Math 201-003 Fall 2019 (Darij Grinberg): midterm 2

1. Reminders

Definition 1.1. Let A be an $n \times n$-matrix. Then, the **determinant** $\det A$ of A is defined to be the sum

$$\sum_{\sigma \text{ is a permutation of } [n]} \operatorname{sign}(\sigma) \cdot A_{1,\sigma(1)} A_{2,\sigma(2)} \cdots A_{n,\sigma(n)}.$$

Here, $[n]$ means $\{1, 2, \ldots, n\}$.

Theorem 1.2 (Laplace expansion)**.** Let A be an $n \times n$-matrix. For each $p, q \in [n]$, we let $M_{p,q}$ be the $(n-1) \times (n-1)$-matrix obtained from A by removing row p and column q. Then:

(a) For each $p \in [n]$, we have

$$\det A = \sum_{q=1}^{n} (-1)^{p+q} A_{p,q} \det \left(M_{p,q}\right).$$

(This is called **Laplace expansion along the p-th row**.)

(b) For each $q \in [n]$, we have

$$\det A = \sum_{p=1}^{n} (-1)^{p+q} A_{p,q} \det \left(M_{p,q}\right).$$

(This is called **Laplace expansion along the q-th column**.)

Definition 1.3. Let A be an $n \times n$-matrix. Let λ be a scalar (i.e., a real number).

(a) A **λ-eigenvector of** A means a nonzero vector $v \in \mathbb{R}^n$ such that $Av = \lambda v$.

(b) We say that λ is an **eigenvalue** of A if and only if there exists a λ-eigenvector of A.

(c) The **characteristic polynomial** of A is the polynomial

$$\chi_A(t) = \det(A - tI_n).$$

2. Determinants

Exercise 1. Let a, b, c, x, y, z be six reals.

(a) Find

$$\det \begin{pmatrix} a-x & a-y & a-z \\ b-x & b-y & b-z \\ c-x & c-y & c-z \end{pmatrix}.$$

(b) Find

$$\det \begin{pmatrix} 1+ax & 1+ay & 1+az \\ 1+bx & 1+by & 1+bz \\ 1+cx & 1+cy & 1+cz \end{pmatrix}.$$

(c) Find

$$\det \begin{pmatrix} a & b & c \\ x & y & 0 \\ z & 0 & 0 \end{pmatrix}.$$

Solution to Exercise 1. Let us recall some properties of determinants:

Property 1: If an $n \times n$-matrix A has two equal rows, then $\det A = 0$. (This is Theorem 1.3.3 in the class notes from 2019-10-23.)

Property 2: If we scale a row of an $n \times n$-matrix by a number λ, then $\det A$ gets multiplied by λ. (This is Theorem 1.3.5 in the class notes from 2019-10-23.)

Property 3: We have $\det (XY) = \det X \cdot \det Y$ for any two $n \times n$-matrices X and Y. (This is Theorem 1.5.1 in the class notes from 2019-10-30.)

Property 4: If an $n \times n$-matrix A has a zero row (i.e., a row full of zeroes), then $\det A = 0$. (This is Corollary 1.2.1 in the class notes from 2019-10-30.)

Property 5: Let A be an $n \times n$-matrix, and let p and q be two distinct elements of $[n]$. If we add $\lambda \cdot \operatorname{row}_p A$ to the q-th row of A, then $\det A$ does not change. (This is Corollary 1.2.5 in the class notes from 2019-10-30.)

Property 6: Let A be an $n \times n$-matrix, and let p and q be two distinct elements of $[n]$. If we subtract $\operatorname{row}_p A$ from the q-th row of A, then $\det A$ does not change. (This follows from Property 5 (applied to $\lambda = -1$), because adding $(-1) \cdot \operatorname{row}_p A$ to the q-th row of A is the same as subtracting $\operatorname{row}_p A$ from the q-th row of A.)

Property 7: If an $n \times n$-matrix A is triangular (i.e., upper-triangular or lower-triangular), then its determinant is the product of its diagonal elements:

$$\det A = A_{1,1} A_{2,2} \cdots A_{n,n}.$$

(This is Theorem 1.1.2 in the class notes from 2019-10-30.)

Property 8: If we swap two rows of an $n \times n$-matrix, then its determinant gets multiplied by -1 (that is, it flips its sign but preserves its magnitude). (This is Theorem 1.2.6 in the class notes from 2019-10-30.)

(a) Part **(a)** of the exercise is similar to Exercise 3 on midterm training #2 (but we have minus signs instead of plus signs now, and we have restricted ourselves to a 3×3-matrix to make the solution more intuitive). Let us give two solutions.

