Math 201-003 Fall 2019 (Darij Grinberg): midterm training 1

1. Matrix operations

Exercise 1. **(a)** Let $A_3 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$ and $B_3 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$. (These matrices are filled in "checkerboard patterns": Entries that are 0 alternate with entries that are 1.)

Compute A_3^2, B_3^2, A_3B_3 and B_3A_3.

(b) Let $A_4 = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}$ and $B_4 = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}$. (These follow the same patterns as A_3 and B_3.)

Compute A_4^2, B_4^2, A_4B_4 and B_4A_4.

Now, let us generalize:

For any positive integer n, define two "checkerboard-pattern" $n \times n$-matrices A_n and B_n by

$$A_n = ((i+j)\,\%2)_{1 \le i \le n,\ 1 \le j \le n}, \qquad B_n = ((i+j-1)\,\%2)_{1 \le i \le n,\ 1 \le j \le n},$$

where $k\%2$ denotes the remainder left when k is divided by 2 (so $k\%2 = 1$ when k is odd, and $k\%2 = 0$ when k is even).

(This is just a formal way to define two matrices that are filled in the same checkerboard way as A_3 and B_3 (or as A_4 and B_4).)

(c) Prove that each **even** $n \in \mathbb{N}$ satisfies $A_n^2 = B_n^2$ and $A_nB_n = B_nA_n$.

(d) Prove that each **odd** $n \ge 3$ satisfies $A_nB_n \ne B_nA_n$.

2. Gaussian elimination

Exercise 2. Consider the system

$$\begin{cases} a+b+c+d = e \\ a+2b+3c+4d = 5e \\ a+3b+6c+10d = 15e \end{cases}$$

of linear equations in five unknowns a, b, c, d, e.

(a) Find the augmented matrix corresponding to this system.

(b) Find the RREF of this matrix.

(c) Solve the system.

Exercise 3. **(a)** Let A_7 be the 7×7-matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}.$$

(Its diagonal entries are 1; its entries just below the diagonal are 1; its entry in the top-right corner is 1; all its other entries are 0.)

Find the RREF of A_7.

(b) Let A_8 be the 8×8-matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}.$$

(It is given by the same rule as A_7, except for having one more row and column.)

Find the RREF of A_8.

Exercise 4. Let $U = \begin{pmatrix} 6 & 3 & -2 & 5 \\ 0 & 0 & -1 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$.

(a) Find all column vectors $x \in \mathbb{R}^4$ satisfying $Ux = b$, where $b = \begin{pmatrix} 1 \\ 5 \\ 2 \\ 0 \end{pmatrix}$.

(b) Find all column vectors $x \in \mathbb{R}^4$ satisfying $Ux = b'$, where $b' = \begin{pmatrix} 1 \\ 5 \\ 2 \\ 1 \end{pmatrix}$.

(c) Find all column vectors $x \in \mathbb{R}^4$ satisfying $Ux = x$.

3. Linear combinations, independence and spanning

Exercise 5. Let $n \geq 2$ be an integer. Recall the vectors $e_1, e_2, \ldots, e_n$ in $\mathbb{R}^n$ that were defined in Exercise 4 of homework set #2.

Now, consider the n vectors

$$e_1 + e_2, \quad e_2 + e_3, \quad e_3 + e_4, \quad \ldots, \quad e_{n-1} + e_n, \quad e_n + e_1.$$

Let us denote them by $f_1, f_2, \ldots, f_n$. (Thus, $f_i = e_i + e_{i+1}$ for each $i \in \{1, 2, \ldots, n-1\}$, and $f_n = e_n + e_1$.)

(a) Are these n vectors $f_1, f_2, \ldots, f_n$ linearly independent when $n = 7$?

(b) Are these n vectors $f_1, f_2, \ldots, f_n$ linearly independent when $n = 8$?

(c) Do these n vectors $f_1, f_2, \ldots, f_n$ span $\mathbb{R}^n$ when $n = 7$?

(d) Do these n vectors $f_1, f_2, \ldots, f_n$ span $\mathbb{R}^n$ when $n = 8$?

Exercise 6. Fix an integer $n \geq 3$. Consider the following four $n \times n$-matrices:

- The $n \times n$-matrix N has all entries in its 1-st row equal to 1, while all other entries are 0.
- The $n \times n$-matrix E has all entries in its n-th column equal to 1, while all other entries are 0.
- The $n \times n$-matrix S has all entries in its n-th row equal to 1, while all other entries are 0.
- The $n \times n$-matrix W has all entries in its 1-st column equal to 1, while all other entries are 0.

For example, for $n = 3$, these matrices look as follows:

$$N = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad E = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix},$$
$$S = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}, \qquad W = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

(For $n = 2$, these are precisely the matrices N, E, S, W from Exercise 6 on homework set #2. But here we are assuming $n \geq 3$.)

(a) Is the identity matrix I_n a linear combination of N, E, S, W ?

(b) Are N, E, S, W linearly dependent?

4. Matrix inversion and invertibility

Exercise 7. Let A be the 4×4-matrix $\begin{pmatrix} 1 & a & b & c \\ 0 & 1 & b' & c' \\ 0 & 0 & 1 & c'' \\ 0 & 0 & 0 & 1 \end{pmatrix}$, where a, b, c, b', c', c'' are arbitrary reals. Compute the inverse A^{-1}.

Exercise 8. Which of the following matrices are invertible?

(a) $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$. **(b)** $\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$. **(c)** $\begin{pmatrix} 1 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 3 \end{pmatrix}$.

(d) $\begin{pmatrix} 1 & 4 & 9 \\ 5 & -1 & 4 \end{pmatrix}$.

[**Hint:** These can be solved without paper.]

Exercise 9. Let n be a nonnegative integer, and let A be an invertible $n \times n$-matrix. Prove that its transpose A^T is also invertible, and its inverse is $\left(A^T\right)^{-1} = \left(A^{-1}\right)^T$.

[**Hint:** By the definition of "inverse", this means showing that $A^T \left(A^{-1}\right)^T = I_n$ and $\left(A^{-1}\right)^T A^T = I_n$. Show this.]

5. Permutations

Recall that we are using $[n]$ to denote the n-element set $\{1, 2, \ldots, n\}$ whenever n is a nonnegative integer.

Exercise 10. Consider three maps α, β, γ from $[4]$ to $[4]$ given in two-line notation as follows:

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 1 & 3 \end{pmatrix}, \qquad \beta = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 3 \end{pmatrix}, \qquad \gamma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}.$$

(a) Which of α, β and γ are permutations?
(b) Express $\alpha \circ \beta$ and $\beta \circ \alpha$ and $\gamma \circ \gamma$ in two-line notation.
(c) Compute the signs of all permutations among α, β and γ.

6. Solutions

The following solutions are a bit rough at some places, but they have enough detail to get full scores.

Some of these solutions use tricks instead of systematic methods. You are free to use the methods – but the tricks are often faster and reveal some ideas that you would have missed if you just followed the methods.

