Math 201-003 Fall 2019 (Darij Grinberg): midterm 1

1. Reminders

- For any two matrices A and B, if the product AB is well-defined, then

$$(AB)_{i,j} = A_{i,1}B_{1,j} + A_{i,2}B_{2,j} + \cdots + A_{i,m}B_{m,j} \tag{1}$$
$$= \operatorname{row}_i A \cdot \operatorname{col}_j B \tag{2}$$

  for all indices i and j.

- The transpose A^T of an $n \times m$-matrix A is the $m \times n$-matrix whose entries are $\left(A^T\right)_{i,j} = A_{j,i}$.

- A matrix is in *RREF* if and only if it satisfies the following four conditions:
  - **RREF0:** Any zero row (= row full of zeros) is below any nonzero row (= row with at least some nonzero entries).
  - **RREF1:** In any nonzero row, the first nonzero entry is equal to 1. This entry is called the *pivot* of the row.
  - **RREF2:** The pivot of any nonzero row must be further to the right than the pivot of the previous nonzero row.
  - **RREF3:** If a **column** contains a pivot, then all other entries in the column are zero.

For the last exercise:

- If n is a nonnegative integer, then $[n]$ means the n-element set $\{1, 2, \ldots, n\}$.

- The composition $\alpha \circ \beta$ of two maps α and β is defined by $(\alpha \circ \beta)(x) = \alpha(\beta(x))$ for all x.

- A *permutation* of $[n]$ means a map from $[n]$ to $[n]$ that is bijective (i.e., both injective and surjective). Equivalently, it is a map from $[n]$ to $[n]$ that is invertible (i.e., has an inverse map).

- The *transposition* $t_{u,v}$ between two distinct elements u and v of $[n]$ is the permutation of $[n]$ that interchanges u with v and leaves all other elements of $[n]$ unchanged.

- Writing a map $f : [n] \to [n]$ in *two-line notation* means writing it as a table of values: $\begin{pmatrix} 1 & 2 & \cdots & n \\ f(1) & f(2) & \cdots & f(n) \end{pmatrix}$.

2. Matrix operations

Exercise 1. For each $n \in \mathbb{N}$, let

$$A_n = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 0 & \cdots & 0 \\ 1 & 1 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & 1 \end{pmatrix}$$

be the lower-triangular $n \times n$-matrix whose entries on and below the diagonal are all 1. Let furthermore B_n be its transpose (that is, $B_n = (A_n)^T$).

(a) Compute $A_3 B_3$ and $B_3 A_3$.

(b) Compute $A_{10} B_{10}$ and $B_{10} A_{10}$. (Feel free to describe the matrices in words instead of writing down each entry.)

Solution to Exercise 1. **(a)** Multiplying

$$A_3 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} \qquad \text{with} \qquad B_3 = (A_3)^T = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix},$$

we obtain

$$A_3 B_3 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{pmatrix} \qquad \text{and} \qquad B_3 A_3 = \begin{pmatrix} 3 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

(b) We have

$$A_{10} B_{10} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\ 1 & 2 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 \\ 1 & 2 & 3 & 4 & 4 & 4 & 4 & 4 & 4 & 4 \\ 1 & 2 & 3 & 4 & 5 & 5 & 5 & 5 & 5 & 5 \\ 1 & 2 & 3 & 4 & 5 & 6 & 6 & 6 & 6 & 6 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 7 & 7 & 7 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 8 & 8 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 9 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \end{pmatrix}$$

and

$$B_{10}A_{10} = \begin{pmatrix} 10 & 9 & 8 & 7 & 6 & 5 & 4 & 3 & 2 & 1 \\ 9 & 9 & 8 & 7 & 6 & 5 & 4 & 3 & 2 & 1 \\ 8 & 8 & 8 & 7 & 6 & 5 & 4 & 3 & 2 & 1 \\ 7 & 7 & 7 & 7 & 6 & 5 & 4 & 3 & 2 & 1 \\ 6 & 6 & 6 & 6 & 6 & 5 & 4 & 3 & 2 & 1 \\ 5 & 5 & 5 & 5 & 5 & 5 & 4 & 3 & 2 & 1 \\ 4 & 4 & 4 & 4 & 4 & 4 & 4 & 3 & 2 & 1 \\ 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 2 & 1 \\ 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

More generally, we can compute $A_n B_n$ and $B_n A_n$. For this, we need a piece of notation:

- If S is any nonempty finite set of real numbers, then let $\min S$ denote the smallest element of S.