*First solution to part **(a)**:* We have

$$\begin{aligned}
&\det \begin{pmatrix} a-x & a-y & a-z \\ b-x & b-y & b-z \\ c-x & c-y & c-z \end{pmatrix} \\
&= \det \begin{pmatrix} a-x & a-y & a-z \\ b-a & b-a & b-a \\ c-x & c-y & c-z \end{pmatrix} \\
&\qquad \left(\begin{array}{c} \text{here, we have subtracted the 1-st row of our matrix} \\ \text{from the 2-nd row; this did not change the determinant} \\ \text{(by Property 6)} \end{array} \right) \\
&= \det \begin{pmatrix} a-x & a-y & a-z \\ b-a & b-a & b-a \\ c-a & c-a & c-a \end{pmatrix} \\
&\qquad \left(\begin{array}{c} \text{here, we have subtracted the 1-st row of our matrix} \\ \text{from the 3-rd row; this did not change the determinant} \\ \text{(by Property 6)} \end{array} \right) \\
&= (b-a) \det \begin{pmatrix} a-x & a-y & a-z \\ 1 & 1 & 1 \\ c-a & c-a & c-a \end{pmatrix} \\
&\qquad \left(\begin{array}{c} \text{by Property 2, because the matrix before the} \\ \text{equality sign can be obtained from the matrix after} \\ \text{the equality sign by scaling the 2-nd row by } b-a \end{array} \right) \\
&= (b-a)(c-a) \underbrace{\det \begin{pmatrix} a-x & a-y & a-z \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}}_{\substack{=0 \\ \text{(by Property 1, because this matrix has two equal rows} \\ \text{(namely, its 2-nd and 3-rd rows are equal))}}}
\end{aligned}$$

$$= 0.$$

*Second solution to part **(a)**:* Define two $n \times n$-matrices X and Y by

$$X = \begin{pmatrix} a & -1 & 0 \\ b & -1 & 0 \\ c & -1 & 0 \end{pmatrix} \qquad \text{and} \qquad Y = \begin{pmatrix} 1 & 1 & 1 \\ x & y & z \\ 0 & 0 & 0 \end{pmatrix}.$$

The matrix Y has a zero row (namely, its third row). Thus, Property 4 yields $\det Y = 0$.

But it is easy to see that $\begin{pmatrix} a-x & a-y & a-z \\ b-x & b-y & b-z \\ c-x & c-y & c-z \end{pmatrix} = XY$. Hence,

$$\begin{aligned} \det \begin{pmatrix} a-x & a-y & a-z \\ b-x & b-y & b-z \\ c-x & c-y & c-z \end{pmatrix} &= \det(XY) = \det X \cdot \underbrace{\det Y}_{=0} \qquad \text{(by Property 3)} \\ &= 0. \end{aligned}$$

(b) This is similar to part **(a)**. Again, we will give two solutions:

*First solution to part **(b)**:* We have

$$\begin{aligned} &\det \begin{pmatrix} 1+ax & 1+ay & 1+az \\ 1+bx & 1+by & 1+bz \\ 1+cx & 1+cy & 1+cz \end{pmatrix} \\ &= \det \begin{pmatrix} 1+ax & 1+ay & 1+az \\ (b-a)x & (b-a)y & (b-a)z \\ 1+cx & 1+cy & 1+cz \end{pmatrix} \\ &\qquad \left(\begin{array}{c} \text{here, we have subtracted the 1-st row of our matrix} \\ \text{from the 2-nd row; this did not change the determinant} \\ \text{(by Property 6)} \end{array}\right) \\ &= \det \begin{pmatrix} 1+ax & 1+ay & 1+az \\ (b-a)x & (b-a)y & (b-a)z \\ (c-a)x & (c-a)y & (c-a)z \end{pmatrix} \\ &\qquad \left(\begin{array}{c} \text{here, we have subtracted the 1-st row of our matrix} \\ \text{from the 3-rd row; this did not change the determinant} \\ \text{(by Property 6)} \end{array}\right) \\ &= (b-a)\det \begin{pmatrix} 1+ax & 1+ay & 1+az \\ x & y & z \\ (c-a)x & (c-a)y & (c-a)z \end{pmatrix} \end{aligned}$$

$$\left(\begin{array}{c}\text{by Property 2, because the matrix before the}\\ \text{equality sign can be obtained from the matrix after}\\ \text{the equality sign by scaling the 2-nd row by } b-a\end{array}\right)$$

$$= (b-a)(c-a) \underbrace{\det\begin{pmatrix} 1+ax & 1+ay & 1+az \\ x & y & z \\ x & y & z \end{pmatrix}}_{\substack{=0 \\ \text{(by Property 1, because this matrix has two equal rows} \\ \text{(namely, its 2-nd and 3-rd rows are equal))}}}$$

$$= 0.$$

*Second solution to part **(b)**:* Define two $n \times n$-matrices X and Y by

$$X = \begin{pmatrix} 1 & a & 0 \\ 1 & b & 0 \\ 1 & c & 0 \end{pmatrix} \qquad \text{and} \qquad Y = \begin{pmatrix} 1 & 1 & 1 \\ x & y & z \\ 0 & 0 & 0 \end{pmatrix}.$$

The matrix Y has a zero row (namely, its third row). Thus, Property 4 yields $\det Y = 0$.