Solution to Exercise 1. Recall the following fact ([lina, Proposition 2.19 **(b)**]): If A is an $n \times m$-matrix and B is an $m \times p$-matrix, then

$$(AB)_{i,j} = \operatorname{row}_i A \cdot \operatorname{col}_j B \qquad \text{for any } i \in \{1,2,\ldots,n\} \text{ and } j \in \{1,2,\ldots,p\}.$$

Thus, in order to find the product AB, we need to multiply each row of A with each column of B. This is particularly easy when A has few different rows and B has few different columns. And this is exactly the situation we have with our "checkerboard pattern" matrices:

(b) The matrix A_4 has only 2 different rows: Namely, all its odd rows are $\begin{pmatrix} 0 & 1 & 0 & 1 \end{pmatrix}$, and all its even rows are $\begin{pmatrix} 1 & 0 & 1 & 0 \end{pmatrix}$. Likewise, all the odd columns of B_4 are $\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$, and all even columns of B_4 are $\begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}$. Thus, in order to find $A_4 B_4$, we only need to compute 4 different products:

$$\underbrace{\begin{pmatrix} 0 & 1 & 0 & 1 \end{pmatrix}}_{\text{an odd row of } A_4} \cdot \underbrace{\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}}_{\text{an odd column of } B_4} = 0; \qquad \underbrace{\begin{pmatrix} 0 & 1 & 0 & 1 \end{pmatrix}}_{\text{an odd row of } A_4} \cdot \underbrace{\begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}}_{\text{an even column of } B_4} = 2;$$

$$\underbrace{\begin{pmatrix} 1 & 0 & 1 & 0 \end{pmatrix}}_{\text{an even row of } A_4} \cdot \underbrace{\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}}_{\text{an odd column of } B_4} = 2; \qquad \underbrace{\begin{pmatrix} 1 & 0 & 1 & 0 \end{pmatrix}}_{\text{an even row of } A_4} \cdot \underbrace{\begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}}_{\text{an even column of } B_4} = 0,$$

and place them in the appropriate cells of $A_4 B_4$. Hence, we obtain

$$A_4 B_4 = \begin{pmatrix} 0 & 2 & 0 & 2 \\ 2 & 0 & 2 & 0 \\ 0 & 2 & 0 & 2 \\ 2 & 0 & 2 & 0 \end{pmatrix}.$$

Similarly,

$$B_4A_4 = \begin{pmatrix} 0 & 2 & 0 & 2 \\ 2 & 0 & 2 & 0 \\ 0 & 2 & 0 & 2 \\ 2 & 0 & 2 & 0 \end{pmatrix}; \qquad A_4^2 = \begin{pmatrix} 2 & 0 & 2 & 0 \\ 0 & 2 & 0 & 2 \\ 2 & 0 & 2 & 0 \\ 0 & 2 & 0 & 2 \end{pmatrix}; \qquad B_4^2 = \begin{pmatrix} 2 & 0 & 2 & 0 \\ 0 & 2 & 0 & 2 \\ 2 & 0 & 2 & 0 \\ 0 & 2 & 0 & 2 \end{pmatrix}.$$

(a) This is similar to our solution of part **(b)**. The result is

$$A_3B_3 = \begin{pmatrix} 0 & 1 & 0 \\ 2 & 0 & 2 \\ 0 & 1 & 0 \end{pmatrix}; \qquad B_3A_3 = \begin{pmatrix} 0 & 2 & 0 \\ 1 & 0 & 1 \\ 0 & 2 & 0 \end{pmatrix};$$

$$A_3^2 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix}; \qquad B_3^2 = \begin{pmatrix} 2 & 0 & 2 \\ 0 & 1 & 0 \\ 2 & 0 & 2 \end{pmatrix}.$$

(c) Let $n \in \mathbb{N}$ be even. As in part **(b)**, we observe that each odd row of A_n is $\begin{pmatrix} 0 & 1 & 0 & 1 & \cdots & 1 \end{pmatrix}$ (with n entries, and the last entry being 1 because n is even), and each even row of A_n is $\begin{pmatrix} 1 & 0 & 1 & 0 & \cdots & 0 \end{pmatrix}$ (with n entries, and the last entry being 0 because n is even). Something similar holds for the columns of B_n. Thus, we can find A_nB_n by computing only the four products

$$\begin{pmatrix} 0 & 1 & 0 & 1 & \cdots & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \\ \vdots \\ 1 \end{pmatrix} = n/2; \qquad \begin{pmatrix} 0 & 1 & 0 & 1 & \cdots & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = 0;$$

$$\begin{pmatrix} 1 & 0 & 1 & 0 & \cdots & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \\ \vdots \\ 1 \end{pmatrix} = 0; \qquad \begin{pmatrix} 1 & 0 & 1 & 0 & \cdots & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = n/2$$

(where each of the vectors has n entries). We thus obtain

$$A_nB_n = \begin{pmatrix} 0 & n/2 & 0 & n/2 & \cdots & 0 & n/2 \\ n/2 & 0 & n/2 & 0 & \cdots & n/2 & 0 \\ 0 & n/2 & 0 & n/2 & \cdots & 0 & n/2 \\ n/2 & 0 & n/2 & 0 & \cdots & n/2 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & n/2 & 0 & n/2 & \cdots & 0 & n/2 \\ n/2 & 0 & n/2 & 0 & \cdots & n/2 & 0 \end{pmatrix}$$

(note the chessboard pattern again!). Similarly,

$$B_nA_n = \begin{pmatrix} 0 & n/2 & 0 & n/2 & \cdots & 0 & n/2 \\ n/2 & 0 & n/2 & 0 & \cdots & n/2 & 0 \\ 0 & n/2 & 0 & n/2 & \cdots & 0 & n/2 \\ n/2 & 0 & n/2 & 0 & \cdots & n/2 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & n/2 & 0 & n/2 & \cdots & 0 & n/2 \\ n/2 & 0 & n/2 & 0 & \cdots & n/2 & 0 \end{pmatrix};$$

$$A_n^2 = \begin{pmatrix} n/2 & 0 & n/2 & 0 & \cdots & n/2 & 0 \\ 0 & n/2 & 0 & n/2 & \cdots & 0 & n/2 \\ n/2 & 0 & n/2 & 0 & \cdots & n/2 & 0 \\ 0 & n/2 & 0 & n/2 & \cdots & 0 & n/2 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ n/2 & 0 & n/2 & 0 & \cdots & n/2 & 0 \\ 0 & n/2 & 0 & n/2 & \cdots & 0 & n/2 \end{pmatrix};$$

$$B_n^2 = \begin{pmatrix} n/2 & 0 & n/2 & 0 & \cdots & n/2 & 0 \\ 0 & n/2 & 0 & n/2 & \cdots & 0 & n/2 \\ n/2 & 0 & n/2 & 0 & \cdots & n/2 & 0 \\ 0 & n/2 & 0 & n/2 & \cdots & 0 & n/2 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ n/2 & 0 & n/2 & 0 & \cdots & n/2 & 0 \\ 0 & n/2 & 0 & n/2 & \cdots & 0 & n/2 \end{pmatrix}.$$

Thus, $A_nB_n = B_nA_n$ and $A_n^2 = B_n^2$.

