Math 201-003 Fall 2019 (Darij Grinberg): midterm training 2

1. Determinants

Exercise 1. Here is a 4×4-matrix filled with the numbers $1, 2, 3, ..., 4^2$ by going from left to right along the first row, then backwards along the second row, then forward again along the third, etc.:

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 8 & 7 & 6 & 5 \\ 9 & 10 & 11 & 12 \\ 16 & 15 & 14 & 13 \end{pmatrix}.$$

Explain why the determinant of this matrix, but also the determinant of the similarly constructed $n \times n$-matrix for every $n \geq 3$, is 0.

Exercise 2. Let n be a positive integer. Let $x_1, x_2, \ldots, x_n$ be n reals, and let $y_1, y_2, \ldots, y_n$ be n further reals. Let P_n denote the $n \times n$-matrix

$$(x_i y_j)_{1 \leq i \leq n,\ 1 \leq j \leq n} = \begin{pmatrix} x_1 y_1 & x_1 y_2 & \cdots & x_1 y_n \\ x_2 y_1 & x_2 y_2 & \cdots & x_2 y_n \\ \vdots & \vdots & \ddots & \vdots \\ x_n y_1 & x_n y_2 & \cdots & x_n y_n \end{pmatrix}.$$

(a) Find $\det(P_n)$ if $n = 2$. (In this case, $P_n = P_2 = \begin{pmatrix} x_1 y_1 & x_1 y_2 \\ x_2 y_1 & x_2 y_2 \end{pmatrix}$.)

(b) Find $\det(P_n)$ if $n = 3$.

(c) Find $\det(P_n)$ if $n = 4$.

[**Hint:** The answers are very simple, and can be obtained in a simple way for all $n \geq 2$ simultaneously.

Feel free to use simpler notations, such as renaming x_1, x_2, x_3 as x, y, z and renaming y_1, y_2, y_3 as a, b, c.]

Exercise 3. Let n be a positive integer. Let $x_1, x_2, \ldots, x_n$ be n reals, and let $y_1, y_2, \ldots, y_n$ be n further reals. Let S_n denote the $n \times n$-matrix

$$(x_i + y_j)_{1 \leq i \leq n,\ 1 \leq j \leq n} = \begin{pmatrix} x_1 + y_1 & x_1 + y_2 & \cdots & x_1 + y_n \\ x_2 + y_1 & x_2 + y_2 & \cdots & x_2 + y_n \\ \vdots & \vdots & \ddots & \vdots \\ x_n + y_1 & x_n + y_2 & \cdots & x_n + y_n \end{pmatrix}.$$

(a) Find $\det(S_n)$ if $n = 3$.

(b) Find $\det(S_n)$ if $n = 4$.

[**Hint:** There are very simple answers for all $n \geq 3$. If you are getting complicated expressions, you should expand them or use a different approach.]

Exercise 4. For each $n \in \mathbb{N}$, let A_n be the $n \times n$-matrix

$$\begin{pmatrix} 1 & 1 & 0 & \cdots & 0 & 0 & 0 \\ -1 & 1 & 1 & \cdots & 0 & 0 & 0 \\ 0 & -1 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 1 & 0 \\ 0 & 0 & 0 & \cdots & -1 & 1 & 1 \\ 0 & 0 & 0 & \cdots & 0 & -1 & 1 \end{pmatrix}.$$

(This is the $n \times n$-matrix whose entries on the diagonal and just above it are 1, while its entries just below the diagonal are -1, and all remaining entries are 0. No exceptional entries hiding in the corners this time!)

(a) Find $\det(A_1)$. (Note that $A_1 = \begin{pmatrix} 1 \end{pmatrix}$.)

(b) Find $\det(A_2)$. (Note that $A_2 = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$.)

(c) Find $\det(A_3)$. (Note that $A_3 = \begin{pmatrix} 1 & 1 & 0 \\ -1 & 1 & 1 \\ 0 & -1 & 1 \end{pmatrix}$.)

(d) For a general integer $n \geq 2$, find an expression for $\det(A_n)$ in terms of $\det(A_{n-1})$ and $\det(A_{n-2})$.

(e) Find a formula for $\det(A_n)$ in terms of things we have seen in class.

[**Hint:** In **(d)**, use Laplace expansion.]

2. Eigenvalues and eigenvectors

Exercise 5. **(a)** Find the eigenvalues of the matrix $A := \begin{pmatrix} 0 & 0 & 3 \\ 0 & 2 & -2 \\ 1 & -1 & 0 \end{pmatrix}$.

(b) Diagonalize this matrix (i.e., find a diagonalization of A).

[**Hint:** This is one of those rare cases where the roots of $\chi_A(t)$ can be described nicely. Actually, they are rational numbers. See the rational root theorem for how to find the rational roots of a polynomial with rational coefficients.]

Exercise 6. **(a)** Find the eigenvalues and at least two linearly independent eigenvectors of the matrix $\begin{pmatrix} 1 & 1 & 0 \\ -1 & 2 & 1 \\ 0 & 1 & 1 \end{pmatrix}$.

(b) Does this matrix have a diagonalization?

Exercise 7. Let $(h_0, h_1, h_2, \ldots)$ be the sequence of integers defined by

$$h_0 = 0, \qquad h_1 = 1, \qquad h_n = h_{n-1} + 6h_{n-2} \qquad \text{for all } n \geq 2.$$

(Thus, $h_2 = h_1 + 6h_0 = 1$ and $h_3 = h_2 + 6h_1 = 7$.)
Find an explicit formula for h_m for all $m \geq 0$.

3. Complex numbers

As usual, we let i denote the imaginary unit in this section. Thus, $i^2 = -1$.

To "find" a complex number always means to write it as $a + bi$ with a and b being real. You don't have to write out a and b in decimal (e.g., you don't have to expand 2^{10} as 1024).

Exercise 8. **(a)** Find $(1+i)^2$.
(b) Find $(1+i)^3$.
(c) Find $(1+i)^4$.
(d) Find $(1+i)^8$.
(e) Find $(1+i)^{1000}$.

Exercise 9. Let $\omega = \dfrac{1+\sqrt{3}i}{2}$.
(a) Find ω^2.
(b) Find ω^3.
(c) Find ω^6.
(d) Draw the first 6 powers of ω on the Argand diagram.
(e) Find ω^{1000}.

Exercise 10. Prove that $|zw| = |z| \cdot |w|$ for any two complex numbers z and w.

4. Solutions

The following solutions are a bit rough at some places, but they have enough detail to get full scores.

Some of these solutions use tricks instead of systematic methods. You are free to use the methods – but the tricks are often faster and reveal some ideas that you would have missed if you just followed the methods.