But it is easy to see that $\begin{pmatrix} 1+ax & 1+ay & 1+az \\ 1+bx & 1+by & 1+bz \\ 1+cx & 1+cy & 1+cz \end{pmatrix} = XY$. Hence,

$$\det\begin{pmatrix} 1+ax & 1+ay & 1+az \\ 1+bx & 1+by & 1+bz \\ 1+cx & 1+cy & 1+cz \end{pmatrix} = \det(XY) = \det X \cdot \underbrace{\det Y}_{=0} \qquad \text{(by Property 3)}$$
$$= 0.$$

(c) Again, let us give two solutions:

*First solution to part **(c)**:* Recall the explicit formula

$$\det A = A_{1,1}A_{2,2}A_{3,3} + A_{1,2}A_{2,3}A_{3,1} + A_{1,3}A_{2,1}A_{3,2} - A_{1,1}A_{2,3}A_{3,2} - A_{1,2}A_{2,1}A_{3,3} - A_{1,3}A_{2,2}A_{3,1}$$

for the determinant of an arbitrary 3×3-matrix. Applying this formula to $A = \begin{pmatrix} a & b & c \\ x & y & 0 \\ z & 0 & 0 \end{pmatrix}$, we obtain

$$\det\begin{pmatrix} a & b & c \\ x & y & 0 \\ z & 0 & 0 \end{pmatrix} = ay \cdot 0 + b \cdot 0z + cx \cdot 0 - a0 \cdot 0 - bx \cdot 0 - cyz = -cyz.$$

Second solution to part ***(c)****:* We have

$$\begin{aligned}
&\det\begin{pmatrix} a & b & c \\ x & y & 0 \\ z & 0 & 0 \end{pmatrix} \\
&= - \underbrace{\det\begin{pmatrix} z & 0 & 0 \\ x & y & 0 \\ a & b & c \end{pmatrix}}_{\substack{=zyc \\ \text{(by Property 7, since this matrix} \\ \text{is lower-triangular)}}} \\
&\qquad \left(\begin{array}{c} \text{here, we have swapped the 1-st and 3-rd rows of the matrix;} \\ \text{this caused the determinant to get multiplied by } -1 \\ \text{(by Property 8)} \end{array}\right) \\
&= -zyc = -cyz.
\end{aligned}$$

□

Let us remark that all three parts of Exercise 1 can be generalized to $n \times n$-matrices. Indeed, Exercise 1 **(a)** is a particular case of the following fact:

Proposition 2.1. Let $n \geq 3$ be an integer. Let $x_1, x_2, \ldots, x_n$ be n reals, and let $y_1, y_2, \ldots, y_n$ be n further reals. Let D_n denote the $n \times n$-matrix

$$(x_i - y_j)_{1 \leq i \leq n,\ 1 \leq j \leq n} = \begin{pmatrix} x_1 - y_1 & x_1 - y_2 & \cdots & x_1 - y_n \\ x_2 - y_1 & x_2 - y_2 & \cdots & x_2 - y_n \\ \vdots & \vdots & \ddots & \vdots \\ x_n - y_1 & x_n - y_2 & \cdots & x_n - y_n \end{pmatrix}.$$

Then, $\det(D_n) = 0$.

This Proposition 2.1 follows from our answer to Exercise 3 on midterm training #2 (applied to $-y_j$ instead of y_j).

Exercise 1 **(b)** is a particular case of the following fact:

Proposition 2.2. Let $n \geq 3$ be an integer. Let $x_1, x_2, \ldots, x_n$ be n reals, and let $y_1, y_2, \ldots, y_n$ be n further reals. Let T_n denote the $n \times n$-matrix

$$(1 + x_i y_j)_{1 \leq i \leq n,\ 1 \leq j \leq n} = \begin{pmatrix} 1 + x_1 y_1 & 1 + x_1 y_2 & \cdots & 1 + x_1 y_n \\ 1 + x_2 y_1 & 1 + x_2 y_2 & \cdots & 1 + x_2 y_n \\ \vdots & \vdots & \ddots & \vdots \\ 1 + x_n y_1 & 1 + x_n y_2 & \cdots & 1 + x_n y_n \end{pmatrix}.$$

Then, $\det(T_n) = 0$.

This Proposition 2.2 can be proved by adapating any of our above two solutions to Exercise 1 **(b)** in a straightforward manner.

Finally, Exercise 1 **(c)** is a particular case of the following fact:

Proposition 2.3. Let $n \in \mathbb{N}$. Let A be an $n \times n$-matrix. The straight line connecting the upper-right corner of A with the bottom-left corner of A will be called the *anti-diagonal* of A.

Assume that all entries of A below the anti-diagonal are 0. (In other words, assume that $A_{i,j} = 0$ for all (i, j) satisfying $i + j > n + 1$.)

Then,

$$\det A = (-1)^{n(n-1)/2} A_{1,n} A_{2,n-1} \cdots A_{n-1,2} A_{n,1}.$$

Applying Proposition 2.3 to $n = 3$, we recover Exercise 1 **(c)**.

Proposition 2.3 can be proven in the same two ways in which we solved Exercise 1 **(c)**, although each of them requires some more thinking in the general case:

- Imitating the second solution to Exercise 1 **(c)**, we can prove Proposition 2.3 by turning the matrix A into a lower-triangular matrix by a sequence of row swaps. (There are several ways to do so; the conceptually simplest way is probably to swap every row with every row.)

- Imitating the first solution to Exercise 1 **(c)**, we can prove Proposition 2.3 by showing that only 1 of the $n!$ many addends in the definition of $\det A$ has any chance to be nonzero, and that this addend is precisely

$$(-1)^{n(n-1)/2} A_{1,n} A_{2,n-1} \cdots A_{n-1,2} A_{n,1}.$$

  (This addend corresponds to the permutation σ of $[n]$ that sends $1, 2, \ldots, n-1, n$ to $n, n-1, \ldots, 2, 1$ respectively; the sign of this permutation is $\operatorname{sign}(\sigma) = (-1)^{n(n-1)/2}$, since σ has $n(n-1)/2$ inversions.)