(d) Let $n \geq 3$ be odd. Then, the same reasoning that we used in part **(c)** reveals that

$$A_nB_n = \begin{pmatrix} 0 & (n-1)/2 & 0 & (n-1)/2 & \cdots & 0 & (n-1)/2 \\ (n+1)/2 & 0 & (n+1)/2 & 0 & \cdots & (n+1)/2 & 0 \\ 0 & (n-1)/2 & 0 & (n-1)/2 & \cdots & 0 & (n-1)/2 \\ (n+1)/2 & 0 & (n+1)/2 & 0 & \cdots & (n+1)/2 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & (n-1)/2 & 0 & (n-1)/2 & \cdots & 0 & (n-1)/2 \\ (n+1)/2 & 0 & (n+1)/2 & 0 & \cdots & (n+1)/2 & 0 \end{pmatrix}$$

and

$$B_nA_n = \begin{pmatrix} 0 & (n+1)/2 & 0 & (n+1)/2 & \cdots & 0 & (n+1)/2 \\ (n-1)/2 & 0 & (n-1)/2 & 0 & \cdots & (n-1)/2 & 0 \\ 0 & (n+1)/2 & 0 & (n+1)/2 & \cdots & 0 & (n+1)/2 \\ (n-1)/2 & 0 & (n-1)/2 & 0 & \cdots & (n-1)/2 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & (n+1)/2 & 0 & (n+1)/2 & \cdots & 0 & (n+1)/2 \\ (n-1)/2 & 0 & (n-1)/2 & 0 & \cdots & (n-1)/2 & 0 \end{pmatrix}.$$

These two matrices clearly differ in their $(1,2)$-entry (namely, A_nB_n has $(1,2)$-entry $(n-1)/2$, while B_nA_n has $(1,2)$-entry $(n+1)/2$). Thus, they are distinct. In other words, $A_nB_n \neq B_nA_n$. □

Solution to Exercise 2. **(a)** The system

$$\begin{cases} a+b+c+d = e \\ a+2b+3c+4d = 5e \\ a+3b+6c+10d = 15e \end{cases}$$

can be rewritten as

$$\begin{cases} a+b+c+d+(-1)e = 0 \\ a+2b+3c+4d+(-5)e = 0 \\ a+3b+6c+10d+(-15)e = 0 \end{cases}.$$

Hence, its augmented matrix is

$$A := \begin{pmatrix} 1 & 1 & 1 & 1 & -1 & 0 \\ 1 & 2 & 3 & 4 & -5 & 0 \\ 1 & 3 & 6 & 10 & -15 & 0 \end{pmatrix}.$$

(Don't let the look of the system fool you into saying that the augmented matrix is $\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 6 & 10 & 15 \end{pmatrix}$! The expressions e, $5e$ and $15e$ on the right hand sides of the equations are not constants!)

(b) Let us transform A into RREF using [Strickland, Method 6.4]:[1]

$$A = \begin{pmatrix} \boxed{1} & 1 & 1 & 1 & -1 & 0 \\ 1 & 2 & 3 & 4 & -5 & 0 \\ 1 & 3 & 6 & 10 & -15 & 0 \end{pmatrix}$$

$$\xrightarrow{\text{add } (-1)\cdot\text{row}\,1 \text{ to row}\,2} \begin{pmatrix} \boxed{1} & 1 & 1 & 1 & -1 & 0 \\ 0 & 1 & 2 & 3 & -4 & 0 \\ 1 & 3 & 6 & 10 & -15 & 0 \end{pmatrix}$$

[1] As usual, pivots are boxed.

$$\underset{\longrightarrow}{\text{add } (-1)\cdot\text{row } 1 \text{ to row } 3} \begin{pmatrix} \boxed{1} & 1 & 1 & 1 & -1 & 0 \\ 0 & 1 & 2 & 3 & -4 & 0 \\ 0 & 2 & 5 & 9 & -14 & 0 \end{pmatrix}$$

$$\underset{\longrightarrow}{\text{freeze row } 1} \begin{pmatrix} \boxed{1} & 1 & 1 & 1 & -1 & 0 & \leftarrow \text{frozen} \\ 0 & \boxed{1} & 2 & 3 & -4 & 0 & \\ 0 & 2 & 5 & 9 & -14 & 0 & \end{pmatrix}$$

$$\underset{\text{(keep in mind: frozen rows are not counted!)}}{\underset{\longrightarrow}{\text{add } (-1)\cdot\text{row } 1 \text{ to row } 2}} \begin{pmatrix} \boxed{1} & 1 & 1 & 1 & -1 & 0 & \leftarrow \text{frozen} \\ 0 & \boxed{1} & 2 & 3 & -4 & 0 & \\ 0 & 0 & \boxed{1} & 3 & -6 & 0 & \end{pmatrix}$$

$$\underset{\longrightarrow}{\text{freeze row } 1} \begin{pmatrix} \boxed{1} & 1 & 1 & 1 & -1 & 0 & \leftarrow \text{frozen} \\ 0 & \boxed{1} & 2 & 3 & -4 & 0 & \leftarrow \text{frozen} \\ 0 & 0 & \boxed{1} & 3 & -6 & 0 & \end{pmatrix}$$

(now, the unfrozen part of the matrix is in RREF, so we start unfreezing)

$$\underset{\longrightarrow}{\text{unfreeze row } 1} \begin{pmatrix} \boxed{1} & 1 & 1 & 1 & -1 & 0 & \leftarrow \text{frozen} \\ 0 & \boxed{1} & 2 & 3 & -4 & 0 & \\ 0 & 0 & \boxed{1} & 3 & -6 & 0 & \end{pmatrix}$$

$$\underset{\longrightarrow}{\text{add } (-2)\cdot\text{row } 2 \text{ to row } 1} \begin{pmatrix} \boxed{1} & 1 & 1 & 1 & -1 & 0 & \leftarrow \text{frozen} \\ 0 & \boxed{1} & 0 & -3 & 8 & 0 & \\ 0 & 0 & \boxed{1} & 3 & -6 & 0 & \end{pmatrix}$$

$$\underset{\longrightarrow}{\text{unfreeze row } 1} \begin{pmatrix} \boxed{1} & 1 & 1 & 1 & -1 & 0 \\ 0 & \boxed{1} & 0 & -3 & 8 & 0 \\ 0 & 0 & \boxed{1} & 3 & -6 & 0 \end{pmatrix}$$

$$\underset{\longrightarrow}{\text{add } (-1)\cdot\text{row } 2 \text{ to row } 1} \begin{pmatrix} \boxed{1} & 0 & 1 & 4 & -9 & 0 \\ 0 & \boxed{1} & 0 & -3 & 8 & 0 \\ 0 & 0 & \boxed{1} & 3 & -6 & 0 \end{pmatrix}$$

$$\underset{\longrightarrow}{\text{add } (-1)\cdot\text{row } 3 \text{ to row } 1} \begin{pmatrix} \boxed{1} & 0 & 0 & 1 & -3 & 0 \\ 0 & \boxed{1} & 0 & -3 & 8 & 0 \\ 0 & 0 & \boxed{1} & 3 & -6 & 0 \end{pmatrix}.$$

Thus, $\begin{pmatrix} \boxed{1} & 0 & 0 & 1 & -3 & 0 \\ 0 & \boxed{1} & 0 & -3 & 8 & 0 \\ 0 & 0 & \boxed{1} & 3 & -6 & 0 \end{pmatrix}$ is the RREF of our augmented matrix A.