Solution to Exercise 1. Let A be the 4×4-matrix shown in the exercise. Our first goal is to show that $\det A = 0$. Indeed, as direct consequences of the construction of A, we have

$$\operatorname{col}_2 A - \operatorname{col}_1 A = \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix} \qquad \text{and} \qquad \operatorname{col}_3 A - \operatorname{col}_2 A = \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}.$$

Comparing these two equalities, we obtain $\operatorname{col}_2 A - \operatorname{col}_1 A = \operatorname{col}_3 A - \operatorname{col}_2 A$. This can be rewritten as

$$(-1) \operatorname{col}_1 A + 2 \operatorname{col}_2 A + (-1) \operatorname{col}_3 A = 0.$$

This is a nontrivial relation between the columns of A. Thus, the columns of A are linearly dependent. By the Non-Inverse Matrix Theorem[1], this entails that $\det A = 0$.

Thus, we have shown that the determinant of the 4×4-matrix shown in the exercise is 0. The same argument yields the same statement for the similarly constructed $n \times n$-matrix for every $n \geq 3$. $\square$

First solution to Exercise 2. We claim that $\det (P_n) = 0$ for each $n \geq 2$.

Before we prove this claim, let us notice that it does not hold for $n = 1$ (because $\det (P_1) = \det \begin{pmatrix} x_1 y_1 \end{pmatrix} = x_1 y_1$); nor does it hold for $n = 0$ (since the 0×0-matrix has determinant 1, by definition). Thus, the proof of this claim must necessarily use the condition $n \geq 2$.

So let us prove our claim. Let n be an integer such that $n \geq 2$. We must show that $\det (P_n) = 0$. We shall use the following two properties of determinants:

> *Property 1:* If an $n \times n$-matrix A has two equal rows, then $\det A = 0$. (This is Theorem 1.3.3 in the class notes from 2019-10-23.)
>
> *Property 2:* If we scale a row of an $n \times n$-matrix by a number λ, then $\det A$ gets multiplied by λ. (This is Theorem 1.3.5 in the class notes from 2019-10-23.)

[1] more precisely: by the implication **(b′)** $\Longrightarrow$ **(l′)** in Theorem 2.1.4 from the class notes from 2019-11-04

Now, let Q_n be the $n \times n$-matrix

$$\begin{pmatrix} y_1 & y_2 & \cdots & y_n \\ y_1 & y_2 & \cdots & y_n \\ x_3 y_1 & x_3 y_2 & \cdots & x_3 y_n \\ x_4 y_1 & x_4 y_2 & \cdots & x_4 y_n \\ \vdots & \vdots & \ddots & \vdots \\ x_n y_1 & x_n y_2 & \cdots & x_n y_n \end{pmatrix}.$$

(This is the matrix P_n, except that its first two rows have been replaced by $(y_1, y_2, \ldots, y_n)$.)

The matrix Q_n has two equal rows (namely, its first two rows are equal)[2]. Thus, Property 1 shows that $\det(Q_n) = 0$. But the matrix P_n is obtained from Q_n by scaling the first row by x_1 and scaling the second row by x_2. Thus, Property 2 (applied twice) yields that

$$\det(P_n) = x_1 \cdot x_2 \cdot \underbrace{\det(Q_n)}_{=0} = 0.$$

Thus, our claim is proved. This solves the exercise. □

Second solution to Exercise 2. We claim that $\det(P_n) = 0$ for each $n \geq 2$.

This time, we shall derive it from the following two properties of determinants:

> *Property 3:* We have $\det(XY) = \det X \cdot \det Y$ for any two $n \times n$-matrices X and Y. (This is Theorem 1.5.1 in the class notes from 2019-10-30.)
>
> *Property 4:* If an $n \times n$-matrix A has a zero row (i.e., a row full of zeroes), then $\det A = 0$. (This is Corollary 1.2.1 in the class notes from 2019-10-30.)

Now, let n be an integer such that $n \geq 2$. Let X be the $n \times n$-matrix

$$\begin{pmatrix} x_1 & 0 & \cdots & 0 \\ x_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ x_n & 0 & \cdots & 0 \end{pmatrix}.$$

(The first column of this matrix is $(x_1, x_2, \ldots, x_n)^T$, while all remaining columns are zero.)

Also, let Y be the $n \times n$-matrix

$$\begin{pmatrix} y_1 & y_2 & \cdots & y_n \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}.$$

(The first row of this matrix is $(y_1, y_2, \ldots, y_n)$, while all remaining rows are zero.)

The matrix Y has a zero row (for example, its second row)[3]. Thus, Property 4

[2] Here we are using $n \geq 2$.
[3] Here we are using $n \geq 2$.

yields $\det Y = 0$.

But it is easy to see (just compute XY) that $P_n = XY$. Thus,

$$\begin{aligned} \det(P_n) &= \det(XY) = \det X \cdot \underbrace{\det Y}_{=0} \qquad \text{(by Property 3)} \\ &= 0. \end{aligned}$$

Thus, our exercise is solved again. □

First solution to Exercise 3. We claim that $\det(S_n) = 0$ for each $n \geq 3$.

Before we prove this claim, let us notice that it does not hold for $n = 2$ (because $\det(S_2) = \det\begin{pmatrix} x_1 + y_1 & x_1 + y_2 \\ x_2 + y_1 & x_2 + y_2 \end{pmatrix} = -(x_1 - x_2)(y_1 - y_2)$); nor does it hold for $n = 1$ (because $\det(S_1) = \det\begin{pmatrix} x_1 + y_1 \end{pmatrix} = x_1 + y_1$); nor does it hold for $n = 0$ (since the 0×0-matrix has determinant 1, by definition). Thus, the proof of this claim must necessarily use the condition $n \geq 3$.

So let us prove our claim. Let n be an integer such that $n \geq 3$. We must show that $\det(S_n) = 0$. We shall use the following two properties of determinants:

> *Property 1:* If an $n \times n$-matrix A has two equal rows, then $\det A = 0$. (This is Theorem 1.3.3 in the class notes from 2019-10-23.)

> *Property 2:* If we scale a row of an $n \times n$-matrix by a number λ, then $\det A$ gets multiplied by λ. (This is Theorem 1.3.5 in the class notes from 2019-10-23.)

We shall also use the following property of determinants:

> *Property 5:* Let A be an $n \times n$-matrix, and let p and q be two distinct elements of $[n]$. If we add $\lambda \cdot \operatorname{row}_p A$ to the q-th row of A, then $\det A$ does not change. (This is Corollary 1.2.5 in the class notes from 2019-10-30.)

More precisely, we shall use the following particular case of Property 5:

> *Property 6:* Let A be an $n \times n$-matrix, and let p and q be two distinct elements of $[n]$. If we subtract $\operatorname{row}_p A$ from the q-th row of A, then $\det A$ does not change.