We leave the details to the interested reader.

Exercise 2. For each $n \in \mathbb{N}$, let B_n be the $n \times n$-matrix

$$\begin{pmatrix} 2 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 1 & 2 & 1 & \cdots & 0 & 0 & 0 \\ 0 & 1 & 2 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 2 & 1 & 0 \\ 0 & 0 & 0 & \cdots & 1 & 2 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 1 & 2 \end{pmatrix}.$$

(This is the $n \times n$-matrix whose entries on the diagonal are 2, while all entries just above and just below the diagonal are 1, and all remaining entries are 0.)

(a) Find $\det(B_1)$. (Note that $B_1 = \begin{pmatrix} 2 \end{pmatrix}$.)

(b) Find $\det(B_2)$. (Note that $B_2 = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$.)

(c) Find $\det(B_3)$. (Note that $B_3 = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}$.)

(d) For a general integer $n \geq 2$, find an expression for $\det(B_n)$ in terms of $\det(B_{n-1})$ and $\det(B_{n-2})$.

(e) Find a formula for $\det(B_n)$ in terms of things we have seen in class.

[**Hint:** In **(d)**, use Laplace expansion.]

Solution to Exercise 2. This exercise is analogous to Exercise 4 on midterm training #2. Parts **(a)**, **(b)**, **(c)** and **(d)** are solved in the exact same way (with some obvious changes) as the corresponding parts of the latter exercise. Part **(e)**, however, involves a new twist.

(a) We have $B_1 = \begin{pmatrix} 2 \end{pmatrix}$ and thus $\det(B_1) = \det\begin{pmatrix} 2 \end{pmatrix} = 2$.

(b) We have $B_2 = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ and thus $\det(B_2) = \det\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} = 3$.

(c) We have $B_3 = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}$ and thus $\det(B_3) = \det\begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix} = 4$.

(d) Let n be an integer such that $n \geq 2$.

We have

$$B_n = \begin{pmatrix} 2 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 1 & 2 & 1 & \cdots & 0 & 0 & 0 \\ 0 & 1 & 2 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 2 & 1 & 0 \\ 0 & 0 & 0 & \cdots & 1 & 2 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 1 & 2 \end{pmatrix}_{n \times n}.$$

Here, the subscript "$n \times n$" tells us that the matrix you are seeing is understood to be an $n \times n$-matrix. (Similar notations will be used further below.)

The 1-st row of the matrix B_n has only two nonzero entries: the 2 in position 1, and the 1 in position 2. Hence, Laplace expansion along the 1-st row (see Theorem 1.7.1 in the class notes from 2019-10-30) yields

$$\det(B_n) = 2 \cdot \det \underbrace{\begin{pmatrix} 2 & 1 & \cdots & 0 & 0 & 0 \\ 1 & 2 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 2 & 1 & 0 \\ 0 & 0 & \cdots & 1 & 2 & 1 \\ 0 & 0 & \cdots & 0 & 1 & 2 \end{pmatrix}_{(n-1)\times(n-1)}}_{=B_{n-1}}$$

$$- 1 \cdot \det \begin{pmatrix} 1 & 1 & \cdots & 0 & 0 & 0 \\ 0 & 2 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 2 & 1 & 0 \\ 0 & 0 & \cdots & 1 & 2 & 1 \\ 0 & 0 & \cdots & 0 & 1 & 2 \end{pmatrix}_{(n-1)\times(n-1)}$$

$$+ (\text{several addends that equal } 0 \text{ and thus can be ignored})$$

$$= 2 \cdot \det(B_{n-1}) - \det \begin{pmatrix} 1 & 1 & \cdots & 0 & 0 & 0 \\ 0 & 2 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 2 & 1 & 0 \\ 0 & 0 & \cdots & 1 & 2 & 1 \\ 0 & 0 & \cdots & 0 & 1 & 2 \end{pmatrix}_{(n-1)\times(n-1)} .$$

Let us denote the second matrix on the right hand side of this equality[1] by C_{n-1}. The 1-st column of this matrix C_{n-1} has only one nonzero entry, namely the 1 in position 1. Hence, Laplace expansion along the 1-st column (see Theorem 1.2.1 in

[1] i.e., the matrix $\begin{pmatrix} 1 & 1 & \cdots & 0 & 0 & 0 \\ 0 & 2 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 2 & 1 & 0 \\ 0 & 0 & \cdots & 1 & 2 & 1 \\ 0 & 0 & \cdots & 0 & 1 & 2 \end{pmatrix}_{(n-1)\times(n-1)}$

the class notes from 2019-11-04) yields

$$\det\left(C_{n-1}\right) = 1 \cdot \det \underbrace{\begin{pmatrix} 2 & \cdots & 0 & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & \cdots & 2 & 1 & 0 \\ 0 & \cdots & 1 & 2 & 1 \\ 0 & \cdots & 0 & 1 & 2 \end{pmatrix}_{(n-2)\times(n-2)}}_{=B_{n-2}} = 1 \cdot \det\left(B_{n-2}\right)$$
$$= \det\left(B_{n-2}\right).$$

Thus, our above computation becomes

$$\det\left(B_n\right) = 2 \cdot \det\left(B_{n-1}\right) - \det \underbrace{\begin{pmatrix} -1 & 1 & \cdots & 0 & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 1 & 0 \\ 0 & 0 & \cdots & -1 & 1 & 1 \\ 0 & 0 & \cdots & 0 & -1 & 1 \end{pmatrix}_{(n-1)\times(n-1)}}_{=C_{n-1}}$$
$$= 2 \cdot \det\left(B_{n-1}\right) - \underbrace{\det\left(C_{n-1}\right)}_{=\det\left(B_{n-2}\right)} = 2 \cdot \det\left(B_{n-1}\right) - \det\left(B_{n-2}\right).$$

This solves part **(d)** of the exercise.