Now, fix $n \in \mathbb{N}$. We claim that

$$A_n B_n = (\min\{i,j\})_{1\leq i\leq n,\ 1\leq j\leq n} = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & 2 & 2 & \cdots & 2 \\ 1 & 2 & 3 & \cdots & 3 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 2 & 3 & \cdots & n \end{pmatrix} \qquad (3)$$

and

$$B_n A_n = (\min\{n+1-i, n+1-j\})_{1\leq i\leq n,\ 1\leq j\leq n} = \begin{pmatrix} n & n-1 & n-2 & \cdots & 1 \\ n-1 & n-1 & n-2 & \cdots & 1 \\ n-2 & n-2 & n-2 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & 1 \end{pmatrix}. \qquad (4)$$

Proof of (3): We must prove that $A_n B_n = (\min\{i,j\})_{1\leq i\leq n,\ 1\leq j\leq n}$. In other words, we must prove that

$$(A_n B_n)_{i,j} = \min\{i,j\} \text{ for each } i,j \in \{1,2,\ldots,n\}.$$

So let $i,j \in \{1,2,\ldots,n\}$. Then,

$$\operatorname{row}_i(A_n) = (1,1,\ldots,1,0,0,\ldots,0)$$

(with the first i entries equal to 1 and the remaining $n-i$ entries equal to 0) and

$$\operatorname{row}_j(A_n) = (1,1,\ldots,1,0,0,\ldots,0) \tag{5}$$

(with the first j entries equal to 1 and the remaining $n-j$ entries equal to 0). Also, recall that transposing a matrix turns rows into columns (or, more precisely, the columns of the transpose of a matrix are the transposes of the rows of the original matrix). Hence, from $B_n = (A_n)^T$, we obtain

$$\operatorname{col}_j(B_n) = \left(\operatorname{row}_j(A_n)\right)^T = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

(with the first j entries equal to 1 and the remaining $n-j$ entries equal to 0), because of (5).

Now, (2) (applied to $A = A_n$ and $B = B_n$) yields

$$\begin{aligned}(A_nB_n)_{i,j} &= \underbrace{\operatorname{row}_i(A_n)}_{\substack{=(1,1,\ldots,1,0,0,\ldots,0) \\ \text{(with the first } i \text{ entries} \\ \text{equal to 1)}}} \cdot \underbrace{\operatorname{col}_j(B_n)}_{\substack{=\begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \\ \text{(with the first } j \text{ entries} \\ \text{equal to 1)}}} \\ &= \underbrace{(1,1,\ldots,1,0,0,\ldots,0)}_{\substack{\text{with the first } i \text{ entries} \\ \text{equal to 1}}} \cdot \underbrace{\begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}}_{\substack{\text{with the first } j \text{ entries} \\ \text{equal to 1}}} .\end{aligned} \tag{6}$$

Multiplying out the right hand side yields a sum of n terms, each of which is either $1 \cdot 1$ or $1 \cdot 0$ or $0 \cdot 1$ or $0 \cdot 0$. More precisely, the first $\min\{i,j\}$ addends in this sum are $1 \cdot 1$ (because in order to get a $1 \cdot 1$, you have to multiply one of the first i entries of $\underbrace{(1,1,\ldots,1,0,0,\ldots,0)}_{\substack{\text{with the first } i \text{ entries}\\ \text{equal to } 1}}$ with one of the first j entries of $\underbrace{\begin{pmatrix} 1\\1\\ \vdots \\ 1\\0\\0\\ \vdots \\ 0\end{pmatrix}}_{\substack{\text{with the first } j \text{ entries}\\ \text{equal to } 1}}$, and this is what is happening for the first $\min\{i,j\}$ entries of the product[1]), while all the remaining addends have one of the forms $1 \cdot 0$ and $0 \cdot 1$ and $0 \cdot 0$ (we don't care which one, since all these forms evaluate to 0). Thus, the whole sum is

$$\underbrace{1 \cdot 1 + 1 \cdot 1 + \cdots + 1 \cdot 1}_{\min\{i,j\} \text{ times}} + \underbrace{(\text{a sum of addends of the forms } 1 \cdot 0 \text{ and } 0 \cdot 1 \text{ and } 0 \cdot 0)}_{=0}$$
$$= \underbrace{1 \cdot 1 + 1 \cdot 1 + \cdots + 1 \cdot 1}_{\min\{i,j\} \text{ times}} = \min\{i,j\} \cdot 1 \cdot 1 = \min\{i,j\}.$$

Hence, (6) rewrites as

$$(A_n B_n)_{i,j} = \min\{i,j\}.$$

This completes our proof of (3).]

