Math 201-003 Fall 2019 (Darij Grinberg): homework set 2
due date: Wednesday 2019-10-16 at the beginning of class,
or before that by email or Blackboard
Please solve 5 of the 6 exercises!

Exercise 1. Let $n \in \mathbb{N}$. For any n numbers $d_1, d_2, \ldots, d_n \in \mathbb{R}$, we let $\operatorname{diag}(d_1, d_2, \ldots, d_n)$ denote the $n \times n$-matrix whose diagonal entries are $d_1, d_2, \ldots, d_n$ (in this order from top to bottom), while its off-diagonal entries are all 0. In other words,

$$\operatorname{diag}(d_1, d_2, \ldots, d_n) = \begin{pmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_n \end{pmatrix}.$$

This is called the *diagonal matrix with diagonal entries* $d_1, d_2, \ldots, d_n$.

(a) Given any $d_1, d_2, d_3 \in \mathbb{R}$ and any 3×3-matrix $A = \begin{pmatrix} a & b & c \\ a' & b' & c' \\ a'' & b'' & c'' \end{pmatrix}$, compute $\operatorname{diag}(d_1, d_2, d_3) \cdot A$.

(b) State a rule for computing $\operatorname{diag}(d_1, d_2, \ldots, d_n) \cdot A$, where $d_1, d_2, \ldots, d_n \in \mathbb{R}$ and where A is any $n \times m$-matrix (with m being any nonnegative integer).

(c) State a rule for computing $A \cdot \operatorname{diag}(d_1, d_2, \ldots, d_n)$, where $d_1, d_2, \ldots, d_n \in \mathbb{R}$ and where A is any $m \times n$-matrix (with m being any nonnegative integer).

[You don't have to prove these rules.]

We recall the definition of linear combinations ([Strickland, Definition 7.1]):

Definition 0.1. Let $v_1, v_2, \ldots, v_k$ be some vectors in $\mathbb{R}^n$. Then, a *linear combination* of $v_1, v_2, \ldots, v_k$ means a vector that can be written in the form

$$\lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_k v_k \qquad \text{for some } \lambda_1, \lambda_2, \ldots, \lambda_k \in \mathbb{R}.$$

The numbers $\lambda_1, \lambda_2, \ldots, \lambda_k$ in this definition are called *coefficients* of this linear combination; but they are not necessarily unique (in fact, they are unique if $v_1, v_2, \ldots, v_k$ are linearly independent; we shall see this later in class). For example, if $v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $v_2 = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$, then the vector $\begin{pmatrix} 3 \\ 3 \end{pmatrix}$ is a linear combination of v_1 and v_2, and can be written as $\lambda_1 v_1 + \lambda_2 v_2$ in many ways:

$$\begin{pmatrix} 3 \\ 3 \end{pmatrix} = 1v_1 + 1v_2 = 3v_1 + 0v_2 = 0v_1 + \frac{3}{2}v_2 = 2v_1 + \frac{1}{2}v_2 = \cdots.$$

We recall the definition of linear independence ([Strickland, §8]):

Definition 0.2. Let $v_1, v_2, \ldots, v_k$ be some vectors in $\mathbb{R}^n$.

(a) A *relation* (more precisely: *linear relation*) between $v_1, v_2, \ldots, v_k$ means a choice of reals $\lambda_1, \lambda_2, \ldots, \lambda_k$ satisfying

$$\lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_k v_k = 0.$$

(Here, 0 denotes the zero vector $0_{n \times 1} \in \mathbb{R}^n$.)

(b) The *trivial relation* between $v_1, v_2, \ldots, v_k$ is the relation obtained by choosing $\lambda_1 = \lambda_2 = \cdots = \lambda_k = 0$. Clearly, $v_1, v_2, \ldots, v_k$ always have this trivial relation.

(c) We say that the vectors $v_1, v_2, \ldots, v_k$ (or, more precisely, the list $(v_1, v_2, \ldots, v_k)$ of these vectors) are *independent* (more precisely: *linearly independent*) if the only relation between $v_1, v_2, \ldots, v_k$ is the trivial relation. Otherwise, we say that these vectors are *dependent*.

For example, if we set

$$v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \qquad v_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \qquad \text{and} \qquad v_3 = \begin{pmatrix} 1 \\ 3 \end{pmatrix},$$

then the vectors v_1, v_2, v_3 are linearly dependent, since they have the nontrivial relation $1v_1 + (-2)\, v_2 + 1v_3 = 0$.

Exercise 2. Let

$$v_1 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \qquad \text{and} \qquad v_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \qquad \text{and} \qquad v_3 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}.$$

(a) Are v_1, v_2, v_3 dependent? (If yes, show a nontrivial relation between them.)

(b) Is $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ a linear combination of v_1, v_2, v_3 ? (If yes, provide a choice of coefficients $\lambda_1, \lambda_2, \lambda_3$ that demonstrate it.)

(c) Is $\begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix}$ a linear combination of v_1, v_2, v_3 ? (If yes, provide a choice of coefficients $\lambda_1, \lambda_2, \lambda_3$ that demonstrate it.)

Clearly, rearranging a list of vectors does not change the set of its linear combinations: In fact, the rearranged vectors will have the same linear combinations as the original vectors; only the coefficients will change their order. For example, the linear combinations of v_1, v_2, v_3 are the same as the linear combinations of v_3, v_1, v_2, since $\lambda_1 v_1 + \lambda_2 v_2 + \lambda_3 v_3 = \lambda_3 v_3 + \lambda_1 v_1 + \lambda_2 v_2$.