(e) We claim that

$$\det\left(B_n\right) = n+1 \qquad \text{for each } n \geq 1. \tag{1}$$

[*Proof of (1):* We shall prove (1) by strong induction on n:

Induction step: Let $m \geq 1$ be an integer. Assume (as induction hypothesis) that (1) holds for all $n < m$. We must then show that (1) holds for $n = m$. In other words, we must show that $\det\left(B_m\right) = m+1$.

This is clearly true when $m = 1$ (because in part **(a)** of this exercise, we showed that $\det\left(B_1\right) = 2 = 1+1$). It is also true when $m = 2$ (because in part **(b)** of this exercise, we showed that $\det\left(B_2\right) = 3 = 2+1$). Thus, for the rest of this induction step, we can WLOG assume that $m \neq 1$ and $m \neq 2$. Assume this; hence, $m \geq 3$.

The integer $m-1$ satisfies $m-1 \geq 1$ (since $m \geq 3 \geq 2$) and $m-1 < m$. Thus, our induction hypothesis shows that (1) holds for $n = m-1$. In other words, we have $\det\left(B_{m-1}\right) = (m-1)+1$.

The integer $m-2$ satisfies $m-2 \geq 1$ (since $m \geq 3$) and $m-2 < m$. Thus, our induction hypothesis shows that (1) holds for $n = m-2$. In other words, we have $\det\left(B_{m-2}\right) = (m-2)+1$.

But in part **(d)** of the exercise, we have seen that $\det(B_n) = 2 \cdot \det(B_{n-1}) - \det(B_{n-2})$ for each $n \geq 2$. Applying this to $n = m$, we find

$$\det(B_m) = 2 \cdot \underbrace{\det(B_{m-1})}_{=(m-1)+1} - \underbrace{\det(B_{m-2})}_{=(m-2)+1} = 2 \cdot ((m-1)+1) - ((m-2)+1) = m+1.$$

In other words, (1) holds for $n = m$. This completes the induction step. Thus, (1) is proved by strong induction.]

Now that (1) is proved, part **(e)** of the exercise is solved.

[*Remark:* The above proof of (1) by strong induction is rigorous and completely straightforward (once you know what you are proving!), but it is somewhat bland and unmotivated (in the sense that you need to know the formula (1) in order to find this proof). Of course, this is not a serious issue here, since the formula (1) is easy to guess (just look at the values of $\det(B_n)$ for $n = 0, 1, 2, 3, 4$, and extend the obvious pattern). Nevertheless, there is value in having a more motivated proof. Here is an outline of such a proof: In part **(d)** of the exercise, we have seen that $\det(B_n) = 2 \cdot \det(B_{n-1}) - \det(B_{n-2})$ for each $n \geq 2$. We can rewrite this equality as

$$\det(B_n) - \det(B_{n-1}) = \det(B_{n-1}) - \det(B_{n-2}).$$

But this equality is just saying that the differences between consecutive elements of the sequence

$$(\det(B_0), \det(B_1), \det(B_2), \det(B_3), \ldots)$$

don't change from one pair of consecutive elements to the next. In other words, these differences are all the same. In other words, the sequence

$$(\det(B_0), \det(B_1), \det(B_2), \det(B_3), \ldots)$$

is an arithmetic sequence. Which arithmetic sequence? Since an arithmetic sequence is uniquely determined by any two of its values, we can answer this question by finding $\det(B_1)$ and $\det(B_2)$ and extending the pattern. Since $\det(B_1) = 2$ and $\det(B_2) = 3$, we see that the pattern is $\det(B_n) = n+1$ for all $n \geq 0$. This proves (1) again.] $\square$

3. Eigenvalues and eigenvectors

Exercise 3. Find the eigenvalues and the eigenvectors of the 3×3-matrix $\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$.

Solution to Exercise 3. Let A denote this 3×3-matrix. Then, the characteristic polynomial of A is

$$\begin{aligned}
\chi_A(t) &= \det(A - tI_3) = \det \begin{pmatrix} -t & 1 & 0 \\ 1 & -t & 1 \\ 0 & 1 & -t \end{pmatrix} \\
&= 2t - t^3 \qquad \text{(this is easy to check by expanding the determinant)} \\
&= t \underbrace{\left(2 - t^2\right)}_{=\left(\sqrt{2}-t\right)\left(\sqrt{2}+t\right)} = t\left(\sqrt{2} - t\right)\left(\sqrt{2} + t\right).
\end{aligned}$$

Hence, the roots of $\chi_A(t)$ are $0, \sqrt{2}, -\sqrt{2}$.

Recall that the eigenvalues of A are the roots of $\chi_A(t)$ (by Proposition 2.1.7 in the class notes from 2019-11-04). But the roots of $\chi_A(t)$ are $0, \sqrt{2}, -\sqrt{2}$. Hence, the eigenvalues of A are $0, \sqrt{2}, -\sqrt{2}$.