(c) We have just computed the RREF of the augmented matrix. This RREF corresponds to the simple system

$$\begin{cases} a + d + (-3)\,e = 0 \\ b + (-3)\,d + 8e = 0 \\ c + 3d + (-6)\,e = 0 \end{cases} . \qquad (1)$$

We can solve this using [Strickland, Method 5.4]: Since the pivots of our matrix are in columns $1, 2, 3$, we see that the dependent variables will be the 1-st, 2-nd and 3-rd variables, i.e., the variables a, b, c. The remaining two variables d, e will thus be independent variables. Now, the equations in (1) can be solved for the dependent variables simply by moving the independent variables on the right hand sides:

$$\begin{cases} a = -d + 3e \\ b = 3d - 8e \\ c = -3d + 6e \end{cases} .$$

This is the general solution of our system (with d and e being free parameters). □

Solution to Exercise 3. **(a)** Let us start bringing A_7 into RREF using [Strickland, Method 6.4]:

$$A_7 = \begin{pmatrix} \boxed{1} & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

$$\overset{\text{add } (-1)\cdot \text{row } 1 \text{ to row } 2}{\longrightarrow} \begin{pmatrix} \boxed{1} & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

$$\overset{\text{freeze row } 1}{\longrightarrow} \begin{pmatrix} \boxed{1} & 0 & 0 & 0 & 0 & 0 & 1 & \leftarrow \text{frozen} \\ 0 & \boxed{1} & 0 & 0 & 0 & 0 & -1 & \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & \end{pmatrix}$$

$$\overset{\text{add }(-1)\cdot\text{row }1\text{ to row }2}{\longrightarrow}\left(\begin{array}{ccccccccc}\boxed{1} & 0 & 0 & 0 & 0 & 0 & 1 & \leftarrow\text{frozen}\\ 0 & \boxed{1} & 0 & 0 & 0 & 0 & -1 & \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & \end{array}\right)$$

$$\overset{\text{freeze row }1}{\longrightarrow}\left(\begin{array}{ccccccccc}\boxed{1} & 0 & 0 & 0 & 0 & 0 & 1 & \leftarrow\text{frozen}\\ 0 & \boxed{1} & 0 & 0 & 0 & 0 & -1 & \leftarrow\text{frozen}\\ 0 & 0 & \boxed{1} & 0 & 0 & 0 & 1 & \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & \end{array}\right)$$

$$\overset{\text{add }(-1)\cdot\text{row }1\text{ to row }2}{\longrightarrow}\left(\begin{array}{ccccccccc}\boxed{1} & 0 & 0 & 0 & 0 & 0 & 1 & \leftarrow\text{frozen}\\ 0 & \boxed{1} & 0 & 0 & 0 & 0 & -1 & \leftarrow\text{frozen}\\ 0 & 0 & \boxed{1} & 0 & 0 & 0 & 1 & \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 & \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & \end{array}\right)$$

$$\longrightarrow\cdots.$$

At this point it should be clear how the procedure goes on: We pick the topmost 1 on the diagonal as pivot; then we clear out the 1 below it by adding $(-1)\cdot\text{row }1$ to row 2; then we freeze row 1; rinse, repeat. As we keep doing this, the first $n-1$ columns of our matrix turn into the first $n-1$ columns of the identity matrix I_n (that is, all their off-diagonal entries become 0, while the diagonal entries remain 1), whereas the n-th column takes the form

$$\left(\begin{array}{c}1\\ -1\\ 1\\ -1\\ \vdots\end{array}\right)$$

(that is, its entries alternate between 1 and -1, starting with a 1), except for its bottommost entry. To see what happens with the bottommost entry, we take a closer look at the final steps of this procedure:

$$\cdots \longrightarrow \begin{pmatrix} \boxed{1} & 0 & 0 & 0 & 0 & 0 & 1 & \leftarrow \text{frozen} \\ 0 & \boxed{1} & 0 & 0 & 0 & 0 & -1 & \leftarrow \text{frozen} \\ 0 & 0 & \boxed{1} & 0 & 0 & 0 & 1 & \leftarrow \text{frozen} \\ 0 & 0 & 0 & \boxed{1} & 0 & 0 & -1 & \leftarrow \text{frozen} \\ 0 & 0 & 0 & 0 & \boxed{1} & 0 & 1 & \leftarrow \text{frozen} \\ 0 & 0 & 0 & 0 & 0 & \boxed{1} & -1 & \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & \end{pmatrix}$$

$$\overset{\text{add } (-1)\cdot \text{row } 1 \text{ to row } 2}{\longrightarrow} \begin{pmatrix} \boxed{1} & 0 & 0 & 0 & 0 & 0 & 1 & \leftarrow \text{frozen} \\ 0 & \boxed{1} & 0 & 0 & 0 & 0 & -1 & \leftarrow \text{frozen} \\ 0 & 0 & \boxed{1} & 0 & 0 & 0 & 1 & \leftarrow \text{frozen} \\ 0 & 0 & 0 & \boxed{1} & 0 & 0 & -1 & \leftarrow \text{frozen} \\ 0 & 0 & 0 & 0 & \boxed{1} & 0 & 1 & \leftarrow \text{frozen} \\ 0 & 0 & 0 & 0 & 0 & \boxed{1} & -1 & \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & \end{pmatrix}$$

$$\overset{\text{freeze row } 1}{\longrightarrow} \begin{pmatrix} \boxed{1} & 0 & 0 & 0 & 0 & 0 & 1 & \leftarrow \text{frozen} \\ 0 & \boxed{1} & 0 & 0 & 0 & 0 & -1 & \leftarrow \text{frozen} \\ 0 & 0 & \boxed{1} & 0 & 0 & 0 & 1 & \leftarrow \text{frozen} \\ 0 & 0 & 0 & \boxed{1} & 0 & 0 & -1 & \leftarrow \text{frozen} \\ 0 & 0 & 0 & 0 & \boxed{1} & 0 & 1 & \leftarrow \text{frozen} \\ 0 & 0 & 0 & 0 & 0 & \boxed{1} & -1 & \leftarrow \text{frozen} \\ 0 & 0 & 0 & 0 & 0 & 0 & \boxed{2} & \end{pmatrix}.$$

Thus, the bottommost entry of the n-th column becomes a 2 (because it was already 1 before we added $(-1) \cdot (-1) = 1$ to it). After thus freezing the first $n-1$ rows of our matrix, we obtain the matrix

$$\begin{pmatrix} \boxed{1} & 0 & 0 & 0 & 0 & 0 & 1 & \leftarrow \text{frozen} \\ 0 & \boxed{1} & 0 & 0 & 0 & 0 & -1 & \leftarrow \text{frozen} \\ 0 & 0 & \boxed{1} & 0 & 0 & 0 & 1 & \leftarrow \text{frozen} \\ 0 & 0 & 0 & \boxed{1} & 0 & 0 & -1 & \leftarrow \text{frozen} \\ 0 & 0 & 0 & 0 & \boxed{1} & 0 & 1 & \leftarrow \text{frozen} \\ 0 & 0 & 0 & 0 & 0 & \boxed{1} & -1 & \leftarrow \text{frozen} \\ 0 & 0 & 0 & 0 & 0 & 0 & \boxed{2} & \end{pmatrix}.$$