Property 6 follows from Property 5 (applied to $\lambda = -1$), because adding $(-1) \cdot \operatorname{row}_p A$ to the q-th row of A is the same as subtracting $\operatorname{row}_p A$ from the q-th row of A.

Now, the definition of S_n shows that

$$\det(S_n) = \det\begin{pmatrix} x_1 + y_1 & x_1 + y_2 & \cdots & x_1 + y_n \\ x_2 + y_1 & x_2 + y_2 & \cdots & x_2 + y_n \\ \vdots & \vdots & \ddots & \vdots \\ x_n + y_1 & x_n + y_2 & \cdots & x_n + y_n \end{pmatrix}$$

$$= \det \begin{pmatrix} x_1+y_1 & x_1+y_2 & \cdots & x_1+y_n \\ x_2+y_1 & x_2+y_2 & \cdots & x_2+y_n \\ x_3+y_1 & x_3+y_2 & \cdots & x_3+y_n \\ x_4+y_1 & x_4+y_2 & \cdots & x_4+y_n \\ \vdots & \vdots & \ddots & \vdots \\ x_n+y_1 & x_n+y_2 & \cdots & x_n+y_n \end{pmatrix}$$

$$= \det \begin{pmatrix} x_1+y_1 & x_1+y_2 & \cdots & x_1+y_n \\ x_2-x_1 & x_2-x_1 & \cdots & x_2-x_1 \\ x_3+y_1 & x_3+y_2 & \cdots & x_3+y_n \\ x_4+y_1 & x_4+y_2 & \cdots & x_4+y_n \\ \vdots & \vdots & \ddots & \vdots \\ x_n+y_1 & x_n+y_2 & \cdots & x_n+y_n \end{pmatrix}$$

$$\left(\begin{array}{c}\text{here, we have subtracted the 1-st row of our matrix} \\ \text{from the 2-nd row; this did not change the determinant} \\ \text{(by Property 6)}\end{array}\right)$$

$$= \det \begin{pmatrix} x_1+y_1 & x_1+y_2 & \cdots & x_1+y_n \\ x_2-x_1 & x_2-x_1 & \cdots & x_2-x_1 \\ x_3-x_1 & x_3-x_1 & \cdots & x_3-x_1 \\ x_4+y_1 & x_4+y_2 & \cdots & x_4+y_n \\ \vdots & \vdots & \ddots & \vdots \\ x_n+y_1 & x_n+y_2 & \cdots & x_n+y_n \end{pmatrix}$$

$$\left(\begin{array}{c}\text{here, we have subtracted the 1-st row of our matrix} \\ \text{from the 3-rd row; this did not change the determinant} \\ \text{(by Property 6)}\end{array}\right)$$

$$= (x_2-x_1) \det \begin{pmatrix} x_1+y_1 & x_1+y_2 & \cdots & x_1+y_n \\ 1 & 1 & \cdots & 1 \\ x_3-x_1 & x_3-x_1 & \cdots & x_3-x_1 \\ x_4+y_1 & x_4+y_2 & \cdots & x_4+y_n \\ \vdots & \vdots & \ddots & \vdots \\ x_n+y_1 & x_n+y_2 & \cdots & x_n+y_n \end{pmatrix}$$

$$\left(\begin{array}{c}\text{by Property 2, because the matrix before the} \\ \text{equality sign can be obtained from the matrix after} \\ \text{the equality sign by scaling the 2-nd row by } x_2-x_1\end{array}\right)$$

$$= (x_2 - x_1)(x_3 - x_1) \underbrace{\det \begin{pmatrix} x_1 + y_1 & x_1 + y_2 & \cdots & x_1 + y_n \\ 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ x_4 + y_1 & x_4 + y_2 & \cdots & x_4 + y_n \\ \vdots & \vdots & \ddots & \vdots \\ x_n + y_1 & x_n + y_2 & \cdots & x_n + y_n \end{pmatrix}}_{\substack{=0 \\ \text{(by Property 1, because this matrix has two equal rows} \\ \text{(namely, its 2-nd and 3-rd rows are equal))}}}$$

$$\left(\begin{array}{c} \text{by Property 2, because the matrix before the} \\ \text{equality sign can be obtained from the matrix after} \\ \text{the equality sign by scaling the 3-rd row by } x_3 - x_1 \end{array}\right)$$

$$= 0.$$

This proves our claim, thus solving the exercise. □

Second solution to Exercise 3. We claim that $\det(S_n) = 0$ for each $n \geq 3$.

This time, we shall derive it from the following two properties of determinants:

> *Property 3:* We have $\det(XY) = \det X \cdot \det Y$ for any two $n \times n$-matrices X and Y. (This is Theorem 1.5.1 in the class notes from 2019-10-30.)
>
> *Property 4:* If an $n \times n$-matrix A has a zero row (i.e., a row full of zeroes), then $\det A = 0$. (This is Corollary 1.2.1 in the class notes from 2019-10-30.)

Now, let n be an integer such that $n \geq 3$. Let X be the $n \times n$-matrix

$$\begin{pmatrix} x_1 & 1 & 0 & \cdots & 0 \\ x_2 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_n & 1 & 0 & \cdots & 0 \end{pmatrix}.$$

(The first column of this matrix is $(x_1, x_2, \ldots, x_n)^T$; the second column is $(1, 1, \ldots, 1)^T$; all remaining columns are zero.)

Also, let Y be the $n \times n$-matrix

$$\begin{pmatrix} 1 & 1 & \cdots & 1 \\ y_1 & y_2 & \cdots & y_n \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}.$$

(The first row of this matrix is $(y_1, y_2, \ldots, y_n)$; the second row is $(1, 1, \ldots, 1)$; all remaining rows are zero.)

The matrix Y has a zero row (for example, its third row)[4]. Thus, Property 4 yields $\det Y = 0$.

But it is easy to see that $S_n = XY$ (indeed, just compute XY and notice that $x_i \cdot 1 + 1 \cdot y_j = x_i + y_j$). Thus,

$$\det(S_n) = \det(XY) = \det X \cdot \underbrace{\det Y}_{=0} \qquad \text{(by Property 3)}$$
$$= 0.$$

Thus, our exercise is solved again. □

Solution to Exercises 4. **(a)** We have $A_1 = \begin{pmatrix} 1 \end{pmatrix}$ and thus $\det(A_1) = \det \begin{pmatrix} 1 \end{pmatrix} = 1$.

(b) We have $A_2 = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$ and thus $\det(A_2) = \det \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} = 2$.

