Math 201-003 Fall 2019 (Darij Grinberg): final exam training

This is only about the material of the last few weeks. Don't forget the rest of the course; go over the midterm training problems 1 and midterm training problems 2, as well as the actual midterms.

1. Subspaces

Exercise 1. Which of the following sets are subspaces of $\mathbb{R}^2$?

$$\begin{aligned}
A &= \left\{ (x_1, x_2)^T \in \mathbb{R}^2 \ \mid \ x_1 + x_2 = 1 \right\}; \\
B &= \left\{ (x_1, x_2)^T \in \mathbb{R}^2 \ \mid \ x_1 + x_2 = 0 \right\}; \\
C &= \left\{ (x_1, x_2)^T \in \mathbb{R}^2 \ \mid \ x_1 - x_2 = 1 \right\}; \\
D &= \left\{ (x_1, x_2)^T \in \mathbb{R}^2 \ \mid \ x_1 - x_2 = 0 \right\}; \\
E &= \left\{ (x_1, x_2)^T \in \mathbb{R}^2 \ \mid \ x_1 x_2 = 1 \right\}; \\
F &= \left\{ (x_1, x_2)^T \in \mathbb{R}^2 \ \mid \ x_1 x_2 = 0 \right\}; \\
G &= \left\{ (x_1, x_2)^T \in \mathbb{R}^2 \ \mid \ x_1^2 + x_2^2 = 1 \right\}; \\
H &= \left\{ (x_1, x_2)^T \in \mathbb{R}^2 \ \mid \ x_1^2 + x_2^2 = 0 \right\}; \\
I &= \left\{ (a, a+b)^T \ \mid \ a, b \in \mathbb{R} \right\}; \\
J &= \left\{ (a-b, b-a)^T \ \mid \ a, b \in \mathbb{R} \right\}.
\end{aligned}$$

2. Independence, spanning and bases

Exercise 2. Define four vectors a, b, c, d in $\mathbb{R}^3$ as follows:

$$a = \begin{pmatrix} 4 \\ 3 \\ 2 \end{pmatrix}, \qquad b = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \qquad c = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \qquad d = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}.$$

Show that $\operatorname{span}(a, b) = \operatorname{span}(c, d)$ as follows:

(a) Write each of a and b as a linear combination of c and d.

(b) Write $\lambda a + \mu b$ (for any fixed reals λ and μ) as a linear combination of c and d. Conclude that $\lambda a + \mu b \in \operatorname{span}(c, d)$ for each $\lambda, \mu \in \mathbb{R}$, and therefore $\operatorname{span}(a, b) \subseteq \operatorname{span}(c, d)$.

(c) Write each of c and d as a linear combination of a and b.

(d) Write $\lambda c + \mu d$ (for any fixed reals λ and μ) as a linear combination of a and b. Conclude that $\lambda c + \mu d \in \operatorname{span}(a, b)$ for each $\lambda, \mu \in \mathbb{R}$, and therefore $\operatorname{span}(c, d) \subseteq \operatorname{span}(a, b)$.

The results of **(b)** and **(d)** combined yield $\operatorname{span}(a, b) = \operatorname{span}(c, d)$.

Exercise 3. **(a)** Find a list of 3 vectors that spans the subspace

$$K = \left\{ (a, b, c, d)^T \in \mathbb{R}^4 \mid a + b = c + d \right\}$$

of $\mathbb{R}^4$.

(b) Find a list of 4 vectors that spans the subspace

$$L = \left\{ (a_1 + a_2, a_2 + a_3, a_3 + a_4, a_4 + a_1)^T \mid a_1, a_2, a_3, a_4 \in \mathbb{R} \right\}$$

of $\mathbb{R}^4$.

(c) Find a list of 3 vectors that spans L.

Exercise 4. Consider the vector space $\mathbb{R}^3$.

(a) The list $\mathbf{a} = \left(\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right)$ spans $\mathbb{R}^3$. Shrink this list to a basis of $\mathbb{R}^3$ by removing some redundant elements.

(b) The list $\mathbf{b} = \left(\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} \right)$ is linearly independent. Extend this list to a basis of $\mathbb{R}^3$ by appending to it some elements from the list $\mathbf{a}$.

3. Rank and rank normal form

Exercise 5. Let $A = \begin{pmatrix} 1 & 6 & 7 & 12 \\ 2 & 5 & 8 & 11 \\ 3 & 4 & 9 & 10 \end{pmatrix}$.

Find the rank of A and a sequence of row and column operations that transform A into its rank normal form.

