Math 201-003 Fall 2019 (Darij Grinberg): homework set 3 with solutions

Exercise 1. Prove that
$$\det(AB) = \det A \cdot \det B$$
holds for any 2×2-matrices $A = \begin{pmatrix} a & b \\ a' & b' \end{pmatrix}$ and $B = \begin{pmatrix} c & d \\ c' & d' \end{pmatrix}$.

[In class, I said that this holds for square matrices of any size, but I didn't give a proof. You cannot use this general fact without proof here.]

Solution to Exercise 1. Recall that every 2×2-matrix C satisfies
$$\det C = C_{1,1}C_{2,2} - C_{1,2}C_{2,1}. \qquad (1)$$
(This is what the definition of a determinant boils down to when applied to a 2×2-matrix.)

Let A and B be two 2×2-matrices. Write A and B in the forms
$$A = \begin{pmatrix} a & b \\ a' & b' \end{pmatrix} \qquad \text{and} \qquad B = \begin{pmatrix} c & d \\ c' & d' \end{pmatrix}.$$
Thus,
$$AB = \begin{pmatrix} a & b \\ a' & b' \end{pmatrix} \begin{pmatrix} c & d \\ c' & d' \end{pmatrix} = \begin{pmatrix} ac + bc' & ad + bd' \\ a'c + b'c' & a'd + b'd' \end{pmatrix}$$
(by the definition of the product of two matrices). Hence, (1) yields
$$\det(AB) = \left(ac + bc'\right)\left(a'd + b'd'\right) - \left(ad + bd'\right)\left(a'c + b'c'\right). \qquad (2)$$
On the other hand, (1) also yields
$$\det A = ab' - ba' \qquad \text{and} \qquad \det B = cd' - dc'.$$
Multiplying these two equalities, we find
$$\det A \cdot \det B = \left(ab' - ba'\right) \cdot \left(cd' - dc'\right). \qquad (3)$$

Now, straightforward expanding (and cancelling terms) shows that the right hand sides of (2) and (3) are equal (and, in fact, equal to $ab'cd' - ab'dc' - ba'cd' + ba'dc'$). Thus, the left hand sides of (2) and (3) are equal as well. In other words, $\det(AB) = \det A \cdot \det B$. This solves the exercise. $\square$

Exercise 2. Let $a, b, c, d, e, f, g, h, i, j, k, \ell, m, n, o, p$ be any numbers.

(a) Find a simple formula for the determinant
$$\det \begin{pmatrix} a & b & c & d \\ \ell & 0 & 0 & e \\ k & 0 & 0 & f \\ j & i & h & g \end{pmatrix}.$$

(b) Find a simple formula for the determinant

$$\det \begin{pmatrix} a & b & c & d & e \\ f & 0 & 0 & 0 & g \\ h & 0 & 0 & 0 & i \\ j & 0 & 0 & 0 & k \\ \ell & m & n & o & p \end{pmatrix}.$$

(Do not mistake the "o" for a "0".)

[**Hint:** Part **(b)** is simpler than part **(a)**.]

Solution to Exercise 2. Of course, both parts of Exercise 2 can be proved by straightforward application of the definition of a determinant, provided you are willing to write down a sum of 24 products (for part **(a)**) and a sum of 120 products (for part **(b)**). But it is easier to "haggle" the sizes of these sums down before doing so:

(a) Let A be the matrix $\begin{pmatrix} a & b & c & d \\ \ell & 0 & 0 & e \\ k & 0 & 0 & f \\ j & i & h & g \end{pmatrix}$. We want to find $\det A$.

Note that we have $A_{2,2} = 0$ and $A_{2,3} = 0$ and $A_{3,2} = 0$ and $A_{3,3} = 0$. (These are the four 0 entries in the middle of the matrix A.)