(c) We have $A_3 = \begin{pmatrix} 1 & 1 & 0 \\ -1 & 1 & 1 \\ 0 & -1 & 1 \end{pmatrix}$ and thus $\det(A_3) = \det \begin{pmatrix} 1 & 1 & 0 \\ -1 & 1 & 1 \\ 0 & -1 & 1 \end{pmatrix} = 3$.

(d) Let n be an integer such that $n \geq 2$.

We have

$$A_n = \begin{pmatrix} 1 & 1 & 0 & \cdots & 0 & 0 & 0 \\ -1 & 1 & 1 & \cdots & 0 & 0 & 0 \\ 0 & -1 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 1 & 0 \\ 0 & 0 & 0 & \cdots & -1 & 1 & 1 \\ 0 & 0 & 0 & \cdots & 0 & -1 & 1 \end{pmatrix}_{n \times n}.$$

Here, the subscript "$n \times n$" tells us that the matrix you are seeing is understood to be an $n \times n$-matrix. (Similar notations will be used further below.)

The 1-st row of the matrix A_n has only two nonzero entries: the 1 in position 1, and the 1 in position 2. Hence, Laplace expansion along the 1-st row (see Theorem 1.7.1 in the class notes from 2019-10-30) yields

$$\det(A_n) = 1 \cdot \det \underbrace{\begin{pmatrix} 1 & 1 & \cdots & 0 & 0 & 0 \\ -1 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 1 & 0 \\ 0 & 0 & \cdots & -1 & 1 & 1 \\ 0 & 0 & \cdots & 0 & -1 & 1 \end{pmatrix}_{(n-1) \times (n-1)}}_{= A_{n-1}}$$

[4] Here we are using $n \geq 3$.

$$- 1 \cdot \det \begin{pmatrix} -1 & 1 & \cdots & 0 & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 1 & 0 \\ 0 & 0 & \cdots & -1 & 1 & 1 \\ 0 & 0 & \cdots & 0 & -1 & 1 \end{pmatrix}_{(n-1)\times(n-1)}$$

$$+ \text{(several addends that equal 0 and thus can be ignored)}$$

$$= \det (A_{n-1}) - \det \begin{pmatrix} -1 & 1 & \cdots & 0 & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 1 & 0 \\ 0 & 0 & \cdots & -1 & 1 & 1 \\ 0 & 0 & \cdots & 0 & -1 & 1 \end{pmatrix}_{(n-1)\times(n-1)} .$$

Let us denote the second matrix on the right hand side of this equality[5] by B_{n-1}. The 1-st column of this matrix B_{n-1} has only one nonzero entry, namely the -1 in position 1. Hence, Laplace expansion along the 1-st column (see Theorem 1.2.1 in the class notes from 2019-11-04) yields

$$\det (B_{n-1}) = (-1) \cdot \det \underbrace{\begin{pmatrix} 1 & \cdots & 0 & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & \cdots & 1 & 1 & 0 \\ 0 & \cdots & -1 & 1 & 1 \\ 0 & \cdots & 0 & -1 & 1 \end{pmatrix}_{(n-2)\times(n-2)}}_{=A_{n-2}} = (-1) \cdot \det (A_{n-2})$$

$$= - \det (A_{n-2}) .$$

Thus, our above computation becomes

$$\det (A_n) = \det (A_{n-1}) - \det \underbrace{\begin{pmatrix} -1 & 1 & \cdots & 0 & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 1 & 0 \\ 0 & 0 & \cdots & -1 & 1 & 1 \\ 0 & 0 & \cdots & 0 & -1 & 1 \end{pmatrix}_{(n-1)\times(n-1)}}_{=B_{n-1}}$$

[5] i.e., the matrix $\begin{pmatrix} -1 & 1 & \cdots & 0 & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 1 & 0 \\ 0 & 0 & \cdots & -1 & 1 & 1 \\ 0 & 0 & \cdots & 0 & -1 & 1 \end{pmatrix}_{(n-1)\times(n-1)}$

$$= \det(A_{n-1}) - \underbrace{\det(B_{n-1})}_{=-\det(A_{n-2})} = \det(A_{n-1}) - (-\det(A_{n-2}))$$
$$= \det(A_{n-1}) + \det(A_{n-2}).$$

This solves part **(d)** of the exercise.

(e) Recall the Fibonacci numbers $f_0, f_1, f_2, \ldots$ defined in the class notes from 2019-11-06. We claim that

$$\det(A_n) = f_{n+1} \qquad \text{for each } n \geq 0.$$

In other words, we claim that the sequence $(\det(A_0), \det(A_1), \det(A_2), \ldots)$ is identical to the sequence $(f_1, f_2, f_3, \ldots)$.

To see why this is true, all we need to show is that

- both of these sequences begin with the entries 1 and 1 (that is, we have $\det(A_0) = 1$ and $\det(A_1) = 1$ and $f_1 = 1$ and $f_2 = 1$), and
- both of these sequences are constructed by the same rule (that is, we have $\det(A_n) = \det(A_{n-1}) + \det(A_{n-2})$ and $f_{n+1} = f_n + f_{n-1}$ for each $n \geq 2$).

But this is clear: The equalities $\det(A_0) = 1$ and $\det(A_1) = 1$ and $f_1 = 1$ and $f_2 = 1$ can be verified directly[6]. The equality $\det(A_n) = \det(A_{n-1}) + \det(A_{n-2})$ is our answer to part **(d)** of this exercise. The equality $f_{n+1} = f_n + f_{n-1}$ follows from the definition of the Fibonacci numbers.

Thus, our claim is proved, and part **(e)** of the exercise is solved.

[*Remark:* An $n \times n$-matrix whose nonzero entries are limited to the diagonal and the cells just above and just below it is called a *tridiagonal matrix*. There is a recursive formula – similarly to the answer we obtained in part **(d)** of this exercise – for the determinant of an arbitrary tridiagonal matrix. See [Grinbe15, §6.13] for a detailed proof of this formula. (Note that our claim that $\det(A_n) = f_{n+1}$ for each $n \geq 0$ appears in [Grinbe15, Exercise 6.27].)] □

Solution to Exercise 5. **(a)** The characteristic polynomial $\chi_A(t)$ of A is

$$\chi_A(t) = \det(A - tI_3) = \det\begin{pmatrix} 0-t & 0 & 3 \\ 0 & 2-t & -2 \\ 1 & -1 & 0-t \end{pmatrix} = -(t+2)(t-1)(t-3).$$

(Obviously, checking the equality $\det\begin{pmatrix} 0-t & 0 & 3 \\ 0 & 2-t & -2 \\ 1 & -1 & 0-t \end{pmatrix} = -(t+2)(t-1)(t-3)$ is straightforward, but how would you find it? I'm afraid there is no better way than to expand $\det\begin{pmatrix} 0-t & 0 & 3 \\ 0 & 2-t & -2 \\ 1 & -1 & 0-t \end{pmatrix}$ as $-t^3 + 2t^2 + 5t - 6$, and then to search for rational roots using the rational root theorem.)