(See §1.3 in the class notes from 2019-12-04 for the meaning of column operations and of rank. But you don't need to proceed in the exact same way as in the example given there; you may find it easier to start applying some convenient column operations before bringing the matrix into RREF.)

4. Solution outlines

Solution to Exercise 1. **(a)** The set A is not a subspace of $\mathbb{R}^2$. Indeed, it fails to contain $\overrightarrow{0} = (0,0)^T$.

(b) The set B is a subspace of $\mathbb{R}^2$.

[*Proof.* We need to check that B contains the zero vector, is closed under addition, and is closed under scaling.

- Let us show that B contains the zero vector: We have $\overrightarrow{0} = (0,0)^T \in B$, since $0+0=0$.
- Let us show that B is closed under addition: Let $v \in B$ and $w \in B$. We must prove that $v + w \in B$.

  We have $v \in B = \left\{ (x_1, x_2)^T \in \mathbb{R}^2 \mid x_1 + x_2 = 0 \right\}$; thus, we can write v as $v = (v_1, v_2)^T \in \mathbb{R}^2$ for some reals v_1, v_2 that satisfy $v_1 + v_2 = 0$. Consider these v_1, v_2.

  We have $w \in B = \left\{ (x_1, x_2)^T \in \mathbb{R}^2 \mid x_1 + x_2 = 0 \right\}$; thus, we can write w as $w = (w_1, w_2)^T \in \mathbb{R}^2$ for some reals w_1, w_2 that satisfy $w_1 + w_2 = 0$. Consider these w_1, w_2.

  Now,
  $$\underbrace{v}_{=(v_1,v_2)^T} + \underbrace{w}_{=(w_1,w_2)^T} = (v_1, v_2)^T + (w_1, w_2)^T = (v_1 + w_1, v_2 + w_2)^T .$$
  Thus, in order to prove that $v + w \in B$, we need to show that $(v_1 + w_1) + (v_2 + w_2) = 0$ (by the definition of B). But this is easy to show:
  $$(v_1 + w_1) + (v_2 + w_2) = \underbrace{(v_1 + v_2)}_{=0} + \underbrace{(w_1 + w_2)}_{=0} = 0.$$
  Thus, we have proved that $v + w \in B$. This completes the proof that B is closed under addition.
- Let us show that B is closed under scaling: Let $\lambda \in \mathbb{R}$ and $v \in B$. We must prove that $\lambda v \in B$.

  We have $v \in B = \left\{ (x_1, x_2)^T \in \mathbb{R}^2 \mid x_1 + x_2 = 0 \right\}$; thus, we can write v as $v = (v_1, v_2)^T \in \mathbb{R}^2$ for some reals v_1, v_2 that satisfy $v_1 + v_2 = 0$. Consider these v_1, v_2.

  Now,
  $$\lambda \underbrace{v}_{=(v_1,v_2)^T} = \lambda (v_1, v_2)^T = (\lambda v_1, \lambda v_2)^T .$$
  Thus, in order to prove that $\lambda v \in B$, we need to show that $\lambda v_1 + \lambda v_2 = 0$ (by the definition of B). But this is easy to show:
  $$\lambda v_1 + \lambda v_2 = \lambda \underbrace{(v_1 + v_2)}_{=0} = 0.$$
  Thus, we have proved that $\lambda v \in B$. This completes the proof that B is closed under scaling.

Combining these three facts, we conclude that B is a subspace of $\mathbb{R}^2$.]

(c) The set C is not a subspace of $\mathbb{R}^2$. Indeed, it fails to contain $\overrightarrow{0} = (0,0)^T$.

(d) The set D is a subspace of $\mathbb{R}^2$.

[*Proof.* The proof of this is analogous to the proof of the fact that B is a subspace of $\mathbb{R}^2$. (The only difference is that some "$+$" signs have to be replaced by "$-$" signs now.)]

(e) The set E is not a subspace of $\mathbb{R}^2$. Indeed, it fails to contain $\overrightarrow{0} = (0,0)^T$.

(f) The set F is not a subspace of $\mathbb{R}^2$. Indeed, it is not closed under addition, since the two elements $(0,1)^T \in F$ and $(1,0)^T \in F$ have sum $(0,1)^T + (1,0)^T = (1,1)^T \notin F$.

(g) The set G is not a subspace of $\mathbb{R}^2$. Indeed, it is not closed under addition, since the two elements $(0,1)^T \in G$ and $(1,0)^T \in G$ have sum $(0,1)^T + (1,0)^T = (1,1)^T \notin G$.