The definition of a determinant yields

$$\det A = \sum_{\sigma \text{ is a permutation of } [4]} \operatorname{sign}(\sigma) \cdot A_{1,\sigma(1)} A_{2,\sigma(2)} A_{3,\sigma(3)} A_{4,\sigma(4)}. \qquad (4)$$

The sum on the right hand side of (4) has 24 addends. However, some of them are 0. Namely, every addend corresponding to a permutation σ of $[4]$ satisfying $\sigma(2) \notin \{1,4\}$ must be 0 [1]. Hence, all such addends can be removed from the sum (without changing the value of this sum). Similarly, all addends corresponding to permutations σ of $[4]$ satisfying $\sigma(3) \notin \{1,4\}$ must be 0, and can therefore also be removed from the sum. The addends that survive these two removals are the ones that correspond to permutations σ of $[4]$ satisfying $\sigma(2) \in \{1,4\}$ and $\sigma(3) \in \{1,4\}$. It is easy to see that there are exactly four such permutations: In two-line notation, these permutations are

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix}, \qquad \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 1 & 3 \end{pmatrix}, \qquad \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 4 & 2 \end{pmatrix}, \qquad \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}.$$

[1] *Proof.* Let σ be a permutaiton of $[4]$ such that $\sigma(2) \notin \{1,4\}$. We must then show that the addend on the right hand side of (4) corresponding to this σ must be 0. In other words, we have to show that $\operatorname{sign}(\sigma) \cdot A_{1,\sigma(1)} A_{2,\sigma(2)} A_{3,\sigma(3)} A_{4,\sigma(4)} = 0$.

We have $\sigma(2) \notin \{1,4\}$, and thus $\sigma(2) \in \{2,3\}$. Hence, $A_{2,\sigma(2)} = 0$ (because $A_{2,2} = 0$ and $A_{2,3} = 0$), and thus $\operatorname{sign}(\sigma) \cdot A_{1,\sigma(1)} \underbrace{A_{2,\sigma(2)}}_{=0} A_{3,\sigma(3)} A_{4,\sigma(4)} = 0$, qed.

The addends corresponding to these permutations are $A_{1,2}A_{2,1}A_{3,4}A_{4,3}$, $-A_{1,2}A_{2,4}A_{3,1}A_{4,3}$, $-A_{1,3}A_{2,1}A_{3,4}A_{4,2}$ and $A_{1,3}A_{2,4}A_{3,1}A_{4,2}$. Hence, (4) simplifies to

$$\begin{aligned}
&\det A \\
&= \underbrace{A_{1,2}}_{=b}\underbrace{A_{2,1}}_{=\ell}\underbrace{A_{3,4}}_{=f}\underbrace{A_{4,3}}_{=h} - \underbrace{A_{1,2}}_{=b}\underbrace{A_{2,4}}_{=e}\underbrace{A_{3,1}}_{=k}\underbrace{A_{4,3}}_{=h} - \underbrace{A_{1,3}}_{=c}\underbrace{A_{2,1}}_{=\ell}\underbrace{A_{3,4}}_{=f}\underbrace{A_{4,2}}_{=i} + \underbrace{A_{1,3}}_{=c}\underbrace{A_{2,4}}_{=e}\underbrace{A_{3,1}}_{=k}\underbrace{A_{4,2}}_{=i} \\
&= b\ell fh - bekh - c\ell fi + ceki.
\end{aligned}$$

This is a simple enough formula to consider an answer to Exercise 2 **(a)**, but we can simplify it even further. Namely,

$$\det A = b\ell fh - bekh - c\ell fi + ceki = (bh - ci)(\ell f - ek).$$

Exercise 2 **(a)** is solved.

(b) Let A be the matrix $\begin{pmatrix} a & b & c & d & e \\ f & 0 & 0 & 0 & g \\ h & 0 & 0 & 0 & i \\ j & 0 & 0 & 0 & k \\ \ell & m & n & o & p \end{pmatrix}$. We want to find $\det A$.

