_{D,C} \nabla_{B,A} = \omega_{B,C} \sum_{\substack{w \in S_n; \\ |w(A) \cap D| = |B \cap C|}} w.$$

The formal Nabla-algebra: definition and conjecture

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  What happens if we create linearly independent "abstract ∇'s" (call them Δ's) and define their product using the product rule?
- **Definition.** For any two subsets A and B of $[n]$ satisfying $|A| = |B|$, introduce a formal symbol $\Delta_{B,A}$. Let $\mathcal{D}$ be the free $\mathbf{k}$-module with basis $(\Delta_{B,A})_{A,B\subseteq[n] \text{ with } |A|=|B|}$. Define a multiplication on $\mathcal{D}$ by

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- **Theorem.** This makes $\mathcal{D}$ into a nonunital $\mathbf{k}$-algebra.
- **Conjecture.** If $n!$ is invertible in $\mathbf{k}$, then this algebra $\mathcal{D}$ has a unity.

- **Example.** For $n = 1$, the nonunital algebra $\mathcal{D}$ has basis (u, v) with $u = \Delta_{\varnothing,\varnothing}$ and $v = \Delta_{\{1\},\{1\}}$, and multiplication

$$uu = uv = vu = u, \qquad vv = v.$$

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- **Example.** For $n = 2$, the nonunital algebra $\mathcal{D}$ has basis $(u, v_{11}, v_{12}, v_{21}, v_{22}, w)$ with $u = \Delta_{\varnothing,\varnothing}$ and $v_{ij} = \Delta_{\{i\},\{j\}}$ and $w = \Delta_{[2],[2]}$. The multiplication on $\mathcal{D}$ is

$$\begin{aligned}
uu = uw = wu &= 2u, \qquad & uv_{ij} = v_{ij}u = u,\\
v_{dc}v_{ba} &= u - v_{da} \qquad & \text{if } b \neq c;\\
v_{dc}v_{ba} &= v_{da} \qquad & \text{if } b = c,\\
v_{ij}w &= v_{i1} + v_{i2}, \qquad & wv_{ij} = v_{1j} + v_{2j},\\
ww &= 2w.
\end{aligned}$$

This nonunital $\mathbf{k}$-algebra $\mathcal{D}$ has a unity if and only if 2 is invertible in $\mathbf{k}$. This unity is $\dfrac{1}{4}(v_{11} + v_{22} - v_{12} - v_{21} + 2w)$.

The formal Nabla-algebra: questions

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- **Question.** Is $\mathcal{D}$ a known object?
- **Question.** Barring that, is there a nice proof of the above theorem?
- Over $\mathbf{k} = \mathbb{Q}$, we have

	$n = 2$	$n = 3$	$n = 4$
$\dim \mathcal{D}$	6	20	70
$\dim Z(\mathcal{D})$	3	4	5
$\dim J(\mathcal{D})$	3	5	39
CIs	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 2 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 2 & 0 \\ 0 & 0 & 1 & 0 & 2 \end{pmatrix}$

("CIs" = Cartan invariants).

Row-to-row sums

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- **Definition.** A **set composition** of $[n]$ is a tuple $\mathbf{U} = (U_1, U_2, \ldots, U_k)$ of disjoint nonempty subsets of $[n]$ such that $U_1 \cup U_2 \cup \cdots \cup U_k = [n]$. We set $\ell(\mathbf{U}) = k$ and call k the **length** of $\mathbf{U}$.

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- **Definition.** Let $\operatorname{SC}(n)$ be the set of all set compositions of $[n]$.
- **Definition.** If $\mathbf{A} = (A_1, A_2, \ldots, A_k)$ and $\mathbf{B} = (B_1, B_2, \ldots, B_k)$ are two set compositions of $[n]$ having the same length, then we define the **row-to-row sum**

$$\nabla_{\mathbf{B},\mathbf{A}} := \sum_{\substack{w \in S_n; \\ w(A_i) = B_i \text{ for all } i}} w \qquad \text{in } \mathbf{k}[S_n].$$

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- **Example.** We have

$$\nabla_{B,A} = \nabla_{\mathbf{B},\mathbf{A}} \qquad \text{for } \mathbf{B} = (B,\ [n] \setminus B) \text{ and } \mathbf{A} = (A,\ [n] \setminus A).$$