The proof of (4) is similar, except that we now have to analyze the product

$$\operatorname{row}_i(B_n) \cdot \operatorname{col}_j(A_n)$$
$$= \underbrace{(0,0,\ldots,0,1,1,\ldots,1)}_{\substack{\text{with the last } n+1-i \text{ entries}\\ \text{equal to } 1}} \cdot \underbrace{\begin{pmatrix} 0\\0\\ \vdots \\ 0\\1\\1\\ \vdots \\ 1\end{pmatrix}}_{\substack{\text{with the last } n+1-j \text{ entries}\\ \text{equal to } 1}}.$$

□

[1] whereas from entry $\min\{i,j\}+1$ on, at least one of your factors will be 0

3. Gaussian elimination

Note: When row-reducing a matrix, you have to either show all row operations or describe them verbally (if they follow a common pattern) and show the result. You don't need to follow the Gaussian elimination algorithm; you can use any sequence of row operations that leads to a matrix in RREF.[2] In particular, you don't need to freeze rows. When solving a system of linear equations, you are free to use any method.

Exercise 2. Solve the system

$$\begin{cases} a+b=c+d \\ a+c=b+d \\ a+d=b+c \end{cases}$$

of linear equations in four unknowns a, b, c, d.

Solution to Exercise 2. Here is the straightforward way of solving this system: Its augmented matrix is

$$\begin{pmatrix} 1 & 1 & -1 & -1 & 0 \\ 1 & -1 & 1 & -1 & 0 \\ 1 & -1 & -1 & 1 & 0 \end{pmatrix}.$$

We transform this matrix into RREF by the usual row reduction algorithm:

$$\begin{pmatrix} \boxed{1} & 1 & -1 & -1 & 0 \\ 1 & -1 & 1 & -1 & 0 \\ 1 & -1 & -1 & 1 & 0 \end{pmatrix}$$

$$\overset{\text{add } (-1)\cdot\text{row 1 to row 2}}{\longrightarrow} \begin{pmatrix} \boxed{1} & 1 & -1 & -1 & 0 \\ 0 & 2 & -2 & 0 & 0 \\ 1 & -1 & -1 & 1 & 0 \end{pmatrix}$$

$$\overset{\text{add } (-1)\cdot\text{row 1 to row 3}}{\longrightarrow} \begin{pmatrix} \boxed{1} & 1 & -1 & -1 & 0 \\ 0 & 2 & -2 & 0 & 0 \\ 0 & -2 & 0 & 2 & 0 \end{pmatrix}$$

$$\overset{\text{freeze row 1}}{\longrightarrow} \begin{pmatrix} \boxed{1} & 1 & -1 & -1 & 0 & \leftarrow \text{frozen} \\ 0 & \boxed{2} & -2 & 0 & 0 & \\ 0 & -2 & 0 & 2 & 0 & \end{pmatrix}$$

$$\overset{\text{scale row 1 by } 1/2}{\longrightarrow} \begin{pmatrix} \boxed{1} & 1 & -1 & -1 & 0 & \leftarrow \text{frozen} \\ 0 & \boxed{1} & -1 & 0 & 0 & \\ 0 & -2 & 0 & 2 & 0 & \end{pmatrix}$$

[2] Any given matrix has only one RREF; thus, no matter which way you get to a RREF, you will always get to the same RREF.

$$\xrightarrow{\text{add } 2\cdot\text{row } 1 \text{ to row } 2} \begin{pmatrix} \boxed{1} & 1 & -1 & -1 & 0 & \leftarrow \text{frozen} \\ 0 & \boxed{1} & -1 & 0 & 0 & \\ 0 & 0 & -2 & 2 & 0 & \end{pmatrix}$$

$$\xrightarrow{\text{freeze row } 1} \begin{pmatrix} \boxed{1} & 1 & -1 & -1 & 0 & \leftarrow \text{frozen} \\ 0 & \boxed{1} & -1 & 0 & 0 & \leftarrow \text{frozen} \\ 0 & 0 & \boxed{-2} & 2 & 0 & \end{pmatrix}$$

$$\xrightarrow{\text{scale row } 1 \text{ by } -1/2} \begin{pmatrix} \boxed{1} & 1 & -1 & -1 & 0 & \leftarrow \text{frozen} \\ 0 & \boxed{1} & -1 & 0 & 0 & \leftarrow \text{frozen} \\ 0 & 0 & \boxed{1} & -1 & 0 & \end{pmatrix}$$