Then, we scale the last row by 1/2 in order to make its pivot equal to 1:

$$\begin{pmatrix} \boxed{1} & 0 & 0 & 0 & 0 & 0 & 1 & \leftarrow \text{frozen} \\ 0 & \boxed{1} & 0 & 0 & 0 & 0 & -1 & \leftarrow \text{frozen} \\ 0 & 0 & \boxed{1} & 0 & 0 & 0 & 1 & \leftarrow \text{frozen} \\ 0 & 0 & 0 & \boxed{1} & 0 & 0 & -1 & \leftarrow \text{frozen} \\ 0 & 0 & 0 & 0 & \boxed{1} & 0 & 1 & \leftarrow \text{frozen} \\ 0 & 0 & 0 & 0 & 0 & \boxed{1} & -1 & \leftarrow \text{frozen} \\ 0 & 0 & 0 & 0 & 0 & 0 & \boxed{1} & \end{pmatrix}.$$

Thus, the unfrozen part of our matrix is in RREF, so we unfreeze rows and clear out the nonzero entries above the pivots. (These nonzero entries only exist in the n-th column, and can be cleared out by adding appropriate multiples of the last row to the rows above it.) At the end of this procedure, we obtain the identity matrix

$$\begin{pmatrix} \boxed{1} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \boxed{1} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \boxed{1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \boxed{1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \boxed{1} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \boxed{1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \boxed{1} \end{pmatrix},$$

which of course is in RREF. So this is the RREF of A_7.

(b) The row reduction proceeds as it did for A_7 in part **(a)** of this exercise, but a surprise happens as we reach the last row:

$$\cdots \longrightarrow \begin{pmatrix} \boxed{1} & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \leftarrow \text{frozen} \\ 0 & \boxed{1} & 0 & 0 & 0 & 0 & 0 & -1 & \leftarrow \text{frozen} \\ 0 & 0 & \boxed{1} & 0 & 0 & 0 & 0 & 1 & \leftarrow \text{frozen} \\ 0 & 0 & 0 & \boxed{1} & 0 & 0 & 0 & -1 & \leftarrow \text{frozen} \\ 0 & 0 & 0 & 0 & \boxed{1} & 0 & 0 & 1 & \leftarrow \text{frozen} \\ 0 & 0 & 0 & 0 & 0 & \boxed{1} & 0 & -1 & \leftarrow \text{frozen} \\ 0 & 0 & 0 & 0 & 0 & 0 & \boxed{1} & 1 & \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & \end{pmatrix}$$

$$\overset{\text{add } (-1)\cdot\text{row 1 to row 2}}{\longrightarrow} \begin{pmatrix} \boxed{1} & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \leftarrow \text{frozen} \\ 0 & \boxed{1} & 0 & 0 & 0 & 0 & 0 & -1 & \leftarrow \text{frozen} \\ 0 & 0 & \boxed{1} & 0 & 0 & 0 & 0 & 1 & \leftarrow \text{frozen} \\ 0 & 0 & 0 & \boxed{1} & 0 & 0 & 0 & -1 & \leftarrow \text{frozen} \\ 0 & 0 & 0 & 0 & \boxed{1} & 0 & 0 & 1 & \leftarrow \text{frozen} \\ 0 & 0 & 0 & 0 & 0 & \boxed{1} & 0 & -1 & \leftarrow \text{frozen} \\ 0 & 0 & 0 & 0 & 0 & 0 & \boxed{1} & 1 & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \end{pmatrix}$$

$$\overset{\text{freeze row 1}}{\longrightarrow} \begin{pmatrix} \boxed{1} & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \leftarrow \text{frozen} \\ 0 & \boxed{1} & 0 & 0 & 0 & 0 & 0 & -1 & \leftarrow \text{frozen} \\ 0 & 0 & \boxed{1} & 0 & 0 & 0 & 0 & 1 & \leftarrow \text{frozen} \\ 0 & 0 & 0 & \boxed{1} & 0 & 0 & 0 & -1 & \leftarrow \text{frozen} \\ 0 & 0 & 0 & 0 & \boxed{1} & 0 & 0 & 1 & \leftarrow \text{frozen} \\ 0 & 0 & 0 & 0 & 0 & \boxed{1} & 0 & -1 & \leftarrow \text{frozen} \\ 0 & 0 & 0 & 0 & 0 & 0 & \boxed{1} & 1 & \leftarrow \text{frozen} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \end{pmatrix}.$$

As you see, the bottommost entry in the n-th column is now 0 rather than 2 (because it was initially 1, but now we added $(-1) \cdot 1$ to it rather than $(-1) \cdot (-1)$). This means that the last remaining unfrozen row has no pivot at all, and so it is already in RREF. We thus start unfreezing the frozen rows. As we do this, we realize that the columns containing pivots have already been cleared, so we don't need to perform any further row operations; our matrix at this point is already in RREF. Thus, the RREF of A_8 is

$$\begin{pmatrix} \boxed{1} & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & \boxed{1} & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & \boxed{1} & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & \boxed{1} & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & \boxed{1} & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & \boxed{1} & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & \boxed{1} & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

[*Remark:* The different behaviors in parts **(a)** and **(b)** come from the fact that 7 is odd while 8 is even. By the same logic, we can find the RREF of the analogous matrices of any given size:

Let $n \geq 2$ be an integer. Let A_n be the $n \times n$-matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 1 & 1 \end{pmatrix}.$$

(Its diagonal entries are 1; its entries just below the diagonal are 1; its entry in the top-right corner is 1; all its other entries are 0.) Then:

- If n is odd, then the RREF of A_n is the identity matrix I_n.
- If n is even, then the RREF of A_n is the $n \times n$-matrix

$$\begin{pmatrix} \boxed{1} & 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & \boxed{1} & 0 & 0 & \cdots & 0 & -1 \\ 0 & 0 & \boxed{1} & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & \boxed{1} & \cdots & 0 & -1 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \boxed{1} & 1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}$$

  (whose first $n-1$ columns are the corresponding columns of I_n, while its last column is $\begin{pmatrix} 1 & -1 & 1 & -1 & \cdots & 1 & 0 \end{pmatrix}^T$).]

□

Solution to Exercise 4. **(a)** Write the unknown vector x as $x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$. Then, the equation $Ux = b$ rewrites as the system $\begin{cases} 6x_1 + 3x_2 + (-2)\,x_3 + 5x_4 = 1 \\ (-1)\,x_3 + 2x_4 = 5 \\ 1x_4 = 2 \\ 0 = 0 \end{cases}$. This system can be solved by Gaussian elimination, or simpler by back-substitution: The fourth equation ($0 = 0$) is automatically satisfied; the third equation can be solved for x_4 (yielding $x_4 = 2$); the second equation can then be solved for x_3 using our already-obtained value of x_4 (yielding $x_3 = -1$); the lack of an equation with "leading variable" x_2 shows that x_2 will be a free variable; finally, the first equation can be solved for x_1 using our already-obtained values for x_2, x_3, x_4 (this yields $x_1 = -\frac{1}{2}x_2 - \frac{11}{6}$). Thus, the solution is

$$\begin{cases} x_1 = -\frac{1}{2}x_{22} - \frac{11}{6}, \\ x_3 = -1 \\ x_4 = 2 \end{cases}, \qquad \text{that is,} \qquad x = \begin{pmatrix} -\frac{1}{2}x_2 - \frac{11}{6} \\ x_2 \\ -1 \\ 2 \end{pmatrix}.$$

(b) Write the unknown vector x as $x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$. Then, the equation $Ux = b'$ rewrites as the system $\begin{cases} 6x_1 + 3x_2 + (-2)\,x_3 + 5x_4 = 1 \\ (-1)\,x_3 + 2x_4 = 5 \\ 1x_4 = 2 \\ 0 = 1 \end{cases}$. This system can be solved by back-substitution: The fourth equation ($0 = 1$) is unsatisfiable, so **there are no solutions**.