It remains to find the eigenvectors. This is an easy matter of solving systems of linear equations:

- The 0-eigenvectors of A are the nonzero vectors $v \in \mathbb{R}^3$ satisfying $Av = 0v$. In other words, they are the nonzero vectors $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3$ such that $A \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0 \begin{pmatrix} x \\ y \\ z \end{pmatrix}$. This is a system of 3 linear equations in the unknowns x, y, z; solving it by Gaussian elimination, we obtain $\begin{cases} x = -z \\ y = 0 \end{cases}$ (where z is a free variable). Thus, they are the nonzero scalar multiples of the vector $\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$.

- Likewise, the $\sqrt{2}$-eigenvectors of A are the nonzero scalar multiples of the vector $\begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix}$.

- Likewise, the $\left(-\sqrt{2}\right)$-eigenvectors of A are the nonzero scalar multiples of the vector $\begin{pmatrix} 1 \\ -\sqrt{2} \\ 1 \end{pmatrix}$.

$\square$

Exercise 4. Consider the matrix $A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 2 & 3 & 0 \\ 0 & 0 & 3 & 4 \end{pmatrix}$. Here are four eigenvectors of A:

$$\begin{pmatrix} 0 \\ 0 \\ -1 \\ 3 \end{pmatrix}, \qquad \begin{pmatrix} 0 \\ 1 \\ -2 \\ 3 \end{pmatrix}, \qquad \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \qquad \begin{pmatrix} -1 \\ 1 \\ -1 \\ 1 \end{pmatrix}.$$

(a) What are the eigenvalues corresponding to these four eigenvectors?

(b) Find a diagonalization of A – that is, a pair (U, D) of an invertible matrix U and a diagonal matrix D such that $A = UDU^{-1}$.

Solution to Exercise 4. **(a)** Let us denote these four eigenvectors by u_1, u_2, u_3, u_4, in the order in which they are given. Let $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ be the corresponding eigenvalues of A.

What is the quickest way to find λ_1 ? We must have $Au_1 = \lambda_1 u_1$ (since u_1 is a λ_1-eigenvector of A). In view of $A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 2 & 3 & 0 \\ 0 & 0 & 3 & 4 \end{pmatrix}$ and $u_1 = \begin{pmatrix} 0 \\ 0 \\ -1 \\ 3 \end{pmatrix}$, this rewrites as

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 2 & 3 & 0 \\ 0 & 0 & 3 & 4 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ -1 \\ 3 \end{pmatrix} = \lambda_1 \begin{pmatrix} 0 \\ 0 \\ -1 \\ 3 \end{pmatrix}.$$

Multiplying the left hand side of this equation out, we rewrite it as

$$\begin{pmatrix} 0 \\ 0 \\ -3 \\ 9 \end{pmatrix} = \lambda_1 \begin{pmatrix} 0 \\ 0 \\ -1 \\ 3 \end{pmatrix}.$$

By comparing the 4-th entries on the vectors of both sides of this equality, we obtain $9 = \lambda_1 \cdot 3$. Solving this for λ_1, we find $\lambda_1 = 3$.

[Note that you don't even need to fully multiply the left hand side out to see this! Just find one nonzero entry of the product, and compare it with the corresponding entry on the right hand side.]

Similarly, we can find $\lambda_2 = 2$ and $\lambda_3 = 4$ and $\lambda_4 = 1$. Thus, the eigenvalues corresponding to our four eigenvectors are $3, 2, 4, 1$ (in this order).

(b) The four eigenvectors u_1, u_2, u_3, u_4 of A correspond to the four distinct eigenvalues $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ (indeed, these four eigenvalues are distinct, because they equal $3, 2, 4, 1$, as we have just observed). Thus, Proposition 1.3.3 in the class notes from 2019-11-06 shows that u_1, u_2, u_3, u_4 form a basis of $\mathbb{R}^4$.

Hence, the 4-tuple (u_1, u_2, u_3, u_4) is a basis of $\mathbb{R}^4$ that consists of eigenvectors of A, and $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ are the corresponding eigenvalues. Hence, we can find a diagonalization of A using Proposition 1.2.3 **(a)** in the class notes from 2019-11-11: We set

$$U = [u_1 \mid u_2 \mid u_3 \mid u_4] = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 1 \\ -1 & -2 & 0 & -1 \\ 3 & 3 & 1 & 1 \end{pmatrix} \qquad \text{and}$$

$$D = \operatorname{diag}(\lambda_1, \lambda_2, \lambda_3, \lambda_4) = \operatorname{diag}(3, 2, 4, 1) = \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The pair U, D is a diagonalization of A (that is, U is invertible, and D is diagonal, and we have $A = UDU^{-1}$). $\square$

Exercise 5. Let a, b, c, x, y, z be six reals. Let A be the 3×3-matrix

$$\begin{pmatrix} ax & ay & az \\ bx & by & bz \\ cx & cy & cz \end{pmatrix}.$$

(a) Explain why $ax + by + cz$ is an eigenvalue of A, and find a corresponding eigenvector.

(b) Assume that a, b, c, x, y, z are nonzero. Find a 0-eigenvector of A.