- **Proposition.** Let $\mathbf{A} = (A_1, A_2, \ldots, A_k)$ and $\mathbf{B} = (B_1, B_2, \ldots, B_k)$.
  - **(a)** We have $\nabla_{\mathbf{B},\mathbf{A}} = 0$ unless $|A_i| = |B_i|$ for all i.
  - **(b)** We have $\nabla_{\mathbf{B},\mathbf{A}} = \nabla_{\mathbf{B}\sigma,\mathbf{A}\sigma}$ for any $\sigma \in S_k$ (acting on set compositions by permuting the blocks).
  - **(c)** We have $S(\nabla_{\mathbf{B},\mathbf{A}}) = \nabla_{\mathbf{A},\mathbf{B}}$, where $S(w) = w^{-1}$ for all $w \in S_n$ as before.

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- The minimal polynomial of $\nabla_{\mathbf{B},\mathbf{A}}$ does not always split over $\mathbb{Z}$ unless $\ell(\mathbf{A}) \leq 2$.
- The $\nabla_{\mathbf{B},\mathbf{A}}$ are not entirely new:
  The **Murphy basis** of $\mathbf{k}[S_n]$ consists of the elements $\nabla_{\mathbf{B},\mathbf{A}}$ for the **standard** set compositions $\mathbf{A}$ and $\mathbf{B}$ of $[n]$. Here, "standard" means that the blocks are the rows of a standard Young tableau (in particular, they must be of partition shape).
  See G. E. Murphy, *On the Representation Theory of the Symmetric Groups and Associated Hecke Algebras*, 1991.

- **Theorem.** Let $\mathcal{A} = \mathbf{k}[S_n]$. Let $k \in \mathbb{N}$. We define two $\mathbf{k}$-submodules $\mathcal{I}_k$ and $\mathcal{J}_k$ of $\mathcal{A}$ by

$$\mathcal{I}_k := \operatorname{span}\{\nabla_{\mathbf{B},\mathbf{A}} \mid \mathbf{A}, \mathbf{B} \in \operatorname{SC}(n) \text{ with } \ell(\mathbf{A}) = \ell(\mathbf{B}) \leq k\}$$

and

$$\mathcal{J}_k := \mathcal{A} \cdot \operatorname{span}\left\{\boldsymbol{\alpha}_U^- \mid U \subseteq [n] \text{ of size } k+1\right\} \cdot \mathcal{A},$$

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Then:

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Then:

(a) Both $\mathcal{I}_k$ and $\mathcal{J}_k$ are ideals of $\mathcal{A}$, and are preserved under S.

- **Theorem (cont'd).**

  (b) We have

$$\mathcal{I}_k = \mathcal{J}_k^\perp = \operatorname{LAnn} \mathcal{J}_k = \operatorname{RAnn} \mathcal{J}_k \qquad \text{and}$$
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  Here, $\mathcal{U}^\perp$ means orthogonal complement wrt the standard bilinear form on $\mathcal{A}$, whereas LAnn and RAnn mean left and right annihilators.

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  (c) The **k**-module $\mathcal{I}_k$ is free of rank $= \#$ of $(1, 2, \ldots, k+1)$-avoiding permutations in S_n.

  (d) The **k**-module $\mathcal{J}_k$ is free of rank $= \#$ of $(1, 2, \ldots, k+1)$-nonavoiding permutations in S_n.

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  (e) The quotients $\mathcal{A}/\mathcal{J}_k$ and $\mathcal{A}/\mathcal{I}_k$ are also free, with bases

$$\left(\overline{w}\right)_{w \in S_n \text{ avoids } (1,2,\ldots,k+1)} \ \text{(for } \mathcal{A}/\mathcal{J}_k\text{)} \qquad \text{and}$$
$$\left(\overline{w}\right)_{w \in S_n \text{ does not avoid } (1,2,\ldots,k+1)} \ \text{(for } \mathcal{A}/\mathcal{I}_k\text{)}.$$

- **Theorem (cont'd).**
  (f) If $n!$ is invertible in $\mathbf{k}$, then $\mathcal{A} = \mathcal{I}_k \oplus \mathcal{J}_k$ (internal direct sum) as $\mathbf{k}$-modules, and $\mathcal{A} \cong \mathcal{I}_k \times \mathcal{J}_k$ as $\mathbf{k}$-algebras.

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- Some of this rehashes results of de Concini, Procesi (1976), Härterich (1999), Raghavan, Samuel, Subrahmanyam (2012), Bowman, Doty, Martin (2018 and 2022), Donkin (2024) and various others. (My proofs are more elementary.)