(c) Write the unknown vector x as $x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$. Then, the equation $Ux = x$ rewrites as the system $\begin{cases} 6x_1 + 3x_2 + (-2)\,x_3 + 5x_4 = x_1 \\ (-1)\,x_3 + 2x_4 = x_2 \\ 1x_4 = x_3 \\ 0 = x_4 \end{cases}$. Bringing the x_1, x_2, x_3, x_4 onto the left hand sides transforms this into $\begin{cases} 5x_1 + 3x_2 + (-2)\,x_3 + 5x_4 = 0 \\ (-1)\,x_2 + (-1)\,x_3 + 2x_4 = 0 \\ (-1)\,x_3 + 1x_4 = 0 \\ (-1)\,x_4 = 0 \end{cases}$.
This system can again be solved by back-substitution, leading to the only solution

$$\begin{cases} x_1 = 0 \\ x_2 = 0 \\ x_3 = 0 \\ x_4 = 0 \end{cases}, \qquad \text{that is,} \qquad x = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

(Or, alternatively, you can take a look at the augmented matrix of the system and immediately see that it has a pivot in each of its first 4 columns; we know already that this forces the system to have a unique solution. Since $\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ is obviously a solution to $Ux = x$, we thus can conclude that $\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ is the only solution.) □

Solution to Exercise 5. **(a)** Yes.

Proof. Let $n = 7$.

Recall the method we learned for checking whether some vectors in $\mathbb{R}^n$ are linearly independent (see [Strickland, Method 8.8] or Theorem 1.1.7 in the notes from 2019-10-07). Following this method, in order to check whether the n vectors $f_1, f_2, \ldots, f_n$ are linearly independent, we need to form the $n \times 7$-matrix $A := [f_1 \mid f_2 \mid \cdots \mid f_n]$, then bring it into RREF and check whether every column of the resulting matrix will have a pivot. But the $n \times 7$-matrix $A = [f_1 \mid f_2 \mid \cdots \mid f_n]$ is precisely the matrix A_7 from Exercise 3 **(a)**[2]. Thus, we already know how it RREF looks like from the solution of Exercise 3 **(a)**. In particular, we know that this RREF is the identity matrix I_7, and thus has a pivot in every column. Hence, the n vectors $f_1, f_2, \ldots, f_n$ are linearly independent.

(b) No.

Proof. Let $n = 8$.

We use the same method as in the solution to part **(a)**. But now, the $n \times 8$-matrix $A = [f_1 \mid f_2 \mid \cdots \mid f_n]$ will be the matrix A_8 from Exercise 3 **(b)** rather than the matrix A_7 from Exercise 3 **(a)**. We have computed the RREF of this matrix A_8 in our solution of Exercise 3 **(b)**. In particular, we know that some column of this RREF has no pivot (namely, the 8-th column has no pivot). Thus, the n vectors $f_1, f_2, \ldots, f_n$ are linearly dependent.

(c) Yes.

Proof. Let $n = 7$.

We know (from the answer to part **(a)**) that the n vectors $f_1, f_2, \ldots, f_n$ in $\mathbb{R}^n$ are linearly independent. Hence, [Strickland, Proposition 10.12 (a)] (or, equivalently,

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[2] because we have $f_i = e_i + e_{i+1} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \\ 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ (with the 1's in positions i and $i+1$) for each $i \in \{1, 2, \ldots, n-1\}$, and because we have $f_n = e_n + e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$

Proposition 1.2.7 **(a)** from the notes from 2019-10-09) yields that the list $f_1, f_2, \ldots, f_n$ is a basis of $\mathbb{R}^n$. Thus, in particular, $f_1, f_2, \ldots, f_n$ span $\mathbb{R}^n$.

(d) No.

Proof. Let $n = 8$.

We must prove that $f_1, f_2, \ldots, f_n$ don't span $\mathbb{R}^n$. Indeed, assume the contrary. Thus, the n vectors $f_1, f_2, \ldots, f_n$ span $\mathbb{R}^n$. Hence, [Strickland, Proposition 10.12 (b)] (or, equivalently, Proposition 1.2.7 **(b)** from the notes from 2019-10-09) yields that the list $f_1, f_2, \ldots, f_n$ is a basis of $\mathbb{R}^n$. Thus, in particular, $f_1, f_2, \ldots, f_n$ are linearly independent. But this contradicts our result from solving part **(b)** of this exercise. This contradiction shows that our assumption was false. Hence, part **(d)** is solved.

[*Remark:* There are alternative ways to solve this exercise. We have solved parts **(a)** and **(b)** first, and then used them to get answers to **(c)** and **(d)**. It could also be done the other way round. In particular, the positive answer to part **(c)** can also be obtained by explicitly writing the standard basis vectors $e_1, e_2, \ldots, e_7$ as linear combinations of $f_1, f_2, \ldots, f_7$, namely as follows:

$$\begin{aligned}
e_1 &= \frac{1}{2}\left(f_1 + f_3 + f_5 + f_7 - f_2 - f_4 - f_6\right); \\
e_2 &= \frac{1}{2}\left(f_2 + f_4 + f_6 + f_1 - f_3 - f_5 - f_7\right); \\
e_3 &= \frac{1}{2}\left(f_3 + f_5 + f_7 + f_2 - f_4 - f_6 - f_1\right); \\
e_4 &= \frac{1}{2}\left(f_4 + f_6 + f_1 + f_3 - f_5 - f_7 - f_2\right); \\
e_5 &= \frac{1}{2}\left(f_5 + f_7 + f_2 + f_4 - f_6 - f_1 - f_3\right); \\
e_6 &= \frac{1}{2}\left(f_6 + f_1 + f_3 + f_5 - f_7 - f_2 - f_4\right); \\
e_7 &= \frac{1}{2}\left(f_7 + f_2 + f_4 + f_6 - f_1 - f_3 - f_5\right).
\end{aligned}$$

(Note that there is a cyclic symmetry inherent in the problem[3], which makes it sufficient to find one of these 7 equations; the others can then be obtained by cyclically rotating the subscripts.) More generally, for each odd n, we have

$$e_1 = \frac{1}{2}\left(f_1 + f_3 + f_5 + \cdots + f_n - f_2 - f_4 - f_6 - \cdots - f_{n-1}\right) = \frac{1}{2}\left(\sum_{i \text{ is odd}} f_i - \sum_{i \text{ is even}} f_i\right)$$

and similar equalities for $e_2, e_3, \ldots, e_n$.

It is even easier to solve part **(b)** by a good guess: Just observe that when n is even, we have

$$f_1 + f_3 + f_5 + \cdots + f_{n-1} = e_1 + e_2 + \cdots + e_n = f_2 + f_4 + f_6 + \cdots + f_n$$

[3] Namely, if you arrange the vectors $e_1, e_2, \ldots, e_n$ on a circle, then each f_i is the sum of two consecutive vectors.

and thus

$$f_1 - f_2 + f_3 - f_4 \pm \cdots + f_{n-1} - f_n = 0,$$

which is a nontrivial relation between $f_1, f_2, \ldots, f_n$. Hence, $f_1, f_2, \ldots, f_n$ are linearly dependent whenever n is even.] □

Solution to Exercise 6. **(a)** No.