The most important property of A is that the 3×3-submatrix in the middle of A is filled with zeroes. In other words,

$$A_{u,v} = 0 \qquad \text{for every } u \in \{2,3,4\} \text{ and } v \in \{2,3,4\}. \tag{5}$$

Now, the definition of a determinant yields

$$\det A = \sum_{\sigma \text{ is a permutation of } [5]} \operatorname{sign}(\sigma) \cdot A_{1,\sigma(1)}A_{2,\sigma(2)}A_{3,\sigma(3)}A_{4,\sigma(4)}A_{5,\sigma(5)}. \tag{6}$$

But every permutation σ of $[5]$ satisfies $A_{2,\sigma(2)}A_{3,\sigma(3)}A_{4,\sigma(4)} = 0$ [2]. Hence, (6) becomes

$$\begin{aligned}
\det A &= \sum_{\sigma \text{ is a permutation of } [5]} \operatorname{sign}(\sigma) \cdot A_{1,\sigma(1)} \underbrace{A_{2,\sigma(2)}A_{3,\sigma(3)}A_{4,\sigma(4)}}_{=0} A_{5,\sigma(5)} \\
&= \sum_{\sigma \text{ is a permutation of } [5]} \operatorname{sign}(\sigma) \cdot A_{1,\sigma(1)} 0 A_{5,\sigma(5)} = 0.
\end{aligned}$$

Exercise 2 **(b)** is thus solved. □

[2] *Proof.* Let σ be a permutation of $[5]$. Then, σ is a bijective map, and hence an injective map. Therefore, the numbers $\sigma(2), \sigma(3), \sigma(4)$ are pairwise distinct.

We now claim that there exists an $u \in \{2,3,4\}$ such that $\sigma(u) \in \{2,3,4\}$. In order to prove this, we assume the contrary. Thus, every $u \in \{2,3,4\}$ satisfies $\sigma(u) \notin \{2,3,4\}$. Hence, every $u \in \{2,3,4\}$ satisfies $\sigma(u) \in \{1,5\}$ (since $\sigma(u) \in \{1,2,3,4,5\}$ but $\sigma(u) \notin \{2,3,4\}$). In other words, the numbers $\sigma(2), \sigma(3), \sigma(4)$ belong to $\{1,5\}$. Hence, $\sigma(2), \sigma(3), \sigma(4)$ are three distinct numbers belonging to the set $\{1,5\}$. But this is absurd, since the set $\{1,5\}$ does not contain three distinct numbers. Hence, we have obtained a contradiction. This shows that our assumption was wrong.

We thus have shown that there exists an $u \in \{2,3,4\}$ such that $\sigma(u) \in \{2,3,4\}$. Consider such a u. Applying (5) to $v = \sigma(u)$, we now obtain $A_{u,\sigma(u)} = 0$. But $u \in \{2,3,4\}$, so that $A_{u,\sigma(u)}$ is a factor in the product $A_{2,\sigma(2)}A_{3,\sigma(3)}A_{4,\sigma(4)}$. Hence, the product $A_{2,\sigma(2)}A_{3,\sigma(3)}A_{4,\sigma(4)}$ is 0 (since its factor $A_{u,\sigma(u)}$ is 0), qed.

Exercise 3. Find

$$\det\begin{pmatrix} 5 & 4 & 3 & 2 & 1 \\ 1 & 0 & 0 & 3 & 0 \\ 2 & 0 & 0 & 0 & 0 \\ 2 & 2 & 4 & 3 & 0 \\ -3 & 0 & 1 & 2 & 0 \end{pmatrix}.$$

First solution to Exercise 3 (sketched). The easiest way to solve this is by repeated Laplace expansion (i.e., expanding the determinant along one row, then expanding the results along a row of theirs, etc.). Indeed, usually, if A is an $n \times n$-matrix, then Laplace expansion along a row will express $\det A$ as a sum of n addends, each of which is (up to sign) a product of an entry of this row times a smaller determinant. However, if this row has many zero entries, then the corresponding addends are zero, and thus the expression greatly simplifies. Here is how this works out for our matrix:

- We expand $\det\begin{pmatrix} 5 & 4 & 3 & 2 & 1 \\ 1 & 0 & 0 & 3 & 0 \\ 2 & 0 & 0 & 0 & 0 \\ 2 & 2 & 4 & 3 & 0 \\ -3 & 0 & 1 & 2 & 0 \end{pmatrix}$ along the 3-rd row. We obtain a sum of 5 addends:

$$\det\begin{pmatrix} 5 & 4 & 3 & 2 & 1 \\ 1 & 0 & 0 & 3 & 0 \\ 2 & 0 & 0 & 0 & 0 \\ 2 & 2 & 4 & 3 & 0 \\ -3 & 0 & 1 & 2 & 0 \end{pmatrix}$$
$$= (-1)^{3+1} 2 \det\begin{pmatrix} 4 & 3 & 2 & 1 \\ 0 & 0 & 3 & 0 \\ 2 & 4 & 3 & 0 \\ 0 & 1 & 2 & 0 \end{pmatrix} + (-1)^{3+2} 0 \det(*)$$
$$+ (-1)^{3+3} 0 \det(*) + (-1)^{3+4} 0 \det(*) + (-1)^{3+5} 0 \det(*),$$

where the symbol "$*$" signifies a matrix that we need not compute because its determinant will be multiplied by 0 anyway. This simplifies to

$$\det\begin{pmatrix} 5 & 4 & 3 & 2 & 1 \\ 1 & 0 & 0 & 3 & 0 \\ 2 & 0 & 0 & 0 & 0 \\ 2 & 2 & 4 & 3 & 0 \\ -3 & 0 & 1 & 2 & 0 \end{pmatrix}$$
$$= \underbrace{(-1)^{3+1}}_{=1} 2 \det\begin{pmatrix} 4 & 3 & 2 & 1 \\ 0 & 0 & 3 & 0 \\ 2 & 4 & 3 & 0 \\ 0 & 1 & 2 & 0 \end{pmatrix} = 2 \det\begin{pmatrix} 4 & 3 & 2 & 1 \\ 0 & 0 & 3 & 0 \\ 2 & 4 & 3 & 0 \\ 0 & 1 & 2 & 0 \end{pmatrix}.$$

- Thus, we need to find $\det\begin{pmatrix} 4 & 3 & 2 & 1 \\ 0 & 0 & 3 & 0 \\ 2 & 4 & 3 & 0 \\ 0 & 1 & 2 & 0 \end{pmatrix}$. We expand this determinant along its 2-nd row:

$$\begin{aligned}
&\det\begin{pmatrix} 4 & 3 & 2 & 1 \\ 0 & 0 & 3 & 0 \\ 2 & 4 & 3 & 0 \\ 0 & 1 & 2 & 0 \end{pmatrix} \\
&= (-1)^{2+1} 0 \det(*) + (-1)^{2+2} 0 \det(*) + (-1)^{2+3} 3 \det\begin{pmatrix} 4 & 3 & 1 \\ 2 & 4 & 0 \\ 0 & 1 & 0 \end{pmatrix} \\
&\qquad + (-1)^{2+4} 0 \det(*) \\
&= \underbrace{(-1)^{2+3}}_{=-1} 3 \det\begin{pmatrix} 4 & 3 & 1 \\ 2 & 4 & 0 \\ 0 & 1 & 0 \end{pmatrix} = -3 \det\begin{pmatrix} 4 & 3 & 1 \\ 2 & 4 & 0 \\ 0 & 1 & 0 \end{pmatrix}.
\end{aligned}$$

