

Math 530 Spring 2023, Lecture 6: Multigraphs and digraphs

website: <https://www.cip.ifi.lmu.de/~grinberg/t/23s>

Note: This is a rough, unedited draft! See the 2022 notes for a more detailed and fleshed-out writeup.

1. Multigraphs (cont'd)

1.1. Eulerian circuits and walks

Recall:

Definition 1.1.1. Let G be a multigraph.

1. A walk $\mathbf{w}$ of G is said to be **Eulerian** if each edge of G appears exactly once in this walk.
2. An **Eulerian circuit** of G means a circuit (i.e., a closed walk) of G that is Eulerian.

Note that Eulerian walks are usually not paths, and Eulerian circuits are usually not cycles.

I showed some examples in class (from Spring 2022 Lecture 8).

Magically, there is a general criterion for the existence of Eulerian circuits and walks, and there is a fairly efficient algorithm to find them. The criterion is called the **Euler–Hierholzer theorem**:

Theorem 1.1.2 (Euler–Hierholzer). Let G be a connected multigraph. Then:

- (a) The multigraph G has an Eulerian circuit if and only if each vertex of G has even degree.
- (b) The multigraph G has an Eulerian walk if and only if all but at most two vertices of G have even degree.

Why connected? If G has at least two components containing at least one edge each, then G certainly cannot have an Eulerian circuit or walk. If G has isolated vertices, then you can ignore those vertices. It thus suffices to consider connected graphs only.

It is easy to see why the “only if” parts of both parts of the Euler–Hierholzer theorem hold. Indeed, if $\mathbf{w}$ is an Eulerian walk, and v is a vertex that is neither the starting nor the ending point of $\mathbf{w}$, then $\mathbf{w}$ must enter and leave v the

same number of times, so that the degree $\deg v$ must be even (since each edge appears exactly once on $\mathbf{w}$). If $\mathbf{w}$ is an Eulerian circuit, then this holds for the starting and ending point as well.

The hard parts are the “if” parts.

The proofs of the “if” parts will use some preparation.

Definition 1.1.3. Let G be a multigraph. A **trail** of G means a walk of G whose edges are distinct.

Notice: $\{\text{paths}\} \subseteq \{\text{trails}\} \subseteq \{\text{walks}\}$.

Clearly, any Eulerian walk is a trail.

So it seems like a reasonable strategy to find a Eulerian walk is to pick a longest possible trail and somehow argue that it must be Eulerian.

First of all, why is there a longest trail?

Lemma 1.1.4. Let G be a multigraph with at least one vertex. Then, G has a longest trail.

Proof. Each trail has length $\leq |E(G)|$. Thus, there are only finitely many trails. But there is at least one trail (since a trivial path (v) is a trail). So there is a trail of largest length. $\square$

Now, some notations.

We say that an edge e of a multigraph G **intersects** a walk $\mathbf{w}$ if at least one endpoint of e is a vertex of $\mathbf{w}$.

Lemma 1.1.5. Let G be a connected multigraph. Let $\mathbf{w}$ be a walk of G . Assume that $\mathbf{w}$ is not Eulerian.

Then, there exists an edge of G that is not an edge of $\mathbf{w}$ but intersects $\mathbf{w}$.

Proof. Since $\mathbf{w}$ is not Eulerian, there is an edge f that is not an edge of $\mathbf{w}$. Pick such an edge f . If f itself intersects $\mathbf{w}$, then we are done.

Otherwise, build a path from some endpoint of f to some vertex of $\mathbf{w}$ (such a path exists since G is connected). Choose the first edge of this path that is not an edge of $\mathbf{w}$. This edge must then intersect $\mathbf{w}$, so we are done. $\square$

Lemma 1.1.6. Let G be a multigraph such that each vertex of G has even degree. Let $\mathbf{w}$ be a longest trail of G . Then, $\mathbf{w}$ is a closed walk.

Proof. Assume the contrary. Let s be the starting point of $\mathbf{w}$. Then, s is not the ending point of $\mathbf{w}$. Hence, $\mathbf{w}$ enters s one fewer time than it leaves s . As a consequence, $\mathbf{w}$ contains an odd number of edges that contain s . But in total, there is an even number of edges that contain s (since $\deg s$ is even by assumption). Hence, at least one edge that contains s is not in $\mathbf{w}$. Pick such an edge, and insert it into $\mathbf{w}$ at the beginning of $\mathbf{w}$. We get a longer trail than $\mathbf{w}$, but this contradicts the maximality of $\mathbf{w}$. $\square$

Now we can finish the proof of the Euler–Hierholzer theorem:

Proof of Euler–Hierholzer. **(a)** $\implies$: We have already explained this.