Proof. Any linear combination of N, E, S, W has the form $aN + bE + cS + dW$ for some $a, b, c, d \in \mathbb{R}$ (by the definition of a linear combination). Thus, its $(2,2)$-th entry is

$$\begin{aligned} &(aN + bE + cS + dW)_{2,2} \\ &= a \underbrace{N_{2,2}}_{=0} + b \underbrace{E_{2,2}}_{=0} + c \underbrace{S_{2,2}}_{=0} + d \underbrace{W_{2,2}}_{=0} \\ &= a \cdot 0 + b \cdot 0 + c \cdot 0 + d \cdot 0 = 0. \end{aligned}$$

But the $(2,2)$-th entry of I_n is not 0 (indeed, it is 1, since it is a diagonal entry). Thus, a linear combination of N, E, S, W cannot be I_2 (since it has the wrong $(2,2)$-th entry). Qed.

(b) No.

Proof. We want to prove that N, E, S, W are linearly independent. In other words, we must prove that every relation between N, E, S, W is trivial.

So let $aN + bE + cS + dW = 0$ be any relation between N, E, S, W. We must prove that this relation is trivial, i.e., that $a = b = c = d = 0$.

From $aN + bE + cS + dW = 0$, we obtain $(aN + bE + cS + dW)_{1,2} = 0$ (since the $(1,2)$-th entry of a zero matrix is 0). Thus,

$$\begin{aligned} 0 &= (aN + bE + cS + dW)_{1,2} \\ &= a \underbrace{N_{1,2}}_{\substack{=1 \\ \text{(since the (1,2)-th} \\ \text{entry of an } n\times n\text{-matrix} \\ \text{belongs to its 1-st row)}}} + b \underbrace{E_{1,2}}_{\substack{=0 \\ \text{(since } n\geq 3\text{, so the (1,2)-th} \\ \text{entry of an } n\times n\text{-matrix} \\ \text{does not belong to its } n\text{-th column)}}} \\ &\qquad + c \underbrace{S_{1,2}}_{\substack{=0 \\ \text{(since } n\geq 3>1\text{, so the (1,2)-th} \\ \text{entry of an } n\times n\text{-matrix} \\ \text{does not belong to its } n\text{-th row)}}} + d \underbrace{W_{1,2}}_{\substack{=0 \\ \text{(since the (1,2)-th} \\ \text{entry of an } n\times n\text{-matrix} \\ \text{does not belong to its 1-st column)}}} \\ &= a \cdot 1 + b \cdot 0 + c \cdot 0 + d \cdot 0 = a. \end{aligned}$$

Thus, $a = 0$.

So we have obtained $a = 0$ by comparing the $(1,2)$-th entries in the equality $aN + bE + cS + dW = 0$. Similarly,

- we can obtain $b = 0$ by comparing the $(2,n)$-th entries in the equality $aN + bE + cS + dW = 0$.

- we can obtain $c = 0$ by comparing the $(n, 2)$-th entries in the equality $aN + bE + cS + dW = 0$.
- we can obtain $d = 0$ by comparing the $(2, 1)$-th entries in the equality $aN + bE + cS + dW = 0$.

Altogether, we now know that $a = 0$ and $b = 0$ and $c = 0$ and $d = 0$. Thus, $a = b = c = d = 0$. And this completes our proof.

[*Remark:* Let us illustrate this argument in the case $n = 3$:

$$\begin{aligned} & aN + bE + cS + dW \\ & = a \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + b \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} + c \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix} + d \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \\ & = \begin{pmatrix} a+d & a & a+b \\ d & 0 & b \\ c+d & c & b+c \end{pmatrix}. \end{aligned}$$

If this matrix is to be 0, then in particular its entries a, b, c, d (which are its $(1, 2)$-th entry, its $(2, n)$-th entry, its $(n, 2)$-th entry and its $(2, 1)$-th entry, respectively) must be 0.] □

Solution to Exercise 7. The inverse of A always exists (i.e., the matrix A is invertible), and equals

$$A^{-1} = \begin{pmatrix} 1 & -a & ab' - b & ac' - c + bc'' - ab'c'' \\ 0 & 1 & -b' & b'c'' - c' \\ 0 & 0 & 1 & -c'' \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Proof. There are many ways to find this. The simplest one is probably using our method for finding inverses via row-reduction (Theorem 1.3.5 in the notes from 2019-10-16):

$$[A \mid I_4] = \begin{pmatrix} \boxed{1} & a & b & c & 1 & 0 & 0 & 0 \\ 0 & 1 & b' & c' & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & c'' & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\overset{\text{add } -a\cdot \text{row } 2 \text{ to row } 1}{\longrightarrow} \begin{pmatrix} \boxed{1} & 0 & b - ab' & c - ac' & 1 & -a & 0 & 0 \\ 0 & 1 & b' & c' & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & c'' & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\longrightarrow \cdots \longrightarrow \begin{pmatrix} \boxed{1} & 0 & 0 & 0 & 1 & -a & ab' - b & ac' - c + bc'' - ab'c'' \\ 0 & \boxed{1} & 0 & 0 & 0 & 1 & -b' & b'c'' - c' \\ 0 & 0 & \boxed{1} & 0 & 0 & 0 & 1 & -c'' \\ 0 & 0 & 0 & \boxed{1} & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \left[I_4 \mid \begin{pmatrix} 1 & -a & ab' - b & ac' - c + bc'' - ab'c'' \\ 0 & 1 & -b' & b'c'' - c' \\ 0 & 0 & 1 & -c'' \\ 0 & 0 & 0 & 1 \end{pmatrix} \right]$$

(where I am leaving out the intermediate steps because you have seen row reduction happen quite a few times by now). □

Solution to Exercise 8. **(a)** This matrix is invertible.

Proof. One way to see it is simply to compute its inverse; namely, its inverse is $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & -1 & 0 \end{pmatrix}$.

But here is a quicker way, which you can do in your head: It is easy to see that the matrix $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ can be turned into an identity matrix by row operations (namely, first swap row 1 with row 3, thus obtaining the lower-triangular matrix $\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$, and then use the 1's on the diagonal to clear out the entries below them, obtaining the identity matrix I_3). In other words, the matrix $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ can be row-reduced to I_3. Hence, the Inverse Matrix Theorem (Theorem 1.2.1 in the notes from 2019-10-16)[4] shows that our matrix is invertible.

(b) This matrix is not invertible.

Proof. Denote the columns of this matrix by v_1, v_2, v_3 (from first to last). Then, $v_1 = v_3$ (since both v_1 and v_3 equal $(1, 0, 1)^T$); in other words, $1v_1 + 0v_2 + (-1)\, v_3 = 0$. This is a nontrivial relation between the columns of this matrix. Thus, the columns of this matrix are linearly dependent. Hence, the matrix cannot be invertible (because if it was invertible, then the Inverse Matrix Theorem (Theorem 1.2.1 in the notes from 2019-10-16)[5] would show that its columns are linearly independent).