- Thus, we need to find $\det\begin{pmatrix} 4 & 3 & 1 \\ 2 & 4 & 0 \\ 0 & 1 & 0 \end{pmatrix}$. We expand this determinant along its 3-rd row:

$$\begin{aligned}
\det\begin{pmatrix} 4 & 3 & 1 \\ 2 & 4 & 0 \\ 0 & 1 & 0 \end{pmatrix} &= (-1)^{3+1} 0 \det(*) + (-1)^{3+2} 1 \det\begin{pmatrix} 4 & 1 \\ 2 & 0 \end{pmatrix} + (-1)^{3+3} 0 \det(*) \\
&= \underbrace{(-1)^{3+2}}_{=-1} 1 \det\begin{pmatrix} 4 & 1 \\ 2 & 0 \end{pmatrix} = -\det\begin{pmatrix} 4 & 1 \\ 2 & 0 \end{pmatrix}.
\end{aligned}$$

- Thus, we need to find $\det\begin{pmatrix} 4 & 1 \\ 2 & 0 \end{pmatrix}$. We can do this by playing the same game further (expanding it along its 2-nd row), but alternatively we can just use the simple formula $\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$ to realize that this determinant is $4 \cdot 0 - 1 \cdot 2 = -2$.

Combining these arguments, we obtain

$$\begin{aligned}
&\det\begin{pmatrix} 5 & 4 & 3 & 2 & 1 \\ 1 & 0 & 0 & 3 & 0 \\ 2 & 0 & 0 & 0 & 0 \\ 2 & 2 & 4 & 3 & 0 \\ -3 & 0 & 1 & 2 & 0 \end{pmatrix} \\
&= 2 \underbrace{\det\begin{pmatrix} 4 & 3 & 2 & 1 \\ 0 & 0 & 3 & 0 \\ 2 & 4 & 3 & 0 \\ 0 & 1 & 2 & 0 \end{pmatrix}}_{=-3\det\begin{pmatrix} 4 & 3 & 1 \\ 2 & 4 & 0 \\ 0 & 1 & 0 \end{pmatrix}} = \underbrace{2(-3)}_{=-6} \underbrace{\det\begin{pmatrix} 4 & 3 & 1 \\ 2 & 4 & 0 \\ 0 & 1 & 0 \end{pmatrix}}_{=-\det\begin{pmatrix} 4 & 1 \\ 2 & 0 \end{pmatrix}} \\
&= (-6)\left(-\det\begin{pmatrix} 4 & 1 \\ 2 & 0 \end{pmatrix}\right) = 6 \underbrace{\det\begin{pmatrix} 4 & 1 \\ 2 & 0 \end{pmatrix}}_{=-2} = 6(-2) = -12.
\end{aligned}$$

□

Second solution to Exercise 3 (sketched). We recall three properties of determinants:

- *Property 1:* If we swap two rows of an $n \times n$-matrix, then its determinant gets multiplied by -1 (that is, it flips its sign but preserves its magnitude). (This is Theorem 1.2.6 in the class notes from 2019-10-30.)
- *Property 2:* If we swap two columns of an $n \times n$-matrix, then its determinant gets multiplied by -1 (that is, it flips its sign but preserves its magnitude). (This is the analogue of Property 1 for columns instead of rows. It can be deduced from Property 1, as we have seen in Theorem 1.6.1 in the class notes from 2019-10-30.)
- *Property 3:* If an $n \times n$-matrix A is triangular (i.e., upper-triangular or lower-triangular), then its determinant is the product of its diagonal elements:

  $$\det A = A_{1,1} A_{2,2} \cdots A_{n,n}.$$

  (This is Theorem 1.1.2 in the class notes from 2019-10-30.)