$\impliedby$: Assume that each vertex of G has even degree.

By our first lemma, G has a longest trail. Let $\mathbf{w}$ be such a longest trail. By the last lemma, $\mathbf{w}$ is a closed walk. Assume (for contradiction) that $\mathbf{w}$ is not Eulerian. Then, by our second lemma, there exists an edge of G that is not an edge of $\mathbf{w}$ but intersects $\mathbf{w}$. Let f be this edge.

By cyclically rotating our closed walk $\mathbf{w}$, we ensure that $\mathbf{w}$ starts at an end-point of f , and we then insert f at the beginning of $\mathbf{w}$. We thus obtain a longer trail than $\mathbf{w}$. But this contradicts the fact that $\mathbf{w}$ is a longest trail.

This contradiction shows that our assumption was false, so that $\mathbf{w}$ is Eulerian. Since $\mathbf{w}$ is closed, this yields that $\mathbf{w}$ is an Eulerian circuit. So we have proved the “ $\impliedby$ ” direction.

[Note that this proof, while being a proof by contradiction, actually contains a pretty good algorithm to find an Eulerian circuit. The way to do so is to read the proof “seriously but not literally”.]

(b) $\implies$: Already explained.

$\impliedby$: Assume that all but at most two vertices of G have even degree. We must prove that G has an Eulerian walk.

If there are 0 vertices with odd degree, then we can use part **(a)** and we are done.

If there is exactly 1 vertex with odd degree, then we get a contradiction to the handshake lemma.

So it remains to consider the case when there are exactly 2 vertices with odd degree. Let u and v be these two vertices. Let G' be the multigraph obtained from G by adding an extra edge joining u with v . This graph G' has the property that all its vertices have even degree, and it is still connected (since G is connected). Thus, part **(a)** yields that G' has an Eulerian circuit. This circuit must use the edge uv exactly once. Rotate it so that it starts with this edge, and cut it at this edge. It thus becomes an Eulerian walk of G . And we are done. $\square$

2. Digraphs and multidigraphs

2.1. Definitions

We have so far seen two kinds of graphs: simple graphs and multigraphs.

For all their differences, they have a major commonality: Their edges are “two-way roads”, i.e., an edge is (or has) an unordered pair of endpoints. So these kinds of graphs are good for modelling mutual (i.e., symmetric) relationships.

For non-mutual relationships, we need **directed graphs**, or, for short, **digraphs**. In such digraphs, each edge (now called an “arc”) has a specified starting point (its “source”) and a specified ending point (its “target”), and is

drawn as an arrow (from its source to its target). Walks can only use arcs in the forward direction. Here are formal definitions:

Definition 2.1.1. A **simple digraph** is a pair (V, A) , where V is a finite set, and where A is a subset of $V \times V$.

Definition 2.1.2. Let $D = (V, A)$ be a simple digraph.

1. The set V is called the **vertex set** of D , and is denoted $V(D)$.
Its elements are called the **vertices** (or **nodes**) of D .
2. The set A is called the **arc set** of D , and is denoted $A(D)$.
Its elements are called the **arcs** (or **directed edges**) of D .
When u and v are two elements of V , we will occasionally abbreviate the pair (u, v) as uv .
3. If (u, v) is an arc of D (or, more generally, a pair in $V \times V$), then we call u the **source** of this arc, and v the **target** of this arc.
4. We draw D as follows: We represent each vertex each point, and each arc by an arrow pointing from its source to its target.

Example 2.1.3. For each $n \in \mathbb{N}$, we define the **divisibility digraph** on $\{1, 2, \dots, n\}$ to be the simple digraph (V, A) , where $V = \{1, 2, \dots, n\}$ and

$$A = \{(i, j) \in V \times V \mid i \text{ divides } j\}.$$

Note that simple digraphs (unlike simple graphs) are allowed to have loops (i.e., arcs of the form (u, u)).

Definition 2.1.4. A **multidigraph** is a triple (V, A, ψ) , where V and A are finite sets, and $\psi : A \rightarrow V \times V$ is a map.

Definition 2.1.5. Let $D = (V, A, \psi)$ be a multidigraph.

1. The set V is called the **vertex set** of D , and is denoted $V(D)$.
Its elements are called the **vertices** (or **nodes**) of D .
 2. The set A is called the **arc set** of D , and is denoted $A(D)$.
Its elements are called the **arcs** (or **directed edges**) of D .
 3. If a is an arc of D , and if $\psi(a) = (u, v)$, then the vertex u is called the **source** of a , and v is called the **target** of a .
 4. We draw D as you would expect.
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Convention 2.1.6. The word “**digraph**” means either “simple digraph” or “multidigraph”, depending on the context.