(now our (non-frozen) matrix is in RREF)

$$\xrightarrow{\text{unfreeze row } 1} \begin{pmatrix} \boxed{1} & 1 & -1 & -1 & 0 & \leftarrow \text{frozen} \\ 0 & \boxed{1} & -1 & 0 & 0 & \\ 0 & 0 & \boxed{1} & -1 & 0 & \end{pmatrix}$$

$$\xrightarrow{\text{add } 1\cdot\text{row } 2 \text{ to row } 1} \begin{pmatrix} \boxed{1} & 1 & -1 & -1 & 0 & \leftarrow \text{frozen} \\ 0 & \boxed{1} & 0 & -1 & 0 & \\ 0 & 0 & \boxed{1} & -1 & 0 & \end{pmatrix}$$

$$\xrightarrow{\text{unfreeze row } 1} \begin{pmatrix} \boxed{1} & 1 & -1 & -1 & 0 \\ 0 & \boxed{1} & 0 & -1 & 0 \\ 0 & 0 & \boxed{1} & -1 & 0 \end{pmatrix}$$

$$\xrightarrow{\text{add } 1\cdot\text{row } 3 \text{ to row } 1} \begin{pmatrix} \boxed{1} & 1 & 0 & -2 & 0 \\ 0 & \boxed{1} & 0 & -1 & 0 \\ 0 & 0 & \boxed{1} & -1 & 0 \end{pmatrix}$$

$$\xrightarrow{\text{add } (-1)\cdot\text{row } 2 \text{ to row } 1} \begin{pmatrix} \boxed{1} & 0 & 0 & -1 & 0 \\ 0 & \boxed{1} & 0 & -1 & 0 \\ 0 & 0 & \boxed{1} & -1 & 0 \end{pmatrix}.$$

The resulting matrix is the augmented matrix of the system

$$\begin{cases} a - d = 0 \\ b - d = 0 \\ c - d = 0 \end{cases}.$$

The solutions of this system can be read off from it immediately: The variable d is a free variable, and the other three variables are given by $a = d$, $b = d$ and $c = d$. In other words, the solutions of our system are precisely the 4-tuples (a, b, c, d) consisting of four equal numbers.

[*Remark:* Here is an easier way to see this, without computing an RREF: Consider any solution (a, b, c, d) of the system $\begin{cases} a+b=c+d \\ a+c=b+d \\ a+d=b+c \end{cases}$. Then, adding together the first two equations of this system, we find $(a+b)+(a+c) = (c+d)+(b+d)$. This simplifies to $2a+b+c = 2d+b+c$. After subtracting $b+c$ from both sides of this, we obtain $2a = 2d$. Hence, $a = d$. Likewise, we find $a = b$ (by adding together the last two equations of the system) and $a = c$ (by adding together the first and the last equations of the system). Combining $a = b$ with $a = c$ and $a = d$, we obtain $a = b = c = d$. In other words, the 4-tuple (a, b, c, d) consists of four equal numbers. Thus, we have shown that any solution (a, b, c, d) of our system must be a 4-tuple consisting of four equal numbers. Conversely, it is clear that every 4-tuple consisting of four equal numbers is a solution of our system. Combining these observations, we conclude that the solutions of our system are precisely the 4-tuples (a, b, c, d) consisting of four equal numbers.] $\square$

Exercise 3. Let B_6 be the 6×6-matrix

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

(Its diagonal entries are 1; its entries just above the diagonal are 1; its entry in the bottom-left corner is 1; all its other entries are 0.)

Find the RREF of B_6.

Solution to Exercise 3. Let us start bringing B_6 into RREF using [Strickland, Method 6.4]:

$$B_6 = \begin{pmatrix} \boxed{1} & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\overset{\text{add } (-1)\cdot \text{row } 1 \text{ to row } 6}{\longrightarrow} \begin{pmatrix} \boxed{1} & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & -1 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\overset{\text{freeze row 1}}{\longrightarrow} \begin{pmatrix} \boxed{1} & 1 & 0 & 0 & 0 & 0 & \leftarrow \text{frozen} \\ 0 & \boxed{1} & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & -1 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\overset{\text{add } 1 \cdot \text{row 1 to row 5}}{\longrightarrow} \begin{pmatrix} \boxed{1} & 1 & 0 & 0 & 0 & 0 & \leftarrow \text{frozen} \\ 0 & \boxed{1} & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix}$$

$$\overset{\text{freeze row 1}}{\longrightarrow} \begin{pmatrix} \boxed{1} & 1 & 0 & 0 & 0 & 0 & \leftarrow \text{frozen} \\ 0 & \boxed{1} & 1 & 0 & 0 & 0 & \leftarrow \text{frozen} \\ 0 & 0 & \boxed{1} & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix}$$

$$\overset{\text{add } (-1) \cdot \text{row 1 to row 4}}{\longrightarrow} \begin{pmatrix} \boxed{1} & 1 & 0 & 0 & 0 & 0 & \leftarrow \text{frozen} \\ 0 & \boxed{1} & 1 & 0 & 0 & 0 & \leftarrow \text{frozen} \\ 0 & 0 & \boxed{1} & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & -1 & 0 & 1 \end{pmatrix}$$

