Math 201-003 Fall 2019 (Darij Grinberg): homework set 1
due date: Monday 2019-10-07 at the beginning of class, or before that by email or Blackboard

Exercise 1. **(a)** Write down the augmented matrix corresponding to the following system of linear equations:

$$\begin{cases} 3x - 6y + z = 5 \\ y - 2z = 8 \\ y - 3z = 1 \\ 2z = 5 \end{cases}.$$

(b) Write down the system of linear equations that corresponds to the following augmented matrix:

$$\begin{pmatrix} 1 & 2 & 3 & \mathbf{1} \\ 2 & 3 & 4 & \mathbf{2} \\ 3 & 4 & 5 & \mathbf{3} \end{pmatrix}.$$

(Here, as in class, we are putting the entries of the last column in boldface instead of drawing a vertical bar to the left of them.)

(c) Find the RREF of the matrix in part **(a)**.

(d) Solve the system of linear equations in part **(a)**.

(e) Find the RREF of the matrix in part **(b)**.

(f) Solve the system of linear equations in part **(b)**.

Exercise 2. Consider the two matrices

$$C = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix} \qquad \text{and} \qquad D = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix}.$$

Let $v = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$ be any column vector of size 4.

(a) Compute Cv.

(b) Compute Dv.

(c) Compute CD.

(d) Compute DC.

Definition 0.1. Let A be any square matrix. Then, the *trace* of A is defined to be the sum of all diagonal entries of A. In other words, it is defined to be $A_{1,1} + A_{2,2} + \cdots + A_{n,n}$, where $n \in \mathbb{N}$ is such that A is an $n \times n$-matrix.

The trace of A is denoted by $\operatorname{Tr} A$.

For example, here is what the traces of 2×2-matrices and 3×3-matrices are:

$$\operatorname{Tr}\begin{pmatrix} a & b \\ a' & b' \end{pmatrix} = a + b'; \qquad \operatorname{Tr}\begin{pmatrix} a & b & c \\ a' & b' & c' \\ a'' & b'' & c'' \end{pmatrix} = a + b' + c''.$$

Exercise 3. **(a)** Prove that $\operatorname{Tr}(AB) = \operatorname{Tr}(BA)$ for any two 2×2-matrices A and B.

(b) Prove that $\operatorname{Tr}(AB) = \operatorname{Tr}(BA)$ for any two 3×3-matrices A and B.

[**Hint:** In part **(a)**, you can set $A = \begin{pmatrix} a & b \\ a' & b' \end{pmatrix}$ and $B = \begin{pmatrix} x & y \\ x' & y' \end{pmatrix}$ and compute both $\operatorname{Tr}(AB)$ and $\operatorname{Tr}(BA)$ and check their equality. Likewise for part **(b)**. Alternatively, you can show that $\operatorname{Tr}(AB) = \operatorname{Tr}(BA)$ for any $n \times n$-matrices A and B of any size n. In the latter case, you will want to be familiar with the basics of summation signs [lina, §2.9].)

Exercise 4. Let $n \in \mathbb{N}$. For each $k \in \{1, 2, \ldots, n\}$, we let e_k denote the $n \times 1$-matrix (i.e., the column vector of size n) whose k-th entry is 1 and whose all other entries are 0.

(For example: If $n = 3$, then $e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ and $e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ and $e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$.)

(a) If $n = 3$ and if $A = \begin{pmatrix} a & b & c \\ a' & b' & c' \end{pmatrix}$ is any 2×3-matrix, then compute Ae_1, Ae_2 and Ae_3.

(b) Express Ae_k in very simple terms (no matrix multiplication involved!) for every $m \times n$-matrix A (of arbitrary dimensions m and n).

(c) If $n = 2$ and if $A = \begin{pmatrix} a & b & c \\ a' & b' & c' \end{pmatrix}$ is any 2×3-matrix, then compute $e_1^T A$ and $e_2^T A$. (Note: e_k^T means $(e_k)^T$, that is, the transpose of e_k. This is a row vector.)

(d) Express $e_k^T A$ in very simple terms (no matrix multiplication involved!) for every $n \times m$-matrix A (of arbitrary dimensions n and m).

[In parts **(b)** and **(d)**, you don't need to prove your answer.]

Exercise 5. Which of the following matrices are in RREF? (See [Strickland, Definition 5.1] for the definition of an RREF.)

(a) $\begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$. **(b)** $\begin{pmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$. **(c)** $\begin{pmatrix} 1 & 0 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$.

(d) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$. **(e)** $\begin{pmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$. **(f)** $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

(g) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. **(h)** $\begin{pmatrix} 1 & 0 & 3 \\ 0 & 0 & 2 \\ 0 & 1 & 1 \end{pmatrix}$ **(i)** $\begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

(j) $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

[No proofs are required.]

Exercise 6. **(a)** Let $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Show that the matrices 2×2-matrices B that satisfy $AB = BA$ are precisely the matrices of the form $\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}$ for $a, b \in \mathbb{R}$.

[**Hint:** Let $B = \begin{pmatrix} x & y \\ x' & y' \end{pmatrix}$ be any 2×2-matrix. Your goal is to prove that $AB = BA$ holds if and only if $x = y'$ and $x' = 0$.]

(b) Let $A' = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$. Characterize the 2×2-matrices B that satisfy $A'B = BA'$.

[That is, find a similar characterization to the one given in part **(a)**. Show your reasoning!]

References

[lina] Darij Grinberg, *Notes on linear algebra*, version of 13 December 2016. `https://github.com/darijgr/lina`

[Strickland] Neil Strickland, *MAS201 Linear Mathematics for Applications*, lecture notes, 28 September 2013. `http://neil-strickland.staff.shef.ac.uk/courses/MAS201/`