2.2. Outdegrees and indegrees

Digraphs have a concept analogous to degrees for graphs. Actually they have two of these concepts:

Definition 2.2.1. Let D be a digraph with vertex set V and arc set A . (This can be a simple digraph or a multidigraph.) Let $v \in V$ be any vertex. Then:

1. The **outdegree** of v denotes the number of arcs of D whose source is v . It is denoted by $\deg^+ v$.
2. The **indegree** of v denotes the number of arcs of D whose target is v . It is denoted by $\deg^- v$.

Example 2.2.2. In the divisibility digraph on $\{1, 2, 3, 4, 5, 6\}$, we have

$$\begin{array}{ll} \deg^+ 1 = 6, & \deg^- 1 = 1, \\ \deg^+ 2 = 3, & \deg^- 2 = 2, \\ \deg^+ 3 = 2, & \deg^- 3 = 2, \\ \deg^+ 4 = 1, & \deg^- 4 = 3, \\ \deg^+ 5 = 1, & \deg^- 5 = 2, \\ \deg^+ 6 = 1, & \deg^- 6 = 4. \end{array}$$

Recall that in a graph, the sum of all degrees is twice the number of edges. For digraphs, an analogous fact holds:

Proposition 2.2.3 (diEuler). Let D be a digraph with vertex set V and arc set A . Then,

$$\sum_{v \in V} \deg^+ v = \sum_{v \in V} \deg^- v = |A|.$$

Proof. Same idea as for graphs, but even easier: Each arc has exactly one source, so you can count the arcs source by source. Thus you get $\sum_{v \in V} \deg^+ v = |A|$.

Similarly $\sum_{v \in V} \deg^- v = |A|$. Details in the 2022 notes (Lecture 9). $\square$

2.3. Conversions

We now have four different types of “graph”: simple graphs, multigraphs, simple digraphs, multidigraphs. There are some ways to convert between these types.

2.3.1. Multidigraphs to multigraphs

Any multidigraph D can be turned into an (undirected) graph G by “removing the arrowheads” (aka “forgetting the directions of the arcs”):

Definition 2.3.1. Let D be a multidigraph. Then, D^{und} will mean the multigraph obtained from D by replacing each arc with an edge whose endpoints are the source and the target of the arc. In other words, if $D = (V, A, \psi)$, then $D^{\text{und}} = (V, A, \varphi)$, where $\varphi : A \rightarrow \mathcal{P}_{1,2}(V)$ is defined as follows: If $\psi(a) = (u, v)$, then $\varphi(a) = \{u, v\}$.

2.3.2. Multigraphs to multidigraphs

Conversely, each multigraph G can be turned into a multidigraph G^{bidir} by “duplicating” each edge (turning it into two arcs going both ways):

Definition 2.3.2. Let $G = (V, E, \varphi)$ be a multigraph. For each edge $e \in E$, let us choose one of the endpoints of e and call it s_e ; the other endpoint will then be called t_e . (If e is a loop, then $t_e = s_e$.)

We then define G^{bidir} to be the multidigraph $(V, E \times \{1, 2\}, \psi)$, where $\psi : E \times \{1, 2\} \rightarrow V \times V$ is defined as follows:

$$\psi(e, 1) = (s_e, t_e) \quad \text{and} \quad \psi(e, 2) = (t_e, s_e)$$

for each edge $e \in E$.

We call G^{bidir} the **bidirectionalized multidigraph** of G .

Question: Is there a way to make this definition canonical?

Note that the map ψ depends on our choice of s_e 's, but different choices lead to isomorphic multidigraphs G^{bidir} . (The notion of **isomorphism** for multidigraphs is defined in a pretty natural way.)

Note that the operation that sends G to G^{bidir} is injective – i.e., you can reconstruct G from G^{bidir} . In contrast, the operation that sends a multidigraph D to D^{und} is not injective, since it forgets the orientations of the arcs.

2.3.3. Simple digraphs to multidigraphs

We learned previously how to make a simple graph into a multigraph. A similar technique works for directed graphs:

Definition 2.3.3. Let $D = (V, A)$ be a simple digraph. Then, the **corresponding multidigraph** D^{mult} is defined to be the multidigraph

$$(V, A, \iota),$$

where $\iota : A \rightarrow V \times V$ is the map sending each arc $a \in A$ to a itself.