[6] Keep in mind that the 0×0-matrix has determinant 1 (by definition); this is why $\det(A_0) = 1$.

Recall that the eigenvalues of A are the roots of $\chi_A(t)$ (by Proposition 2.1.7 in the class notes from 2019-11-04). But the roots of $\chi_A(t)$ are $-2, 1, 3$ (since $\chi_A(t) = -(t+2)(t-1)(t-3)$). Hence, the eigenvalues of A are $-2, 1, 3$.

(b) We have just seen that the eigenvalues of A are $-2, 1, 3$. Denote these eigenvalues by $\lambda_1, \lambda_2, \lambda_3$, respectively. Let us find eigenvectors for them:

- The λ_1-eigenvectors (i.e., the (-2)-eigenvectors) of A are the nonzero vectors $v \in \mathbb{R}^3$ satisfying $Av = (-2)v$. In other words, they are the nonzero vectors $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3$ such that $A \begin{pmatrix} x \\ y \\ z \end{pmatrix} = (-2) \begin{pmatrix} x \\ y \\ z \end{pmatrix}$. This is a system of 3 linear equations in the unknowns x, y, z; solving it by Gaussian elimination, we obtain $\begin{cases} x = -\frac{3}{2}z \\ y = \frac{1}{2}z \end{cases}$ (where z is a free variable). Thus, they are the nonzero scalar multiples of the vector $\begin{pmatrix} -\frac{3}{2} \\ \frac{1}{2} \\ 1 \end{pmatrix}$. For convenience, let us scale this vector by 2, so it becomes $\begin{pmatrix} -3 \\ 1 \\ 2 \end{pmatrix}$.
- Likewise, the λ_2-eigenvectors (i.e., the 1-eigenvectors) of A are the nonzero scalar multiples of the vector $\begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$.
- Likewise, the λ_3-eigenvectors (i.e., the 3-eigenvectors) of A are the nonzero scalar multiples of the vector $\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$.

We have now found three eigenvectors for A: namely,

$$u_1 = \begin{pmatrix} -3 \\ 1 \\ 2 \end{pmatrix} \qquad (\text{a } \lambda_1\text{-eigenvector});$$

$$u_2 = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} \qquad (\text{a } \lambda_2\text{-eigenvector});$$

$$u_3 = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \qquad (\text{a } \lambda_3\text{-eigenvector}).$$

These 3 vectors u_1, u_2, u_3 form a basis of $\mathbb{R}^3$ (indeed, this can either be derived from Proposition 1.3.3 in the class notes from 2019-11-06, or just checked by hand).

Thus, (u_1, u_2, u_3) is a basis of $\mathbb{R}^3$ that consists of eigenvectors of A, and $\lambda_1, \lambda_2, \lambda_3$ are the corresponding eigenvalues. Hence, we can find a diagonalization of A using Proposition 1.2.3 **(a)** in the class notes from 2019-11-11: We set

$$U = [u_1 \mid u_2 \mid u_3] = \begin{pmatrix} -3 & 3 & 1 \\ 1 & 2 & -2 \\ 2 & 1 & 1 \end{pmatrix} \qquad \text{and}$$

$$D = \operatorname{diag}(\lambda_1, \lambda_2, \lambda_3) = \operatorname{diag}(-2, 1, 3) = \begin{pmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}.$$

The pair U, D is a diagonalization of A (that is, U is invertible, and D is diagonal, and we have $A = UDU^{-1}$).

[*Remark:* There are many diagonalizations of A. For example, you can obtain a different diagonalization of A by listing the eigenvalues of A in a different order, or by scaling the eigenvectors u_1, u_2, u_3 differently.] $\square$

Solution to Exercise 6. Let us denote the matrix shown in this exercise by A.

(a) The characteristic polynomial $\chi_A(t)$ of A is

$$\chi_A(t) = \det(A - tI_3) = \det \begin{pmatrix} 1-t & 1 & 0 \\ -1 & 2-t & 1 \\ 0 & 1 & 1-t \end{pmatrix} = -(t-1)^2(t-2).$$

Recall that the eigenvalues of A are the roots of $\chi_A(t)$ (by Proposition 2.1.7 in the class notes from 2019-11-04). But the roots of $\chi_A(t)$ are 1 and 2 (since $\chi_A(t) = -(t-1)^2(t-2)$). Hence, the eigenvalues of A are 1 and 2.

(b) No.

Proof. We have just found the eigenvalues of A: they are 1 and 2. Let us find eigenvectors for them:

- The 1-eigenvectors of A are the nonzero scalar multiples of the vector $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$.

  (This can be found by Gaussian elimination, just as in the above solution to Exercise 5.)

- The 2-eigenvectors of A are the nonzero scalar multiples of the vector $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

This shows that we cannot find 3 linearly independent eigenvectors for A. Thus, there is no basis of $\mathbb{R}^3$ consisting of eigenvectors of A. But Proposition 1.2.3 **(b)** in the class notes from 2019-11-11 shows that every diagonalization of A can be obtained from such a basis. Thus, there is no diagonalization of A. $\square$

Solution to Exercise 7. We proceed in the same way as we analyzed the Fibonacci numbers (in Example 1.4.1 in the class notes from 2019-11-06).

Instead of computing h_m, let us look at the vectors $\begin{pmatrix} h_m \\ h_{m+1} \end{pmatrix}$ for $m \geq 0$. Each of these vectors determines the next one:

$$\begin{pmatrix} h_{m+1} \\ h_{m+2} \end{pmatrix} = \begin{pmatrix} h_{m+1} \\ h_{m+1} + 6h_m \end{pmatrix} \qquad \left(\text{since the definition of the } h_n \text{ yields } h_{m+2} = h_{m+1} + 6h_m\right)$$
$$= \begin{pmatrix} 0 & 1 \\ 6 & 1 \end{pmatrix} \begin{pmatrix} h_m \\ h_{m+1} \end{pmatrix} \qquad \text{for each } m \geq 0.$$

Let us define the 2×2-matrix $A = \begin{pmatrix} 0 & 1 \\ 6 & 1 \end{pmatrix}$. Thus, this becomes

$$\begin{pmatrix} h_{m+1} \\ h_{m+2} \end{pmatrix} = A \begin{pmatrix} h_m \\ h_{m+1} \end{pmatrix} \qquad \text{for each } m \geq 0.$$

Hence, by induction on m, we can show that

$$\begin{pmatrix} h_m \\ h_{m+1} \end{pmatrix} = A^m \begin{pmatrix} h_0 \\ h_1 \end{pmatrix} \tag{1}$$

for each $m \geq 0$. Thus, in order to compute h_m, it suffices to compute A^m.