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- **Question.** Is there a product rule for the $\nabla_{\mathbf{B},\mathbf{A}}$'s?
- **Question.** How much of the representation theory of S_n can be developed using the $\nabla_{\mathbf{B},\mathbf{A}}$'s?

- **Conjecture (Donkin 2024).** Let $\operatorname{Par}(n)$ be the set of all partitions of n, partially ordered by dominance.
  Let X be an up-set ($=$ order filter) of $\operatorname{Par}(n)$. Let

  $$\mathcal{I}_X = \operatorname{span}\{\nabla_{\mathbf{B},\mathbf{A}} \mid \mathbf{A}, \mathbf{B} \text{ are } n\text{-tableaux with a shape } \lambda \in X\}.$$

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- Proved by Donkin when X is a principal up-set (= all partitions dominating a given μ).
  Checked with Sage for all X when $n \leq 7$.

- **Conjecture (Donkin 2024).** Let $\operatorname{Par}(n)$ be the set of all partitions of n, partially ordered by dominance.
  Let X be an up-set (= order filter) of $\operatorname{Par}(n)$. Let

  $$\mathcal{I}_X = \operatorname{span}\{\nabla_{\mathbf{B},\mathbf{A}} \mid \mathbf{A}, \mathbf{B} \text{ are } n\text{-tableaux with a shape } \lambda \in X\}.$$

  Then, the $\mathbf{k}$-module $\mathcal{A}/\mathcal{I}_X$ is free with basis

  $$(\overline{w})_{w \in S_n \text{ such that } \operatorname{shape} w \notin X},$$

  where $\operatorname{shape} w$ denotes the shape of the RSK-tableaux $P(w)$ and $Q(w)$ of w.
- Proved by Donkin when X is a principal up-set (= all partitions dominating a given μ).
  Checked with Sage for all X when $n \leq 7$.
- **Note:** $\mathcal{I}_X$ has a basis consisting of the $\nabla_{\mathbf{B},\mathbf{A}}$ where the tableaux $\mathbf{A}$ and $\mathbf{B}$ are standard. This is a known result (Murphy basis).

Thank you,

- **Per Alexandersson** and **Theo Douvropoulos** for conversations in 2023 that motivated some of this project.
- **Nadia Lafrenière, Jon Novak, Vic Reiner, Richard P. Stanley** for helpful comments.
- **Travis Scrimshaw** for the invitation.
- **you for any ideas, suggestions and answers!**

- The following is not an open problem but a result of Karp and Purbhoo (arXiv:2309.04645v1 §3), but the proof is so long and indirect that I am looking for a new one.

Bonus: Karp–Purbhoo commutativity

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Bonus: Karp–Purbhoo commutativity

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- Let $z_1, z_2, \ldots, z_n \in \mathbf{k}$ be constants.
- For any class function φ on S_n (= function $S_n \to \mathbf{k}$ that sends conjugate permutations to the same value), we set

$$\mathbf{p}^{\varphi} := \sum_{\sigma \in S_n} z_{\operatorname{Fix} \sigma} \cdot \varphi(\sigma) \, \sigma \in \mathcal{A},$$

where $\operatorname{Fix} \sigma := \{ i \in [n] \mid \sigma(i) = i \}$ and $z_Y := \prod_{y \in Y} z_y$.

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- **Theorem (Karp/Purbhoo 2023, after a few Möbius inversions):** Each $\mathbf{p}^{\varphi}$ commutes with each $\mathbf{p}^{\psi}$ (whenever φ and ψ are two class functions).
- WLOG φ and ψ are conjugacy class indicators, i.e.,

$$\varphi(\sigma) = [\sigma \text{ has cycle type } \lambda] \qquad \text{and}$$
$$\psi(\sigma) = [\sigma \text{ has cycle type } \mu] \qquad \text{for given partitions } \lambda, \mu \vdash n.$$

- **Combinatorial restatement of Karp–Purbhoo commutativity:** There is a bijection

$$\Phi : S_n \times S_n \to S_n \times S_n$$

with the property that when it sends $(\sigma, \tau) \mapsto (\alpha, \beta)$, we always have

$$\alpha\beta = \sigma\tau \qquad \text{and} \qquad \alpha \sim \tau \qquad \text{and} \qquad \beta \sim \sigma$$
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- Maybe we shouldn't expect this bijection to be very canonical. Maybe it should be a si- or multijection.