(h) The set H is a subspace of $\mathbb{R}^2$. Indeed, it is the one-element set $\left\{\overrightarrow{0}\right\} = \left\{(0,0)^T\right\}$, since the only two reals x_1 and x_2 satisfying $x_1^2 + x_2^2 = 0$ are 0 and 0.

(i) The set I is a subspace of $\mathbb{R}^2$. Indeed, it is the full $\mathbb{R}^2$, since each $(x_1, x_2)^T \in \mathbb{R}^2$ can be written as $(a, a+b)^T$ for some $a, b \in \mathbb{R}$ (namely, for $a = x_1$ and $b = x_2 - x_1$).

(j) The set J is a subspace of $\mathbb{R}^2$. Indeed, it is the same set as B.

[*Proof.* Each element of J has the form $(a-b, b-a)^T$ for some $a, b \in \mathbb{R}$ (by the definition of J), and thus is a vector in $\mathbb{R}^2$ whose two entries sum to 0 (because its two entries $a-b$ and $b-a$ sum to $(a-b)+(b-a) = 0$).

The definition of B can be restated as follows: The set B consists of all vectors in $\mathbb{R}^2$ whose two entries sum to 0. Thus, $J \subseteq B$ (since each element of J is a vector in $\mathbb{R}^2$ whose two entries sum to 0). Conversely, it is easy to see that $B \subseteq J$. (Indeed, any element of B has the form $(x_1, x_2)^T \in \mathbb{R}^2$ for some reals x_1, x_2 satisfying $x_1 + x_2 = 0$ (by the definition of B); thus, it has the form $(x_1, -x_1)$ (since $x_1 + x_2 = 0$ entails $x_2 = -x_1$), and this shows that it has the form $(a-b, b-a)^T$ for some $a, b \in \mathbb{R}$ (namely, for $a = x_1$ and $b = 0$). But the latter says precisely that it belongs to J. So we have shown that every element of B belongs to J. In other words, $B \subseteq J$.)

Combining $J \subseteq B$ with $B \subseteq J$, we obtain $J = B$. Thus, J is a subspace of $\mathbb{R}^2$ (since we already know that B is a subspace of $\mathbb{R}^2$.)] □

Solution to Exercise 2. **(a)** Let us first write a as a linear combination of c and d. In other words, we are seeking two real numbers γ and δ such that $a = \gamma c + \delta d$. In other words, we are solving the equation $a = \gamma c + \delta d$ in two real unknowns γ and δ.

Since $a = \begin{pmatrix} 4 \\ 3 \\ 2 \end{pmatrix}$, $c = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$ and $d = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$, this equation rewrites as $\begin{pmatrix} 4 \\ 3 \\ 2 \end{pmatrix} = \gamma \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + \delta \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$. This is equivalent to the system $\begin{cases} 2\gamma + (-1)\,\delta = 4 \\ 1\gamma + 0\delta = 3 \\ 0\gamma + 1\delta = 2 \end{cases}$ of linear equations. Solving this system in any way, we find $\gamma = 3$ and $\delta = 2$. Thus, $a = \gamma c + \delta d$ becomes $a = 3c + 2d$.

So we have written a as a linear combination of c and d. Similarly, we can do the same for b, obtaining $b = 2c + 3d$.

(b) For any two reals λ and μ, we have

$$\begin{aligned} \lambda \underbrace{a}_{=3c+2d} + \mu \underbrace{b}_{=2c+3d} &= \lambda\,(3c + 2d) + \mu\,(2c + 3d) \\ &= 3\lambda c + 2\lambda d + 2\mu c + 3\mu d \\ &= (3\lambda + 2\mu)\,c + (2\lambda + 3\mu)\,d. \end{aligned}$$

This is a representation of $\lambda a + \mu b$ as a linear combination of c and d. Thus, $\lambda a + \mu b \in \operatorname{span}(c, d)$ for each $\lambda, \mu \in \mathbb{R}$. In other words,

$$\{\lambda a + \mu b \mid \lambda, \mu \in \mathbb{R}\} \subseteq \operatorname{span}(c, d).$$

But the definition of $\operatorname{span}(a, b)$ yields $\operatorname{span}(a, b) = \{\lambda a + \mu b \mid \lambda, \mu \in \mathbb{R}\}$. Hence,

$$\operatorname{span}(a, b) = \{\lambda a + \mu b \mid \lambda, \mu \in \mathbb{R}\} \subseteq \operatorname{span}(c, d).$$

This solves part **(b)**.