From here on, we proceed as in Example 1.2.4 in the class notes from 2019-11-11 (except that our matrix A is a different one now). We seek a diagonalization of A. Since we have already seen how to diagonalize a matrix (see, e.g., the solution to Exercise 5 **(b)** above), let me be brief: The characteristic polynomial of A is $\chi_A(t) = \det \begin{pmatrix} 0-t & 1 \\ 6 & 1-t \end{pmatrix} = (t-3)(t+2)$; thus, the eigenvalues of A are 3 and -2; the corresponding eigenvectors are nonzero scalar multiples of $\begin{pmatrix} 1 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 2 \end{pmatrix}$, respectively; thus, we obtain a diagonalization U, D with

$$U = [u_1 \mid u_2] = \begin{pmatrix} 1 & -1 \\ 3 & 2 \end{pmatrix} \qquad \text{and}$$
$$D = \operatorname{diag}(3, -2) = \begin{pmatrix} 3 & 0 \\ 0 & -2 \end{pmatrix}.$$

Now, let $m \geq 0$. Then, Proposition 1.2.1 in the class notes from 2019-11-11 (applied to $n = 2$, $d_1 = 3$ and $d_2 = -2$) yields

$$\left(UDU^{-1}\right)^m = U \operatorname{diag}\left(3^m, (-2)^m\right) U^{-1} \qquad \left(\text{since } D = \operatorname{diag}(3, -2)\right).$$

In view of $UDU^{-1} = A$, this rewrites as

$$A^m = U \operatorname{diag}\left(3^m, (-2)^m\right) U^{-1} = \begin{pmatrix} 1 & -1 \\ 3 & 2 \end{pmatrix} \operatorname{diag}\left(3^m, (-2)^m\right) \begin{pmatrix} 1 & -1 \\ 3 & 2 \end{pmatrix}^{-1}$$

$$\left(\text{since } U = \begin{pmatrix} 1 & -1 \\ 3 & 2 \end{pmatrix} \text{ and } n = 2 \text{ and } d_1 = 3 \text{ and } d_2 = -2\right)$$

$$= \begin{pmatrix} 1 & -1 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} 3^m & 0 \\ 0 & (-2)^m \end{pmatrix} \underbrace{\begin{pmatrix} 1 & -1 \\ 3 & 2 \end{pmatrix}^{-1}}_{\substack{= \frac{1}{2-(-1)3} \begin{pmatrix} 2 & 1 \\ -3 & 1 \end{pmatrix} \\ \text{(by the formula } \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{1} = \frac{1}{ad-bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \\ \text{for the inverse of a } 2\times 2\text{-matrix)}}}$$

$$= \underbrace{\frac{1}{2-(-1)3}}_{=\frac{1}{5}} \begin{pmatrix} 1 & -1 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} 3^m & 0 \\ 0 & (-2)^m \end{pmatrix} \begin{pmatrix} 2 & 1 \\ -3 & 1 \end{pmatrix}$$

$$= \frac{1}{5} \begin{pmatrix} 1 & -1 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} 3^m & 0 \\ 0 & (-2)^m \end{pmatrix} \begin{pmatrix} 2 & 1 \\ -3 & 1 \end{pmatrix}.$$

We could multiply this out. But we want h_m, not A^m. There is a faster way to get h_m: From (1), we obtain

$$\begin{pmatrix} h_m \\ h_{m+1} \end{pmatrix} = \underbrace{A^m}_{= \frac{1}{5} \begin{pmatrix} 1 & -1 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} 3^m & 0 \\ 0 & (-2)^m \end{pmatrix} \begin{pmatrix} 2 & 1 \\ -3 & 1 \end{pmatrix}} \underbrace{\begin{pmatrix} h_0 \\ h_1 \end{pmatrix}}_{\substack{= \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ \text{(since } h_0 = 0 \text{ and } h_1 = 1)}}$$

$$= \frac{1}{5} \begin{pmatrix} 1 & -1 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} 3^m & 0 \\ 0 & (-2)^m \end{pmatrix} \underbrace{\begin{pmatrix} 2 & 1 \\ -3 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix}}_{= \begin{pmatrix} 1 \\ 1 \end{pmatrix}}$$

$$= \frac{1}{5} \begin{pmatrix} 1 & -1 \\ 3 & 2 \end{pmatrix} \underbrace{\begin{pmatrix} 3^m & 0 \\ 0 & (-2)^m \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix}}_{= \begin{pmatrix} 3^m \\ (-2)^m \end{pmatrix}} = \frac{1}{5} \underbrace{\begin{pmatrix} 1 & -1 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} 3^m \\ (-2)^m \end{pmatrix}}_{\substack{= \begin{pmatrix} 3^m - (-2)^m \\ * \end{pmatrix} \\ \text{(where the asterisk ``}*\text{'' stands} \\ \text{for an entry that we} \\ \text{are not interested in)}}}$$

$$= \frac{1}{5} \begin{pmatrix} 3^m - (-2)^m \\ * \end{pmatrix} = \begin{pmatrix} \frac{1}{5} \left(3^m - (-2)^m\right) \\ * \end{pmatrix}.$$

Thus, by comparing the $(1,1)$-entries on both sides, we obtain

$$h_m = \frac{1}{5} \left(3^m - (-2)^m\right).$$

This solves the exercise.

[*Remark:* In general, if a 2×2-matrix has rational entries, then its eigenvalues will be quadratic irrationalities (i.e., numbers of the form $a + \sqrt{b}$ with $a, b \in \mathbb{Q}$); they don't have to be rational numbers (and they don't have to be real numbers either; b can be negative). The reason why the matrix A above has integer eigenvalues is that I have picked such a matrix deliberately.] □

Before we solve the next two exercises, let me mention some basic properties of powers of complex numbers:

Proposition 4.1. (a) We have $\alpha^{n+1} = \alpha\alpha^n$ for all $\alpha \in \mathbb{C}$ and $n \in \mathbb{N}$.

(b) We have $\alpha^{n+m} = \alpha^n\alpha^m$ for all $\alpha \in \mathbb{C}$ and $n, m \in \mathbb{N}$.

(c) We have $(\alpha\beta)^n = \alpha^n\beta^n$ for all $\alpha, \beta \in \mathbb{C}$ and $n \in \mathbb{N}$.

(d) We have $(\alpha^n)^m = \alpha^{nm}$ for all $\alpha \in \mathbb{C}$ and $n, m \in \mathbb{N}$.

(e) We have $1^n = 1$ for all $n \in \mathbb{N}$.