$$\longrightarrow \cdots .$$

At this point it should be clear how the procedure goes on: We pick the topmost 1 on the diagonal as pivot; then we clear out the 1 or the -1 in the last cell of its column by adding either $(-1) \cdot \text{row } 1$ or $1 \cdot \text{row } 1$ to the last row; then we freeze row 1; rinse, repeat. As we keep doing this, the entries of our matrix on the diagonal and above the diagonal do not change, whereas the entries below the diagonal become 0. Let us take a closer look at the final step of this procedure:[3]

$$\cdots \longrightarrow \left(\begin{array}{cccccc} \boxed{1} & 1 & 0 & 0 & 0 & 0 \\ 0 & \boxed{1} & 1 & 0 & 0 & 0 \\ 0 & 0 & \boxed{1} & 1 & 0 & 0 \\ 0 & 0 & 0 & \boxed{1} & 1 & 0 \\ 0 & 0 & 0 & 0 & \boxed{1} & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{array}\right) \begin{array}{l} \leftarrow \text{frozen} \\ \leftarrow \text{frozen} \\ \leftarrow \text{frozen} \\ \leftarrow \text{frozen} \\ \\ \\ \end{array}$$

$$\underset{\longrightarrow}{\text{add } (-1) \cdot \text{row } 1 \text{ to row } 2} \left(\begin{array}{cccccc} \boxed{1} & 1 & 0 & 0 & 0 & 0 \\ 0 & \boxed{1} & 1 & 0 & 0 & 0 \\ 0 & 0 & \boxed{1} & 1 & 0 & 0 \\ 0 & 0 & 0 & \boxed{1} & 1 & 0 \\ 0 & 0 & 0 & 0 & \boxed{1} & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right) \begin{array}{l} \leftarrow \text{frozen} \\ \leftarrow \text{frozen} \\ \leftarrow \text{frozen} \\ \leftarrow \text{frozen} \\ \\ \\ \end{array}.$$

At this point, the unfrozen part of our matrix is in RREF, so we unfreeze rows and clear out the nonzero entries above the pivots. At the end of this procedure, we obtain the matrix

$$\left(\begin{array}{cccccc} \boxed{1} & 0 & 0 & 0 & 0 & 1 \\ 0 & \boxed{1} & 0 & 0 & 0 & -1 \\ 0 & 0 & \boxed{1} & 0 & 0 & 1 \\ 0 & 0 & 0 & \boxed{1} & 0 & -1 \\ 0 & 0 & 0 & 0 & \boxed{1} & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right),$$

which differs from the identity matrix only in its last column. The last column has a 0 at the very bottom and then alternates between 1's and -1's (starting with a 1) as you move upwards. This matrix is in RREF. So this is the RREF of B_6.

[*Remark:* For any $n \in \mathbb{N}$, we can define an $n \times n$-matrix B_n similarly to B_6. We then get a similar answer for the RREF of B_n whenever n is even: namely, the RREF differs from I_n only in its last column, which has a 0 at the very bottom and then

[3] It is important to keep track of the last row of the matrix. Every time we pick a new pivot, the column that contains this pivot has either a 1 or a -1 in the bottommost position, whereas all other columns except for the last column of the matrix have 0's in their bottommost position. (Thus, the last row of the matrix has only two nonzero entries.) We clear out the 1 or the -1 in the bottommost position of the column with the current pivot by adding $(-1) \cdot \text{row } 1$ or $1 \cdot \text{row } 1$ to the last row. Thus, this 1 or -1 disappears, but the neighboring entry to its right (previously a 0) becomes a -1 or a 1 (which will have to be cleared in the next step). Thus, we can imagine that there is a "wandering" 1 in the last row of our matrix, which starts in the leftmost position and then moves a step to the right every time we perform a row operation; it also flips its sign when doing so. Thus, after k row operations, it will have flipped its sign k times (so it will equal $(-1)^k$) and will have taken k steps to the right (so it will be in the $(k+1)$-st column of the matrix).

alternates between 1's and -1's (starting with a 1) as you move upwards. When n is odd, the RREF of B_n is simply the identity matrix I_n.] □

4. Linear combinations, independence and spanning

Exercise 4. Let

$$v_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \qquad v_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \qquad w = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

(a) Are the vectors v_1, v_2 linearly dependent?

(b) Is w a linear combination of v_1, v_2 ? (If yes, then find the coefficients λ_1, λ_2 such that $w = \lambda_1 v_1 + \lambda_2 v_2$.)

Solution to Exercise 4. **(a)** No. They are linearly **independent**.

Proof. We must show that any linear relation between v_1 and v_2 is trivial.

So let $\lambda_1 v_1 + \lambda_2 v_2 = 0$ be any linear relation between v_1 and v_2. We must show that it is trivial, i.e., that $\lambda_1 = \lambda_2 = 0$.