(c) Similarly to part **(a)**, we find $c = \frac{3}{5}a + \frac{-2}{5}b$ and $d = \frac{-2}{5}a + \frac{3}{5}b$.

(d) Similarly to part **(b)**, we can represent $\lambda c + \mu d$ as a linear combination of a and b as follows:

$$\lambda c + \mu d = \left(\frac{3}{5}\lambda + \frac{-2}{5}\mu\right) a + \left(\frac{-2}{5}\lambda + \frac{3}{5}\mu\right) b.$$

□

In order to solve Exercise 3, we shall use the following simple fact:

Proposition 4.1. Let $\mathbb{K}$ be a field. Let V be a $\mathbb{K}$-vector space. Let $v_1, v_2, \ldots, v_k$ be some vectors in V. Let $W = \operatorname{span}(v_1, v_2, \ldots, v_k)$ (this is a subspace of V). Let $w \in W$. Then, $\operatorname{span}(v_1, v_2, \ldots, v_k, w) = W$.

(Roughly speaking, Proposition 4.1 says that the span of a list of vectors does not change if we append a new vector to the list, as long as this new vector already lies in the span of the old vectors.)

Proof of Proposition 4.1. We have $w \in W = \operatorname{span}(v_1, v_2, \ldots, v_k)$. In other words, w is a linear combination of $v_1, v_2, \ldots, v_k$ (since $\operatorname{span}(v_1, v_2, \ldots, v_k)$ is the set of all such combinations). In other words, there exist scalars $\alpha_1, \alpha_2, \ldots, \alpha_k \in \mathbb{K}$ such that $w = \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_k v_k$. Consider these $\alpha_1, \alpha_2, \ldots, \alpha_k$.

Now, we shall prove that $\operatorname{span}(v_1, v_2, \ldots, v_k, w) \subseteq W$ and $W \subseteq \operatorname{span}(v_1, v_2, \ldots, v_k, w)$ separately:

- *Proof of* $\operatorname{span}(v_1, v_2, \ldots, v_k, w) \subseteq W$*:*

  Let $p \in \operatorname{span}(v_1, v_2, \ldots, v_k, w)$. Thus, p is a linear combination of $v_1, v_2, \ldots, v_k, w$. In other words, there exist scalars $\beta_1, \beta_2, \ldots, \beta_k, \gamma \in \mathbb{K}$ such that $p = \beta_1 v_1 + \beta_2 v_2 + \cdots + \beta_k v_k + \gamma w$. Consider these $\beta_1, \beta_2, \ldots, \beta_k, \gamma$. We have

  $$\begin{aligned} p &= \beta_1 v_1 + \beta_2 v_2 + \cdots + \beta_k v_k + \gamma \underbrace{w}_{=\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_k v_k} \\ &= \beta_1 v_1 + \beta_2 v_2 + \cdots + \beta_k v_k + \gamma (\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_k v_k) \\ &= \beta_1 v_1 + \beta_2 v_2 + \cdots + \beta_k v_k + \gamma\alpha_1 v_1 + \gamma\alpha_2 v_2 + \cdots + \gamma\alpha_k v_k \\ &= (\beta_1 v_1 + \gamma\alpha_1 v_1) + (\beta_2 v_2 + \gamma\alpha_2 v_2) + \cdots + (\beta_k v_k + \gamma\alpha_k v_k) \\ &= (\beta_1 + \gamma\alpha_1) v_1 + (\beta_2 + \gamma\alpha_2) v_2 + \cdots + (\beta_k + \gamma\alpha_k) v_k. \end{aligned}$$

  Hence, p is a linear combination of $v_1, v_2, \ldots, v_k$. In other words, $p \in \operatorname{span}(v_1, v_2, \ldots, v_k)$. In other words, $p \in W$ (since $W = \operatorname{span}(v_1, v_2, \ldots, v_k)$).

  We thus have showed that every $p \in \operatorname{span}(v_1, v_2, \ldots, v_k, w)$ satisfies $p \in W$. In other words, $\operatorname{span}(v_1, v_2, \ldots, v_k, w) \subseteq W$.