(f) We have $\alpha^{n+1} = \alpha\alpha^n$ for all nonzero $\alpha \in \mathbb{C}$ and all $n \in \mathbb{Z}$.

(g) We have $\alpha^{-n} = \left(\alpha^{-1}\right)^n$ for all nonzero $\alpha \in \mathbb{C}$ and all $n \in \mathbb{Z}$.

(h) We have $\alpha^{n+m} = \alpha^n\alpha^m$ for all nonzero $\alpha \in \mathbb{C}$ and all $n, m \in \mathbb{Z}$.

(i) We have $(\alpha\beta)^n = \alpha^n\beta^n$ for all nonzero $\alpha, \beta \in \mathbb{C}$ and all $n \in \mathbb{Z}$.

(j) We have $1^n = 1$ for all $n \in \mathbb{Z}$.

(k) We have $(\alpha^n)^{-1} = \alpha^{-n}$ for all nonzero $\alpha \in \mathbb{C}$ and all $n \in \mathbb{Z}$. (In particular, α^n is nonzero, so that $(\alpha^n)^{-1}$ is well-defined.)

(l) We have $(\alpha^n)^m = \alpha^{nm}$ for all nonzero $\alpha \in \mathbb{C}$ and all $n, m \in \mathbb{Z}$. (In particular, α^n is nonzero, so that $(\alpha^n)^m$ is well-defined for all $m \in \mathbb{Z}$.)

(m) Complex numbers satisfy the binomial formula: That is, if $\alpha, \beta \in \mathbb{C}$, then

$$(\alpha + \beta)^n = \sum_{k=0}^{n} \binom{n}{k} \alpha^k \beta^{n-k} \qquad \text{for } n \in \mathbb{N}.$$

This proposition is [19s, Proposition 4.1.20]; see [19s, solution to Exercise 4.1.1] for a detailed proof. As for us, we will only need its parts **(a)**, **(b)**, **(c)** and **(d)**, which are both easy to check directly using the definition of nonnegative integer powers (i.e., by using the fact that $\alpha^k = \underbrace{\alpha\alpha\cdots\alpha}_{k \text{ times}}$ for each $\alpha \in \mathbb{C}$ and each $k \in \mathbb{N}$, where the empty product $\underbrace{\alpha\alpha\cdots\alpha}_{0 \text{ times}}$ is understood to be 1).

Solution to Exercise 8. **(a)** We have

$$(1+i)^2 = (1+i)(1+i) = 1+i+i+\underbrace{i^2}_{=-1} = 1+i+i+(-1) = 2i.$$

(b) Part **(a)** of this exercise shows that $(1+i)^2 = 2i$. But Proposition 4.1 **(a)** (applied to $\alpha = 1+i$ and $n=2$) yields

$$(1+i)^3 = (1+i)\underbrace{(1+i)^2}_{=2i} = (1+i)\cdot(2i) = 2i + \underbrace{i\cdot 2i}_{=2i^2} = 2i + 2\underbrace{i^2}_{=-1}$$
$$= 2i + 2(-1) = 2i - 2.$$

(c) Part **(a)** of this exercise shows that $(1+i)^2 = 2i$. But Proposition 4.1 **(d)** (applied to $\alpha = 1+i$, $n=2$ and $m=2$) yields $\left((1+i)^2\right)^2 = (1+i)^{2\cdot 2} = (1+i)^4$. Thus,

$$(1+i)^4 = \left(\underbrace{(1+i)^2}_{=2i}\right)^2 = (2i)^2 = 2^2\underbrace{i^2}_{=-1} \qquad \text{(by Proposition 4.1 (c))}$$
$$= 2^2(-1) = -4.$$

(d) Part **(c)** of this exercise shows that $(1+i)^4 = -4$. But Proposition 4.1 **(d)** (applied to $\alpha = 1+i$, $n=4$ and $m=2$) yields $\left((1+i)^4\right)^2 = (1+i)^{4\cdot 2} = (1+i)^8$. Thus,

$$(1+i)^8 = \left(\underbrace{(1+i)^4}_{=-4}\right)^2 = (-4)^2 = 4^2 = 16.$$

(e) By dividing 1000 by 4 with remainder, we obtain $1000 = 4\cdot 250$.

Part **(c)** of this exercise shows that $(1+i)^4 = -4$. Now, Proposition 4.1 **(d)** (applied to $\alpha = 1+i$, $n=4$ and $m=250$) yields $\left((1+i)^4\right)^{250} = (1+i)^{4\cdot 250} = (1+i)^{1000}$. Thus,

$$(1+i)^{1000} = \left(\underbrace{(1+i)^4}_{=-4}\right)^{250} = (-4)^{250} = 4^{250} \qquad \text{(since 250 is even)}.$$

□

Solution to Exercise 9. **(a)** From $\omega = \dfrac{1+\sqrt{3}i}{2}$, we obtain

$$\omega^2 = \left(\frac{1+\sqrt{3}i}{2}\right)^2 = \frac{1+\sqrt{3}i}{2}\cdot\frac{1+\sqrt{3}i}{2} = \frac{1}{4}\underbrace{\left(1+\sqrt{3}i\right)\left(1+\sqrt{3}i\right)}_{=1+\sqrt{3}i+\sqrt{3}i+\left(\sqrt{3}i\right)^2}$$

$$= \frac{1}{4} \left(1 + \underbrace{\sqrt{3}i + \sqrt{3}i}_{=2\sqrt{3}i} + \underbrace{\left(\sqrt{3}i\right)^2}_{\substack{=\left(\sqrt{3}\right)^2 i^2 \\ \text{(by Proposition 4.1 (c))}}} \right)$$

$$= \frac{1}{4} \left(1 + 2\sqrt{3}i + \underbrace{\left(\sqrt{3}\right)^2}_{=3} \underbrace{i^2}_{=-1} \right) = \frac{1}{4} \underbrace{\left(1 + 2\sqrt{3}i + 3(-1)\right)}_{=-2+2\sqrt{3}i}$$

$$= \frac{1}{4} \left(-2 + 2\sqrt{3}i\right) = \frac{-1+\sqrt{3}i}{2}.$$

(b) Part **(a)** of this exercise shows that $\omega^2 = \dfrac{-1+\sqrt{3}i}{2}$. But Proposition 4.1 **(a)** (applied to $\alpha = \omega$ and $n = 2$) yields

$$\omega^3 = \underbrace{\omega}_{=\frac{1+\sqrt{3}i}{2}} \underbrace{\omega^2}_{=\frac{-1+\sqrt{3}i}{2}} = \frac{1+\sqrt{3}i}{2} \cdot \frac{-1+\sqrt{3}i}{2}$$