2.3.4. Multidigraphs to simple digraphs

In analogy to the $G \mapsto G^{\text{simp}}$ operation for graphs (which drops loops and collapses parallel edges), we can define a $D \mapsto D^{\text{simp}}$ operation for digraphs:

Definition 2.3.4. Let $D = (V, A, \psi)$ be a multidigraph. Then, the **underlying simple digraph** D^{simp} of D means the simple digraph

$$(V, \{\psi(a) \mid a \in A\}).$$

In other words, D^{simp} is obtained by “collapsing” parallel arcs (note that loops are not collapsed).

2.3.5. Multidigraphs as a big tent

As a consequence, every notion of “graph” we have seen so far can be transformed losslessly into a multidigraph:

- Each simple graph becomes a multigraph via $G \mapsto G^{\text{mult}}$.
- Each multigraph, in turn, becomes a multidigraph via $D \mapsto D^{\text{bidir}}$.
- Each simple digraph becomes a multidigraph via $D \mapsto D^{\text{mult}}$.

Thus, multidigraphs are the most general notion of “graphs”. If a theorem holds for multidigraphs, you can then automatically conclude properties of the other types of “graphs” from it.

2.4. Walks, paths, closed walks, cycles

Let us define walks (and their various types) for digraphs. We begin with the case of simple digraphs:

Definition 2.4.1. Let D be a simple digraph. Then:

1. A **walk** (in D) means a finite sequence $(v_0, v_1, \dots, v_k)$ of vertices of D (with $k \geq 0$) such that $v_0v_1, v_1v_2, v_2v_3, \dots, v_{k-1}v_k$ are arcs of D .
2. If $\mathbf{w} = (v_0, v_1, \dots, v_k)$ is a walk of D , then:

- a) The **vertices** of $\mathbf{w}$ are defined to be $v_0, v_1, \dots, v_k$.
 - b) The **arcs** of $\mathbf{w}$ are defined to be the pairs $v_0v_1, v_1v_2, v_2v_3, \dots, v_{k-1}v_k$.
 - c) The nonnegative integer k is called the **length** of $\mathbf{w}$.
 - d) The vertex v_0 is called the **starting point** of $\mathbf{w}$, and the vertex v_k is called the **ending point** of $\mathbf{w}$.
 - e) The walk $\mathbf{w}$ is said to **start** at v_0 and to **end** at v_k .
3. A **path** (in D) means a walk whose vertices are distinct.
 4. A **walk from p to q** means a walk that starts at p and ends at q . A **path from p to q** means a path that starts at p and ends at q .
 5. A **closed walk** means a walk whose starting point is its ending point. It is also called a **circuit**.
 6. A **cycle** of D means a closed walk $(w_0, w_1, \dots, w_k)$ such that $k \geq 1$ and such that the vertices $w_0, w_1, \dots, w_{k-1}$ are distinct.

See the 2022 notes for examples.

We can also define all of these things for multidigraphs:

Definition 2.4.2. Let $D = (V, A, \psi)$ be a multidigraph. Then:

1. A **walk** (in D) means a finite sequence $(v_0, a_1, v_1, a_2, v_2, \dots, a_k, v_k)$ of vertices and arcs of D (with $k \geq 0$) such that $v_0, v_1, \dots, v_k$ are vertices and $a_1, a_2, \dots, a_k$ are arcs and $\psi(a_i) = (v_{i-1}, v_i)$ for each $i \in \{1, 2, \dots, k\}$.
2. If $\mathbf{w} = (v_0, a_1, v_1, a_2, v_2, \dots, a_k, v_k)$ is a walk of D , then:
 - a) The **vertices** of $\mathbf{w}$ are defined to be $v_0, v_1, \dots, v_k$.
 - b) The **arcs** of $\mathbf{w}$ are defined to be the arcs $a_1, a_2, \dots, a_k$.
 - c) The nonnegative integer k is called the **length** of $\mathbf{w}$.
 - d) The vertex v_0 is called the **starting point** of $\mathbf{w}$, and the vertex v_k is called the **ending point** of $\mathbf{w}$.
 - e) The walk $\mathbf{w}$ is said to **start** at v_0 and to **end** at v_k .
3. A **path** (in D) means a walk whose vertices are distinct.
4. A **walk from p to q** means a walk that starts at p and ends at q . A **path from p to q** means a path that starts at p and ends at q .
5. A **closed walk** means a walk whose starting point is its ending point. It is also called a **circuit**.

6. A **cycle** of D means a closed walk $(w_0, a_1, w_1, \dots, a_k, w_k)$ such that $k \geq 1$ and such that the vertices $w_0, w_1, \dots, w_{k-1}$ are distinct. (This yields that the arcs are distinct – check it!)