- *Proof of* $W \subseteq \operatorname{span}(v_1, v_2, \ldots, v_k, w)$*:*

  Let $p \in W$. Thus, $p \in W = \operatorname{span}(v_1, v_2, \ldots, v_k)$. In other words, p is a linear combination of $v_1, v_2, \ldots, v_k$. In other words, there exist scalars $\delta_1, \delta_2, \ldots, \delta_k \in \mathbb{K}$ such that $p = \delta_1 v_1 + \delta_2 v_2 + \cdots + \delta_k v_k$. Consider these $\delta_1, \delta_2, \ldots, \delta_k$. We have

  $$\begin{aligned} p &= \delta_1 v_1 + \delta_2 v_2 + \cdots + \delta_k v_k \\ &= \delta_1 v_1 + \delta_2 v_2 + \cdots + \delta_k v_k + \underbrace{\overrightarrow{0}}_{=0w} \\ &= \delta_1 v_1 + \delta_2 v_2 + \cdots + \delta_k v_k + 0w. \end{aligned}$$

  Hence, p is a linear combination of $v_1, v_2, \ldots, v_k, w$. In other words, $p \in \operatorname{span}(v_1, v_2, \ldots, v_k, w)$.

  We thus have showed that every $p \in W$ satisfies $p \in \operatorname{span}(v_1, v_2, \ldots, v_k, w)$. In other words, $W \subseteq \operatorname{span}(v_1, v_2, \ldots, v_k, w)$.

We have now proved the two relations $\operatorname{span}(v_1, v_2, \ldots, v_k, w) \subseteq W$ and $W \subseteq \operatorname{span}(v_1, v_2, \ldots, v_k, w)$. Combining them, we obtain $\operatorname{span}(v_1, v_2, \ldots, v_k, w) = W$. This proves Proposition 4.1.

(You have seen some very similar arguments in class.) $\square$

Solution to Exercise 3. [**Note:** Throughout the solution of this problem, you have a lot of freedom to make choices. Thus, your answers may be completely different from mine and still correct.]

(a) The set K is the set of all solutions of the system of (one) linear equation $\{a + b = c + d$ in the four unknowns a, b, c, d. Solving this system, we find that its solutions have the form

$$\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} s+t-u \\ u \\ t \\ s \end{pmatrix}$$

with three free variables s, t, u. Thus,

$$\begin{aligned} K &= \left\{ \begin{pmatrix} s+t-u \\ u \\ t \\ s \end{pmatrix} \mid s, t, u \in \mathbb{R} \right\} \\ &= \left\{ s \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} + t \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + u \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \mid s, t, u \in \mathbb{R} \right\} \\ &\qquad \left(\text{since } \begin{pmatrix} s+t-u \\ u \\ t \\ s \end{pmatrix} = s \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} + t \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + u \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \right) \\ &= \operatorname{span} \left(\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \right). \end{aligned}$$

Thus, we have written K as a span of three vectors.

(b) For every $a_1, a_2, a_3, a_4 \in \mathbb{R}$, we have

$$\begin{aligned} &(a_1 + a_2, a_2 + a_3, a_3 + a_4, a_4 + a_1)^T \\ &= \begin{pmatrix} a_1 + a_2 \\ a_2 + a_3 \\ a_3 + a_4 \\ a_4 + a_1 \end{pmatrix} = a_1 \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} + a_2 \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + a_3 \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} + a_4 \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}. \end{aligned}$$

Thus, the definition of L rewrites as follows:

$$L = \left\{ a_1 \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} + a_2 \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + a_3 \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} + a_4 \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} \mid a_1, a_2, a_3, a_4 \in \mathbb{R} \right\}$$

$$= \operatorname{span}\left(\begin{pmatrix}1\\0\\0\\1\end{pmatrix},\begin{pmatrix}1\\1\\0\\0\end{pmatrix},\begin{pmatrix}0\\1\\1\\0\end{pmatrix},\begin{pmatrix}0\\0\\1\\1\end{pmatrix}\right). \tag{1}$$

Hence, we have written L as a span of four vectors.

(c) Set $\alpha = \begin{pmatrix}1\\0\\0\\1\end{pmatrix}$, $\beta = \begin{pmatrix}1\\1\\0\\0\end{pmatrix}$, $\gamma = \begin{pmatrix}0\\1\\1\\0\end{pmatrix}$, and $\delta = \begin{pmatrix}0\\0\\1\\1\end{pmatrix}$. Then, (1) rewrites as

$$L = \operatorname{span}(\alpha, \beta, \gamma, \delta). \tag{2}$$

However, it is easy to observe that $\alpha + \gamma = \beta + \delta$. Hence, $\delta = \alpha + \gamma - \beta$. In particular, δ is a linear combination of α, β, γ. Hence, $\delta \in \operatorname{span}(\alpha, \beta, \gamma)$.