Rearranging a list of vectors also does not change its linear dependence or independence: In fact, any linear relation between the rearranged vectors is a rearranged linear relation between the original vectors, and vice versa. Thus, v_1, v_2, v_3 are dependent if and only if v_3, v_1, v_2 are dependent.

What happens to lists of vectors when we duplicate a vector? The following exercise answers this question (at least when it's the last vector that is duplicated):

Exercise 3. Let $v_1, v_2, \ldots, v_k$ be some vectors in $\mathbb{R}^n$. Assume that $v_{k-1} = v_k$. Justify the following (i.e., sketch the proofs):

(a) The vectors $v_1, v_2, \ldots, v_k$ are linearly dependent.

(b) The linear combinations of $v_1, v_2, \ldots, v_k$ are the same vectors as the linear combinations of $v_1, v_2, \ldots, v_{k-1}$.

Exercise 4. Let $n \in \mathbb{N}$. For each $k \in \{1, 2, \ldots, n\}$, we let e_k denote the column vector of size n whose k-th entry is 1 and whose all other entries are 0.

(For example: If $n = 3$, then $e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ and $e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ and $e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$.)

Explain why each vector in $\mathbb{R}^n$ is a linear combination of $e_1, e_2, \ldots, e_n$.

Definition 0.1 and Definition 0.2 make sense not only for vectors in $\mathbb{R}^n$. For example, we can replace "vectors in $\mathbb{R}^n$" by "polynomials" in both definitions, and obtain definitions for linear combinations and linear independence of polynomials (in one variable x, with real coefficients).[1] Thus:

- The polynomial $x^2 + 2x - 1$ is a linear combination of the polynomials x^2 and $(x-1)^2$, since
$$x^2 + 2x - 1 = 2 \cdot x^2 + (-1) \cdot (x-1)^2 .$$
- The polynomials x^2, $(x-1)^2$, $(x-2)^2$ and $(x-3)^2$ are linearly dependent, since there exists a nontrivial linear relation between them: If we take $\lambda_1 = 1$, $\lambda_2 = -3$, $\lambda_3 = 3$ and $\lambda_4 = -1$, then
$$\begin{aligned} &\lambda_1 x^2 + \lambda_2 (x-1)^2 + \lambda_3 (x-2)^2 + \lambda_4 (x-3)^2 \\ &= 1 \cdot x^2 + (-3) \cdot (x-1)^2 + 3 \cdot (x-2)^2 + (-1) \cdot (x-3)^2 = 0. \end{aligned}$$
- The polynomials x^2, $(x-1)^2$ and $(x-2)^2$ are linearly independent, because the only relation
$$\lambda_1 x^2 + \lambda_2 (x-1)^2 + \lambda_3 (x-2)^2 = 0 \qquad (1)$$
between them is the trivial one. How to be sure of this? One way is to assume that (1) holds, and try to derive $\lambda_1 = \lambda_2 = \lambda_3 = 0$ from it. This can be done, for example, by comparing coefficients: Straightforward expansion yields
$$\begin{aligned} &\lambda_1 x^2 + \lambda_2 (x-1)^2 + \lambda_3 (x-2)^2 \\ &= \lambda_1 x^2 + \lambda_2 \left(x^2 - 2x + 1\right) + \lambda_3 \left(x^2 - 4x + 4\right) \\ &= (\lambda_1 + \lambda_2 + \lambda_3) x^2 + (-2\lambda_2 - 4\lambda_3) x + (\lambda_2 + 4\lambda_3) , \end{aligned}$$

[1] Of course, we then have to interpret 0 as the zero polynomial rather than the zero vector $0_{n \times 1}$.

and thus (1) rewrites as

$$(\lambda_1 + \lambda_2 + \lambda_3) x^2 + (-2\lambda_2 - 4\lambda_3) x + (\lambda_2 + 4\lambda_3) = 0.$$

Since equal polynomials have equal coefficients, this is equivalent to the following system of linear equations:

$$\begin{cases} \lambda_1 + \lambda_2 + \lambda_3 = 0 \\ -2\lambda_2 - 4\lambda_3 = 0 \\ \lambda_2 + 4\lambda_3 = 0 \end{cases},$$

whose only solution is $(\lambda_1, \lambda_2, \lambda_3) = (0, 0, 0)$ (you know how to check this) – and thus the relation must be trivial. So the polynomials x^2, $(x-1)^2$ and $(x-2)^2$ are linearly independent.

Exercise 5. **(a)** Are the four polynomials $x^3, (x-1)^3, (x-2)^3, (x-3)^3$ linearly independent?

(b) Is 1 a linear combination of these four polynomials?

Similarly, we can define linear combinations and linear independence of matrices: Just replace "vectors in $\mathbb{R}^n$" by "$n \times m$-matrices" in Definition 0.1 and Definition 0.2[2]. For example, the matrix $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ is a linear combination of the matrices $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$, since

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = 1 \cdot \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} + (-1) \cdot \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + 1 \cdot \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}.$$

Exercise 6. Consider the four matrices

$$N = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \qquad E = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \qquad S = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}, \qquad W = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}.$$

(The letter E stands for "east", not for the identity matrix, which some also call E.)

(a) Is the identity matrix I_2 a linear combination of N, E, S, W ?

(b) Are N, E, S, W linearly dependent?

References

[Strickland] Neil Strickland, *MAS201 Linear Mathematics for Applications*, lecture notes, 28 September 2013.
`http://neil-strickland.staff.shef.ac.uk/courses/MAS201/`

[2]Of course, we then have to interpret 0 as the zero matrix $0_{n\times m}$ rather than the zero vector $0_{n\times 1}$.