These properties suffice to quickly solve the exercise: Just keep swapping rows and columns until the matrix becomes triangular. To wit:

$$\begin{aligned}
&\det\begin{pmatrix} 5 & 4 & 3 & 2 & 1 \\ 1 & 0 & 0 & 3 & 0 \\ 2 & 0 & 0 & 0 & 0 \\ 2 & 2 & 4 & 3 & 0 \\ -3 & 0 & 1 & 2 & 0 \end{pmatrix} \\
&= -\det\begin{pmatrix} -3 & 0 & 1 & 2 & 0 \\ 1 & 0 & 0 & 3 & 0 \\ 2 & 0 & 0 & 0 & 0 \\ 2 & 2 & 4 & 3 & 0 \\ 5 & 4 & 3 & 2 & 1 \end{pmatrix} \qquad \left(\begin{array}{c}\text{by Property 1, since we have} \\ \text{swapped the 1-st and 5-th rows here}\end{array}\right) \\
&= \det\begin{pmatrix} 2 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 3 & 0 \\ -3 & 0 & 1 & 2 & 0 \\ 2 & 2 & 4 & 3 & 0 \\ 5 & 4 & 3 & 2 & 1 \end{pmatrix} \qquad \left(\begin{array}{c}\text{by Property 1, since we have} \\ \text{swapped the 1-st and 3-rd rows here}\end{array}\right) \\
&= -\underbrace{\det\begin{pmatrix} 2 & 0 & 0 & 0 & 0 \\ 1 & 3 & 0 & 0 & 0 \\ -3 & 2 & 1 & 0 & 0 \\ 2 & 3 & 4 & 2 & 0 \\ 5 & 2 & 3 & 4 & 1 \end{pmatrix}}_{\substack{=2\cdot 3\cdot 1\cdot 2\cdot 1 \\ \text{(by Property 3, since} \\ \text{this matrix is lower-triangular)}}} \qquad \left(\begin{array}{c}\text{by Property 2, since we have} \\ \text{swapped the 2-nd and 4-th columns here}\end{array}\right) \\
&= -2\cdot 3\cdot 1\cdot 2\cdot 1 = -12.
\end{aligned}$$

□

Exercise 4. Here is a 5×5-matrix: $\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 1 & 0 & 2 & 2 & 1 \end{pmatrix}$. Its determinant is 0.

Find a 0 entry which can be replaced by a 1 to give a nonzero determinant. (You can box this entry in the matrix. Note that you cannot replace more than one entry simultaneously.)

Solution to Exercise 4 (sketched). These 0 entries are boxed: $\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ \boxed{0} & 0 & 4 & 0 & \boxed{0} \\ 0 & 0 & 1 & 2 & 0 \\ \boxed{0} & 0 & 3 & 0 & \boxed{0} \\ 1 & 0 & 2 & 2 & 1 \end{pmatrix}$.

Why is it these entries and no others? Well, you can just check every one until you find one that works. But this requires some patience; there are faster ways. Here is one: Recall Properties 1, 2 and 3 from the second solution to Exercise 3. Let us transform our matrix into a lower-triangular one by swapping rows and swapping columns:

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 1 & 0 & 2 & 2 & 1 \end{pmatrix} \overset{\text{swap 1-st and 3-rd columns}}{\longrightarrow} \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 4 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 2 & 0 \\ 3 & 0 & 0 & 0 & 0 \\ 2 & 0 & 1 & 2 & 1 \end{pmatrix}$$
$$\overset{\text{swap 3-rd and 4-th columns}}{\longrightarrow} \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 4 & 0 & 0 & 0 & 0 \\ 1 & 0 & 2 & 0 & 0 \\ 3 & 0 & 0 & 0 & 0 \\ 2 & 0 & 2 & 1 & 1 \end{pmatrix}$$
$$\overset{\text{swap 1-st and 2-nd columns}}{\longrightarrow} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 2 & 2 & 1 & 1 \end{pmatrix}.$$

We have thus obtained a lower-triangular matrix with diagonal entries $1, 4, 2, 0, 1$. Thus, its determinant is 0, but if we replace the 0 on the diagonal by a 1, then it will become $1 \cdot 4 \cdot 2 \cdot 1 \cdot 1 \neq 0$. Thus we have found a 0 that we can replace by a 1 to obtain a nonzero determinant. All we need is to trace it through our swaps in order to learn what cell it occupied in the original matrix. (This is left to the reader.) $\square$