From $v_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $v_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$, we obtain

$$\lambda_1 v_1 + \lambda_2 v_2 = \lambda_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \lambda_2 \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} \lambda_1 \cdot 1 + \lambda_2 \cdot 2 \\ \lambda_1 \cdot 2 + \lambda_2 \cdot 1 \end{pmatrix} = \begin{pmatrix} \lambda_1 + 2\lambda_2 \\ 2\lambda_1 + \lambda_2 \end{pmatrix},$$

so that

$$\begin{pmatrix} \lambda_1 + 2\lambda_2 \\ 2\lambda_1 + \lambda_2 \end{pmatrix} = \lambda_1 v_1 + \lambda_2 v_2 = 0.$$

In other words, the two equalities $\lambda_1 + 2\lambda_2 = 0$ and $2\lambda_1 + \lambda_2 = 0$ hold. We could now treat these two equalities as a system of equations and solve it, but it is easier to use a common-sense strategy: Subtracting the equality $\lambda_1 + 2\lambda_2 = 0$ from the equality $2\lambda_1 + \lambda_2 = 0$, we obtain $\lambda_1 - \lambda_2 = 0$. Thus, $\lambda_1 = \lambda_2$. Hence, the equality $\lambda_1 + 2\lambda_2 = 0$ rewrites as $\lambda_2 + 2\lambda_2 = 0$, i.e., as $3\lambda_2 = 0$. Thus, $\lambda_2 = 0$. Hence, $\lambda_1 = \lambda_2 = 0$, which is precisely what we desired to show.

(b) Yes. Indeed, $w = \dfrac{1}{3} v_1 + \dfrac{1}{3} v_2$.

[This is easily seen by eyeballing: Just observe that $v_1 + v_2 = \begin{pmatrix} 3 \\ 3 \end{pmatrix} = 3w$. Of course, you can also obtain the answer by solving a system of linear equations.] □

Exercise 5. Let $n \geq 2$ be an integer. Recall the vectors $e_1, e_2, \ldots, e_n$ in $\mathbb{R}^n$ that were defined in Exercise 4 of homework set #2. (Thus, $e_i = (0, 0, \ldots, 0, 1, 0, 0, \ldots, 0)^T$ with the 1 in position i.)

Now, consider the n vectors

$$e_1 - e_2, \quad e_2 - e_3, \quad e_3 - e_4, \quad \ldots, \quad e_{n-1} - e_n, \quad e_n - e_1.$$

Let us denote them by $d_1, d_2, \ldots, d_n$. (Thus, $d_i = e_i - e_{i+1}$ for each $i \in \{1, 2, \ldots, n-1\}$, and $d_n = e_n - e_1$.)

(a) Are these n vectors $d_1, d_2, \ldots, d_n$ linearly independent when $n = 7$?

(b) Do these n vectors $d_1, d_2, \ldots, d_n$ span $\mathbb{R}^n$ when $n = 7$?

Solution to Exercise 5. We shall ignore the "when $n = 7$" part, since the answers and the proofs work equally for all values of $n \geq 2$.

(a) The n vectors $d_1, d_2, \ldots, d_n$ are linearly **dependent**.

Proof. The following is a nontrivial linear relation between them:

$$\begin{aligned} & 1d_1 + 1d_2 + \cdots + 1d_n \\ & = d_1 + d_2 + \cdots + d_n \\ & = (e_1 - e_2) + (e_2 - e_3) + \cdots + (e_{n-1} - e_n) + (e_n - e_1) \\ & \qquad \left(\text{since } d_i = e_i - e_{i+1} \text{ for each } i \in \{1, 2, \ldots, n-1\}, \text{ and } d_n = e_n - e_1\right) \\ & = 0 \qquad (\text{since all addends and subtrahends cancel}). \end{aligned}$$

(b) The n vectors $d_1, d_2, \ldots, d_n$ do not span $\mathbb{R}^n$.

First proof. A vector $w \in \mathbb{R}^n$ shall be called *zero-sum* if the sum of its entries is 0. (Thus, a vector $w = (w_1, w_2, \ldots, w_n)^T \in \mathbb{R}^n$ is zero-sum if and only if $w_1 + w_2 + \cdots + w_n = 0$.) Each of the vectors $d_1, d_2, \ldots, d_n$ is zero-sum (since one of its entries is 1, another of its entries is -1, and all the remaining entries are 0). Thus, any linear combination of $d_1, d_2, \ldots, d_n$ is zero-sum as well (because it is easy to see that any linear combination of zero-sum vectors is again zero-sum). But there are certainly vectors in $\mathbb{R}^n$ that are not zero-sum (for example, e_1). Thus, there are vectors in $\mathbb{R}^n$ that are not linear combinations of $d_1, d_2, \ldots, d_n$. In other words, $d_1, d_2, \ldots, d_n$ do not span $\mathbb{R}^n$.