Solution to Exercise 5. **(a)** It clearly suffices to find an $(ax + by + cz)$-eigenvector of A, because if such an eigenvector exists, then $ax + by + cz$ will be an eigenvalue of A (by the definition of "eigenvalue").

If a, b, c are all 0, then an $(ax + by + cz)$-eigenvector of A is easy to find, because **any** nonzero vector in $\mathbb{R}^3$ is an $(ax + by + cz)$-eigenvector of A in this case[2]. Thus, for the rest of this proof, we WLOG assume that a, b, c are not all 0. Hence, the vector $\begin{pmatrix} a \\ b \\ c \end{pmatrix} \in \mathbb{R}^3$ is nonzero.

The vector $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ is an $(ax + by + cz)$-eigenvector of A, since it is nonzero and satisfies

$$\begin{aligned} A \begin{pmatrix} a \\ b \\ c \end{pmatrix} &= \begin{pmatrix} ax & ay & az \\ bx & by & bz \\ cx & cy & cz \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} a^2x + aby + acz \\ b^2y + abx + bcz \\ c^2z + acx + bcy \end{pmatrix} \\ &= \begin{pmatrix} (ax + by + cz)\, a \\ (ax + by + cz)\, b \\ (ax + by + cz)\, c \end{pmatrix} = (ax + by + cz) \begin{pmatrix} a \\ b \\ c \end{pmatrix}. \end{aligned}$$

Hence, we have found an $(ax + by + cz)$-eigenvector of A.

(b) The vector $\begin{pmatrix} 0 \\ z \\ -y \end{pmatrix}$ is nonzero (since y is nonzero) and satisfies

$$A \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} ax & ay & az \\ bx & by & bz \\ cx & cy & cz \end{pmatrix} \begin{pmatrix} 0 \\ z \\ -y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = 0 \begin{pmatrix} 0 \\ z \\ -y \end{pmatrix}.$$

[2] *Proof.* Assume that a, b, c are all 0. Then, A is the zero matrix, and therefore $Av = 0 = 0v$ for each $v \in \mathbb{R}^3$. Hence, any nonzero vector in $\mathbb{R}^3$ is a 0-eigenvector of A. In other words, any nonzero vector is an $(ax + by + cz)$-eigenvector of A (since $\underbrace{a}_{=0} x + \underbrace{b}_{=0} y + \underbrace{c}_{=0} z = 0x + 0y + 0z = 0$).

Qed.

In other words, this vector is a 0-eigenvector of A.

[*Remark:* Many answers are possible in part **(b)**. In particular, all three vectors $\begin{pmatrix} 0 \\ z \\ -y \end{pmatrix}$, $\begin{pmatrix} -z \\ 0 \\ x \end{pmatrix}$ and $\begin{pmatrix} y \\ -x \\ 0 \end{pmatrix}$ as well as all their nonzero linear combinations are 0-eigenvectors of A.

Note that our above argument only used the assumption that y is nonzero (not that the rest of a, b, c, x, z are nonzero). Likewise, we can solve the problem if x is nonzero or if z is nonzero. On the other hand, if all of x, y, z are zero, then $A = 0_{3\times 3}$, and thus every nonzero vector in $\mathbb{R}^3$ is a 0-eigenvector of A.] $\square$

Exercise 6. True or false? **No justifications are required in this exercise. Just write Y or N into the respective box!**

(a) $\boxed{}$ An $n \times n$-matrix A is invertible if and only if 0 is an eigenvalue of A.

(b) $\boxed{}$ An $n \times n$-matrix A is invertible if and only if 0 is not an eigenvalue of A.

(c) $\boxed{}$ The column vector $e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \in \mathbb{R}^n$ is an eigenvector of every upper-triangular $n \times n$-matrix.

(d) $\boxed{}$ The column vector $e_n = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \in \mathbb{R}^n$ is an eigenvector of every upper-triangular $n \times n$-matrix.

(e) $\boxed{}$ If λ is an eigenvalue of two $n \times n$-matrices A and B, then λ is an eigenvalue of $A + B$ as well.

(f) $\boxed{}$ If v is an eigenvector of two $n \times n$-matrices A and B, then v is an eigenvector of $A + B$ as well.

(g) $\boxed{}$ If an $n \times n$-matrix A has n distinct eigenvalues, then it is diagonalizable.

(h) $\boxed{}$ If two $n \times n$-matrices A and B have the same characteristic polynomial (that is, $\chi_A(t) = \chi_B(t)$), then $\det A = \det B$.

(i) $\boxed{}$ If all diagonal entries of an $n \times n$-matrix A are 0, then $\det A = 0$.

(j) $\boxed{}$ If all off-diagonal entries of an $n \times n$-matrix A are 0, then $\det A = 0$.

Solution to Exercise 6. **(a) NO.**

Proof. The identity matrix I_n is invertible, but 0 is not among its eigenvalues (indeed, the only eigenvalue of I_n is 1, since $I_n v = v = 1v$ for each $v \in \mathbb{R}^n$).

(b) YES.

Proof. Let A be an $n \times n$-matrix. Recall that the eigenvalues of A are the roots of $\chi_A(t)$ (by Proposition 2.1.7 in the class notes from 2019-11-04).

We have $\chi_A(t) = \det(A - tI_n)$ (by the definition of χ_A). Substituting 0 for t in this equality, we find $\chi_A(0) = \det \underbrace{(A - 0I_n)}_{=A} = \det A$.