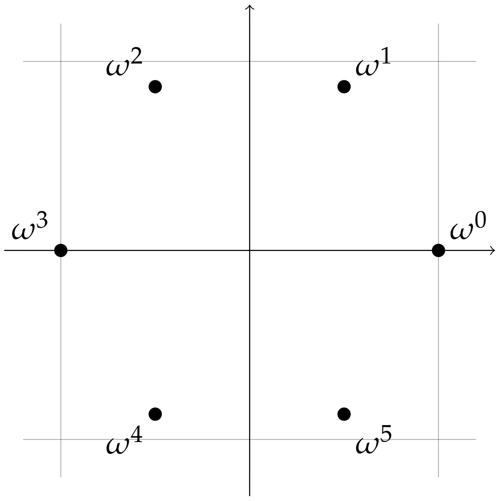
$$= \frac{1}{4} \underbrace{\left(1+\sqrt{3}i\right)\left(-1+\sqrt{3}i\right)}_{=-1+\sqrt{3}i-\sqrt{3}i+\left(\sqrt{3}i\right)^2} = \frac{1}{4} \left(-1 + \underbrace{\sqrt{3}i - \sqrt{3}i}_{=0} + \underbrace{\left(\sqrt{3}i\right)^2}_{\substack{=\left(\sqrt{3}\right)^2 i^2 \\ \text{(by Proposition 4.1 (c))}}} \right)$$

$$= \frac{1}{4} \left(-1 + \underbrace{\left(\sqrt{3}\right)^2}_{=3} \underbrace{i^2}_{=-1} \right) = \frac{1}{4} \left(-1 + 3(-1)\right) = -1.$$

(c) Part **(b)** of this exercise shows that $\omega^3 = -1$. But Proposition 4.1 **(d)** (applied to $\alpha = \omega$, $n = 3$ and $m = 2$) yields $\left(\omega^3\right)^2 = \omega^{3\cdot 2} = \omega^6$. Thus,

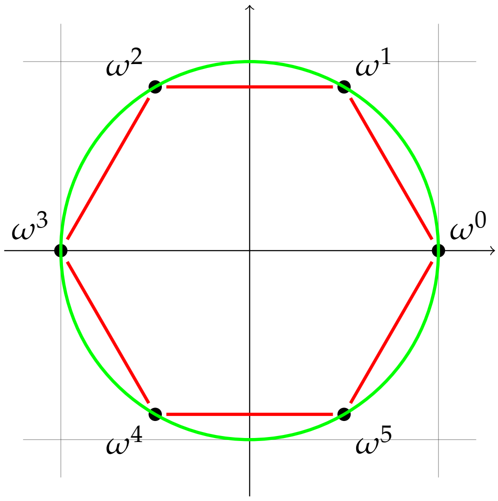
$$\omega^6 = \left(\underbrace{\omega^3}_{=-1} \right)^2 = (-1)^2 = 1.$$

(d) Here is how they look:



.

As we can see, they are the vertices of a regular hexagon inscribed in the unit circle, centered at the origin:



.

(e) By dividing 1000 by 3 with remainder, we obtain $1000 = 3 \cdot 333 + 1$.

Part **(b)** of this exercise shows that $\omega^3 = -1$. Now, Proposition 4.1 **(d)** (applied to $\alpha = \omega$, $n = 3$ and $m = 333$) yields $\left(\omega^3\right)^{333} = \omega^{3 \cdot 333} = \omega^{999}$. Thus,

$$\omega^{999} = \left(\underbrace{\omega^3}_{=-1}\right)^{333} = (-1)^{333} = -1 \qquad (\text{since } 333 \text{ is odd}).$$

Now, Proposition 4.1 **(a)** (applied to $\alpha = \omega$ and $n = 999$) yields

$$\omega^{1000} = \omega \underbrace{\omega^{999}}_{=-1} = \omega(-1) = -\omega = -\frac{1+\sqrt{3}i}{2} \qquad \left(\text{since } \omega = \frac{1+\sqrt{3}i}{2}\right).$$

$\square$

Solution to Exercise 10. Recall that a complex number has been defined as a pair of real numbers. Thus, we can write the complex numbers z and w as $z = (a, b)$ and $w = (c, d)$, respectively, where a, b, c, d are four reals. Consider these a, b, c, d.

From $z = (a, b)$ and $w = (c, d)$, we obtain

$$zw = (a, b)(c, d) = (ac - bd, ad + bc)$$

(by the definition of multiplication of complex numbers).

But the definition of the absolute value of a complex number yields

$$\begin{aligned} |z| &= \sqrt{a^2 + b^2} \qquad (\text{since } z = (a, b)) \qquad \text{and} \\ |w| &= \sqrt{c^2 + d^2} \qquad (\text{since } w = (c, d)) \qquad \text{and} \\ |zw| &= \sqrt{(ac - bd)^2 + (ad + bc)^2} \qquad (\text{since } zw = (ac - bd, ad + bc)). \end{aligned}$$

Multiplying the first two of these three equalities, we find

$$|z| \cdot |w| = \sqrt{a^2 + b^2} \cdot \sqrt{c^2 + d^2} = \sqrt{(a^2 + b^2)(c^2 + d^2)} = \sqrt{a^2c^2 + a^2d^2 + b^2c^2 + b^2d^2}.$$

Comparing this with

$$|zw| = \sqrt{(ac - bd)^2 + (ad + bc)^2} = \sqrt{a^2c^2 + a^2d^2 + b^2c^2 + b^2d^2}$$
$$\left(\begin{array}{l} \text{since } (ac - bd)^2 + (ad + bc)^2 \\ = \left(a^2c^2 - 2acbd + b^2d^2\right) + \left(a^2d^2 + 2adbc + b^2c^2\right) \\ = \left(a^2c^2 - 2abcd + b^2d^2\right) + \left(a^2d^2 + 2abcd + b^2c^2\right) \\ = a^2c^2 + a^2d^2 + b^2c^2 + b^2d^2 \end{array} \right),$$

we obtain $|zw| = |z| \cdot |w|$. This solves the exercise. □

References

[19s] Darij Grinberg, *Introduction to Modern Algebra (UMN Spring 2019 Math 4281 notes)*, 29 June 2019.
`http://www.cip.ifi.lmu.de/~grinberg/t/19s/notes.pdf`

[Grinbe15] Darij Grinberg, *Notes on the combinatorial fundamentals of algebra*, 10 January 2019.
`http://www.cip.ifi.lmu.de/~grinberg/primes2015/sols.pdf`
The numbering of theorems and formulas in this link might shift when the project gets updated; for a "frozen" version whose numbering is guaranteed to match that in the citations above, see `https://github.com/darijgr/detnotes/releases/tag/2019-01-10` .