(c) This matrix is invertible.

Proof. One way to see it is simply to compute its inverse; namely, its inverse is $\begin{pmatrix} -3 & 0 & 2 \\ 0 & \frac{1}{2} & 0 \\ 2 & 0 & -1 \end{pmatrix}$.

[4]More specifically, we are using the implication **(a)** $\Longrightarrow$ **(k)** of the Inverse Matrix Theorem.

[5]More specifically, we are using the implication **(k)** $\Longrightarrow$ **(b)** of the Inverse Matrix Theorem.

But here is a quicker way, which you can do in your head: It is easy to see that the matrix $\begin{pmatrix} 1 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 3 \end{pmatrix}$ can be turned into an identity matrix by row operations (namely, first add (-2) ·row 1 with row 3, thus obtaining the upper-triangular matrix $\begin{pmatrix} 1 & 0 & 2 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{pmatrix}$, and then scale the rows to turn the pivots into 1's, and finally use the pivots to clear out the entries above them, obtaining the identity matrix I_3). In other words, the matrix $\begin{pmatrix} 1 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 3 \end{pmatrix}$ can be row-reduced to I_3. Hence, the Inverse Matrix Theorem (Theorem 1.2.1 in the notes from 2019-10-16)[6] shows that our matrix is invertible.

(d) This matrix is not invertible.

Proof. Corollary 1.2.7 in the notes from 2019-10-16 shows that if an $n \times m$-matrix has an inverse, then $n = m$. In other words, if a matrix has an inverse, then it is a square matrix. Hence, our matrix does not have an inverse (since it is not a square matrix). In other words, it is not invertible. □

For the solution of Exercise 9, we will need the following fact ([lina, Proposition 3.18 **(e)**]):

Proposition 6.1. Let $n \in \mathbb{N}$, $m \in \mathbb{N}$ and $p \in \mathbb{N}$. Let A be an $n \times m$-matrix. Let B be an $m \times p$-matrix. Then, $(AB)^T = B^T A^T$.

Solution to Exercise 9. We need to prove that $\left(A^{-1}\right)^T$ is an inverse of A^T. By the definition of an "inverse", this means that we need to prove the two equalities

$$A^T \left(A^{-1}\right)^T = I_n \qquad \text{and} \qquad \left(A^{-1}\right)^T A^T = I_n \tag{2}$$

(because these equalities are what makes $\left(A^{-1}\right)^T$ into an inverse of A^T).

So let us prove them. The identity matrix I_n is diagonal; thus, it has the same numbers below and above the diagonal (namely, zeroes). Hence, it does not change if we transpose it. In other words, $(I_n)^T = I_n$.

Proposition 6.1 (applied to $m = n$, $p = n$ and $B = A^{-1}$) yields $\left(AA^{-1}\right)^T = \left(A^{-1}\right)^T A^T$. Hence, $\left(A^{-1}\right)^T A^T = \left(\underbrace{AA^{-1}}_{=I_n}\right)^T = (I_n)^T = I_n$.

[6] More specifically, we are using the implication **(a)** $\Longrightarrow$ **(k)** of the Inverse Matrix Theorem.

Also, Proposition 6.1 (applied to n, n, A^{-1} and A instead of m, p, A and B) yields $\left(A^{-1}A\right)^T = A^T\left(A^{-1}\right)^T$. Hence, $A^T\left(A^{-1}\right)^T = \left(\underbrace{A^{-1}A}_{=I_n}\right)^T = \left(I_n\right)^T = I_n$.

Thus, we have proved the two equalities in (2). This shows that $\left(A^{-1}\right)^T$ is an inverse of A^T. Hence, the matrix A^T is invertible, and its inverse is $\left(A^T\right)^{-1} = \left(A^{-1}\right)^T$. This solves Exercise 9. □

Solution to Exercise 10. From $\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 1 & 3 \end{pmatrix}$, we read off all values of α, namely

$$\alpha(1) = 2, \qquad \alpha(2) = 4, \qquad \alpha(3) = 1, \qquad \alpha(4) = 3.$$

Similarly,

$$\beta(1) = 3, \qquad \beta(2) = 4, \qquad \beta(3) = 1, \qquad \beta(4) = 3$$

and

$$\gamma(1) = 3, \qquad \gamma(2) = 4, \qquad \gamma(3) = 1, \qquad \gamma(4) = 2.$$

(a) The maps α and γ are permutations, but β is not.

Proof. The map β is not injective (indeed, it sends the distinct elements 1 and 4 to the same image, because $\beta(1) = 3 = \beta(4)$). Thus, β is not bijective. Hence, β is not a permutation.

The map α sends the elements $1, 2, 3, 4$ to the distinct elements $2, 4, 1, 3$; thus, it is injective. Furthermore, all of the 4 elements of $[4]$ appear as values of α. Thus, α is surjective. Hence, the map α is both injective and surjective. In other words, α is bijective. Hence, α is a permutation (since α is a map from $[4]$ to $[4]$).

A similar argument shows that γ is a permutation.

(b) We have

$$(\alpha \circ \beta)(1) = \alpha\left(\underbrace{\beta(1)}_{=3}\right) = \alpha(3) = 1;$$

$$(\alpha \circ \beta)(2) = \alpha\left(\underbrace{\beta(2)}_{=4}\right) = \alpha(4) = 3;$$

$$(\alpha \circ \beta)(3) = \alpha\left(\underbrace{\beta(3)}_{=1}\right) = \alpha(1) = 2;$$

$$(\alpha \circ \beta)(4) = \alpha\left(\underbrace{\beta(4)}_{=3}\right) = \alpha(3) = 1.$$

Tabulating these values of $\alpha \circ \beta$, we obtain

$$\alpha \circ \beta = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 2 & 1 \end{pmatrix} \qquad \text{in two-line notation.}$$

Similarly,

$$\beta \circ \alpha = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 3 & 1 \end{pmatrix} \qquad \text{in two-line notation;}$$

$$\gamma \circ \gamma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix} \qquad \text{in two-line notation.}$$

(Thus, $\gamma \circ \gamma$ is the identity map $\operatorname{id}_{[4]}$.)

(c) We have $\operatorname{sign}(\alpha) = -1$ and $\operatorname{sign}(\gamma) = 1$.

Proof. The inversions of α are $(1,3)$ (since $\alpha(1) > \alpha(3)$) and $(2,3)$ (since $\alpha(2) > \alpha(3)$) and $(2,4)$ (since $\alpha(2) > \alpha(4)$). Hence, α has 3 inversions. In other words, the Coxeter length of α is $\ell(\alpha) = 3$. Hence, the definition of sign yields $\operatorname{sign}(\alpha) = (-1)^{\ell(\alpha)} = (-1)^3 = -1$.

A similar argument shows that $\operatorname{sign}(\gamma) = 1$. (Alternatively, you can conclude this from the corollary in the notes from 2019-10-23, once you realize that $\gamma = t_{2,3} \circ \alpha$.) $\square$

References

[lina] Darij Grinberg, *Notes on linear algebra*, version of 13 December 2016. https://github.com/darijgr/lina

[Strickland] Neil Strickland, *MAS201 Linear Mathematics for Applications*, lecture notes, 28 September 2013. http://neil-strickland.staff.shef.ac.uk/courses/MAS201/