Theorem 1.4.1 in the class notes from 2019-10-30 shows that A is invertible if and only if $\det A \neq 0$. Thus, we have the following chain of equivalences:

$$\begin{aligned}
(A \text{ is invertible}) &\iff (\det A \neq 0) \\
&\iff (\chi_A(0) \neq 0) \qquad (\text{since } \det A = \chi_A(0)) \\
&\iff (0 \text{ is not a root of the polynomial } \chi_A(t)) \\
&\iff (0 \text{ is not an eigenvalue of } A)
\end{aligned}$$

(since the eigenvalues of A are the roots of $\chi_A(t)$). Thus, A is invertible if and only if 0 is not an eigenvalue of A. This proves the claim of part **(b)**.

(c) YES.

Proof. Let us just prove this in the case when $n = 3$: Let A be any upper-triangular 3×3-matrix. Then, we can write A in the form $A = \begin{pmatrix} a & b & c \\ 0 & b' & c' \\ 0 & 0 & c'' \end{pmatrix}$ for some reals a, b, c, b', c', c''. Thus, we have

$$\begin{aligned}
Ae_1 &= \begin{pmatrix} a & b & c \\ 0 & b' & c' \\ 0 & 0 & c'' \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad \left(\text{since } e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right) \\
&= \begin{pmatrix} a \cdot 1 + b \cdot 0 + c \cdot 0 \\ 0 \cdot 1 + b' \cdot 0 + c' \cdot 0 \\ 0 \cdot 1 + 0 \cdot 0 + 0 \cdot c'' \end{pmatrix} = \begin{pmatrix} a \\ 0 \\ 0 \end{pmatrix} = a \underbrace{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}}_{=e_1} = ae_1.
\end{aligned}$$

This shows that e_1 is an a-eigenvector of A (since e_1 is nonzero). Thus, e_1 is an eigenvector of A.

The same argument works for all n (except that the matrices become larger). Alternative, you may recall that $Ae_1 = \operatorname{col}_1 A$ (as a consequence of Exercise 4 **(b)** on homework set #1), and use this to show that $Ae_1 = A_{1,1} e_1$ whenever A is upper-triangular. Either way, the claim of part **(c)** is proved.

(d) NO.

Proof. Just check that this fails for the upper-triangular matrix $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ when $n = 2$. (Indeed, $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} e_n = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is not a scalar multiple of e_n.)

(e) NO.

Proof. The number 1 is an eigenvalue of the 1×1-matrices $\begin{pmatrix} 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \end{pmatrix}$, but is not an eigenvalue of their sum $\begin{pmatrix} 1 \end{pmatrix} + \begin{pmatrix} 1 \end{pmatrix} = \begin{pmatrix} 2 \end{pmatrix}$.

(f) YES.

Proof. Let A and B be two $n \times n$-matrices. Let $v \in \mathbb{R}$ be an eigenvector of both matrices A and B. Let λ and μ be the corresponding eigenvalues (so that v is a λ-eigenvector of A and a μ-eigenvector of B). Thus, $Av = \lambda v$ and $Bv = \mu v$. Adding these two equalities together, we obtain $Av + Bv = \lambda v + \mu v = (\lambda + \mu) v$. Hence, $(A + B) v = Av + Bv = (\lambda + \mu) v$. Since v is nonzero (because v is an eigenvector of A), this shows that v is a $(\lambda + \mu)$-eigenvector of $A + B$. Hence, v is an eigenvector of $A + B$. This proves the claim of part **(f)** of the exercise.

(g) YES.

Proof. Let A be an $n \times n$-matrix that has n distinct eigenvalues. Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be these eigenvalues. For each $i \in \{1, 2, \ldots, n\}$, there exists a λ_i-eigenvector u_i (by the definition of an eigenvalue); consider this u_i. Thus, Proposition 1.3.3 in the class notes from 2019-11-06 shows that $u_1, u_2, \ldots, u_n$ form a basis of $\mathbb{R}^n$. Thus, $(u_1, u_2, \ldots, u_n)$ is a basis of $\mathbb{R}^n$ that consists of eigenvectors of A. Hence, we can find a diagonalization of A using Proposition 1.2.3 **(a)** in the class notes from 2019-11-11. Thus, A has a diagonalization, i.e., is diagonalizable. This proves the claim of part **(g)**.

(h) YES.

Proof. Let A and B be two matrices that have the same characteristic polynomial (that is, $\chi_A(t) = \chi_B(t)$).

We have $\chi_A(t) = \det(A - tI_n)$ (by the definition of χ_A). Substituting 0 for t in this equality, we find $\chi_A(0) = \det \underbrace{(A - 0I_n)}_{=A} = \det A$. Likewise, $\chi_B(0) = \det B$. But substituting 0 for t in the equality $\chi_A(t) = \chi_B(t)$, we obtain $\chi_A(0) = \chi_B(0)$. In view of $\chi_A(0) = \det A$ and $\chi_B(0) = \det B$, this rewrites as $\det A = \det B$. This proves the claim of part **(h)**.

(i) NO.

Proof. All diagonal entries of the 2×2-matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ are 0, yet the determinant of this matrix is $-1 \neq 0$.

(j) NO.

Proof. All off-diagonal entries of the 2×2-matrix $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ are 0, yet the determinant of this matrix is $1 \neq 0$. $\square$