Second proof. We must prove that the n vectors $d_1, d_2, \ldots, d_n$ do not span $\mathbb{R}^n$. Indeed, assume the contrary. Thus, the n vectors $d_1, d_2, \ldots, d_n$ span $\mathbb{R}^n$. Hence, [Strickland, Proposition 10.12 (b)] (or, equivalently, Proposition 1.2.7 **(b)** from the notes from 2019-10-09) yields that the list $(d_1, d_2, \ldots, d_n)$ is a basis of $\mathbb{R}^n$. Thus, in particular, $d_1, d_2, \ldots, d_n$ are linearly independent. But this contradicts our result from solving part **(a)** of this exercise. This contradiction shows that our assumption was false. Hence, part **(b)** is solved. □

5. Matrix inversion and invertibility

Exercise 6. Let A be the 2×2-matrix $\begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix}$, where a is any real. Compute the inverse A^{-1}.

Solution to Exercise 6. The inverse of A always exists (i.e., the matrix A is invertible), and equals

$$A^{-1} = \begin{pmatrix} 1 & 0 \\ -a & 1 \end{pmatrix}.$$

Proof. There are many ways to find this. The simplest one is probably using our method for finding inverses via row-reduction (Theorem 1.3.5 in the notes from 2019-10-16):

$$[A \mid I_2] = \begin{pmatrix} \boxed{1} & 0 & 1 & 0 \\ a & 1 & 0 & 1 \end{pmatrix} \xrightarrow{\text{add } -a\cdot\text{row } 1 \text{ to row } 2} \begin{pmatrix} \boxed{1} & 0 & 1 & 0 \\ 0 & 1 & -a & 1 \end{pmatrix} = \left[I_2 \mid \begin{pmatrix} 1 & 0 \\ -a & 1 \end{pmatrix} \right]$$

(that's it, we're done).

Alternatively, you can just "guess" that the inverse should be $\begin{pmatrix} 1 & 0 \\ -a & 1 \end{pmatrix}$, and then verify this by checking that $\begin{pmatrix} 1 & 0 \\ -a & 1 \end{pmatrix} \cdot A = I_2$ and $A \cdot \begin{pmatrix} 1 & 0 \\ -a & 1 \end{pmatrix} = I_2$. (By some results we proved in class, it suffices to verify only one of these two equalities.) □

Exercise 7. Which of the following matrices are invertible?

(a) $\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix}$. **(b)** $\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$. **(c)** $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 7 \end{pmatrix}$.

(d) $\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix}$. **(e)** $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$.

Solution to Exercise 7. **(a)** This matrix is **not invertible**.

Proof. Denote the columns of this matrix by v_1, v_2, v_3 (from first to last). Then, $v_3 = 0$; in other words, $0v_1 + 0v_2 + 1v_3 = 0$. This is a nontrivial relation between the columns of this matrix. Thus, the columns of this matrix are linearly dependent. Hence, the matrix cannot be invertible (because if it was invertible, then the Inverse Matrix Theorem (Theorem 1.2.1 in the notes from 2019-10-16)[4] would show that its columns are linearly independent).

(b) This matrix is **invertible**.

Proof. It is easy to see that the RREF of this matrix is I_3 (indeed, row-reduction will get rid of the two 1's below the diagonal without disturbing any other entries of the matrix). Thus, the matrix can be row-reduced to I_3. Hence, the Inverse Matrix Theorem (Theorem 1.2.1 in the notes from 2019-10-16)[5] shows that it is invertible.

(c) This matrix is **not invertible**.

Proof. Let A be this matrix. Denote the columns of the matrix A^T by v_1, v_2, v_3, v_4 (from first to last). Then, $6v_2 = 5v_3$ (since $v_2 = (0,0,0,5)^T$ and $v_3 = (0,0,0,6)^T$).

[4] More specifically, we are using the implication **(k)** $\Longrightarrow$ **(b)** of the Inverse Matrix Theorem.

[5] More specifically, we are using the implication **(a)** $\Longrightarrow$ **(k)** of the Inverse Matrix Theorem.

In other words, $0v_1 + 6v_2 + (-5)v_3 + 0v_4 = 0$. This is a nontrivial relation between the columns of the matrix A^T. Thus, the columns of A^T are linearly dependent. Hence, the matrix A cannot be invertible (because if it was invertible, then the Inverse Matrix Theorem (Theorem 1.2.1 in the notes from 2019-10-16)[6] would show that the columns of A^T are linearly independent).

(d) This matrix is **not invertible**.

Proof. Corollary 1.2.7 in the notes from 2019-10-16 shows that if an $n \times m$-matrix has an inverse, then $n = m$. In other words, if a matrix has an inverse, then it is a square matrix. Hence, our matrix does not have an inverse (since it is not a square matrix). In other words, it is not invertible.

(e) This matrix is **not invertible**.

Proof. Denote the columns of this matrix by v_1, v_2, v_3 (from first to last). Then, $v_1 + v_3 = 2v_2$ (because each entry in the middle column of the matrix is the average of its left and right neighbors); in other words, $1v_1 + (-2)v_2 + 1v_3 = 0$. This is a nontrivial relation between the columns of this matrix. Thus, the columns of this matrix are linearly dependent. Hence, the matrix cannot be invertible (because if it was invertible, then the Inverse Matrix Theorem (Theorem 1.2.1 in the notes from 2019-10-16)[7] would show that its columns are linearly independent). □

6. Permutations

Exercise 8. **(a)** Write the three transpositions $t_{1,2}$, $t_{1,3}$ and $t_{2,3}$ of $[3]$ in two-line notation.

(b) Compute $t_{1,2} \circ t_{2,3}$.

(c) Compute $\operatorname{sign}(t_{1,2} \circ t_{2,3})$.

(d) Which of the maps (written in two-line notation as)

$$\alpha = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}, \qquad \beta = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}, \qquad \gamma = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 3 \end{pmatrix}$$

are permutations of $[3]$?

Solution to Exercise 8. **(a)** We have

$$t_{1,2} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}, \qquad t_{1,3} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}, \qquad t_{2,3} = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}.$$

(b) We have

$$t_{1,2} \circ t_{2,3} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}.$$

[6] More specifically, we are using the implication **(k)** $\Longrightarrow$ **(f)** of the Inverse Matrix Theorem.

[7] More specifically, we are using the implication **(k)** $\Longrightarrow$ **(b)** of the Inverse Matrix Theorem.

This is because

$$\left(t_{1,2}\circ t_{2,3}\right)(1) = t_{1,2}\left(\underbrace{t_{2,3}(1)}_{=1}\right) = t_{1,2}(1) = 2;$$
$$\left(t_{1,2}\circ t_{2,3}\right)(2) = t_{1,2}\left(\underbrace{t_{2,3}(2)}_{=3}\right) = t_{1,2}(3) = 3;$$
$$\left(t_{1,2}\circ t_{2,3}\right)(3) = t_{1,2}\left(\underbrace{t_{2,3}(3)}_{=2}\right) = t_{1,2}(2) = 1.$$

(c) One way to do this is by counting the inversions: The inversions of $t_{1,2}\circ t_{2,3}$ are $(1,2)$ and $(1,3)$ (as you can easily see from the two-line notation of $t_{1,2}\circ t_{2,3}$). Thus, $t_{1,2}\circ t_{2,3}$ has 2 inversions. In other words, $\ell\left(t_{1,2}\circ t_{2,3}\right) = 2$. Hence, the definition of sign yields $\operatorname{sign}\left(t_{1,2}\circ t_{2,3}\right) = (-1)^{\ell(t_{1,2}\circ t_{2,3})} = (-1)^2 = 1$.

An alternative way to do this is by recalling the following two facts:

1. We have $\operatorname{sign}\left(\sigma\circ t_{p,q}\right) = -\operatorname{sign}(\sigma)$ for any permutation σ of $[n]$ and any two distinct elements p and q of $[n]$. (This is part of the Corollary in the notes from 2019-10-23.)

2. We have $\operatorname{sign}\left(t_{p,q}\right) = -1$ for any two distinct elements p and q of $[n]$. (This is a Proposition in the notes from 2019-10-23.)

Applying fact 2 to $p=1$ and $q=2$, we find $\operatorname{sign}\left(t_{1,2}\right) = -1$. Now, applying fact 1 to $\sigma = t_{1,2}$, $p=2$ and $q=3$, we find $\operatorname{sign}\left(t_{1,2}\circ t_{2,3}\right) = -\underbrace{\operatorname{sign}\left(t_{1,2}\right)}_{=-1} = -(-1) = 1$.

Thus, part **(c)** is solved again.

(d) The maps α and β are permutations (indeed, they are both injective and surjective), whereas the map γ is not a permutation (indeed, $\gamma(1) = 3 = \gamma(3)$ shows that γ is not injective, and thus γ is not bijective, so that γ cannot be a permutation). □

References

[lina] Darij Grinberg, *Notes on linear algebra*, version of 13 December 2016. `https://github.com/darijgr/lina`

[Strickland] Neil Strickland, *MAS201 Linear Mathematics for Applications*, lecture notes, 28 September 2013. `http://neil-strickland.staff.shef.ac.uk/courses/MAS201/`