Now, set $V = \mathbb{R}^4$; let $v_1, v_2, \ldots, v_k$ be the vectors α, β, γ (so $k = 3$); let $W = \operatorname{span}(\alpha, \beta, \gamma)$; and let $w = \delta$. Then, the condition of Proposition 4.1 is satisfied (because we have $\delta \in \operatorname{span}(\alpha, \beta, \gamma)$). Thus, Proposition 4.1 (applied to $\mathbb{K} = \mathbb{R}$) yields $\operatorname{span}(\alpha, \beta, \gamma, \delta) = \operatorname{span}(\alpha, \beta, \gamma)$. Hence, (2) becomes $L = \operatorname{span}(\alpha, \beta, \gamma, \delta) = \operatorname{span}(\alpha, \beta, \gamma)$. Thus, we have represented L as a span of three vectors. □

Solution to Exercise 4. (I am going to be very detailed here. You don't need to write half as much when solving this kind of problem!)

(a) We proceed using the standard algorithm[1]: We scan the list **a** from left to right. Each time we read an entry of **a**, we check if this entry is a linear combination of the entries before it. If it is, then we remove this entry from **a** and start from scratch with the new (shorter) **a**. If it is not, then we proceed to the next entry. If we have arrived at the end of the list, then our list has no redundant entries, and thus is a basis of $\mathbb{R}^3$.

Let us execute this algorithm step by step:

- We scan the list **a** from left to right. Thus, we begin at its first entry, which is $(1, 2, -1)^T$.

- Is this first entry $(1, 2, -1)^T$ a linear combination of the entries before it? There are no entries before it, and thus the only linear combination of the entries before it is $\overrightarrow{0}$ (because the only linear combination of no vectors is $\overrightarrow{0}$). But our entry $(1, 2, -1)^T$ is not $\overrightarrow{0}$; thus, $(1, 2, -1)^T$ is not a linear combination of the entries before it. Hence, we proceed to the second entry.

- Is this second entry $(1, 1, 0)^T$ a linear combination of the entries before it? There is only one entry before it, namely $(1, 2, -1)^T$. Hence, we are asking whether $(1, 1, 0)^T$ is a linear combination of the vector $(1, 2, -1)^T$. In other words, we are asking whether $(1, 1, 0)^T = \lambda_1 (1, 2, -1)^T$ for some $\lambda_1 \in \mathbb{R}$.

[1] In this algorithm, we treat **a** as a mutable variable.

Equivalently, we want to know whether $\begin{cases} 1 = 1\lambda_1 \\ 1 = 2\lambda_1 \\ 0 = -1\lambda_1 \end{cases}$ for some $\lambda_1 \in \mathbb{R}$ (because the equation $(1,1,0)^T = \lambda_1 (1,2,-1)^T$ is equivalent to the system of equations $\begin{cases} 1 = 1\lambda_1 \\ 1 = 2\lambda_1 \\ 0 = -1\lambda_1 \end{cases}$). In other words, we want to know whether the system $\begin{cases} 1 = 1\lambda_1 \\ 1 = 2\lambda_1 \\ 0 = -1\lambda_1 \end{cases}$ of linear equations (in the unknown λ_1) has a solution. But this question is easy to answer (e.g., by Gaussian elimination), and the answer is "no". Thus, our entry $(1,1,0)^T$ is not a linear combination of the entries before it. Hence, we proceed to the third entry.

- Is this third entry $(0,1,-1)^T$ a linear combination of the entries before it? The entries before it are $(1,2,-1)^T$ and $(1,1,0)^T$. Hence, we are asking whether $(0,1,-1)^T$ is a linear combination of the vectors $(1,2,-1)^T$ and $(1,1,0)^T$. In other words, we are asking whether $(0,1,-1)^T = \lambda_1 (1,2,-1)^T + \lambda_2 (1,1,0)^T$ for some $\lambda_1 \in \mathbb{R}$. Equivalently, we want to know whether $\begin{cases} 0 = 1\lambda_1 + 1\lambda_2 \\ 1 = 2\lambda_1 + 1\lambda_2 \\ -1 = -1\lambda_1 + 0\lambda_2 \end{cases}$ for some $\lambda_1, \lambda_2 \in \mathbb{R}$ (because the equation $(0,1,-1)^T = \lambda_1 (1,2,-1)^T + \lambda_2 (1,1,0)^T$ is equivalent to the system of equations $\begin{cases} 0 = 1\lambda_1 + 1\lambda_2 \\ 1 = 2\lambda_1 + 1\lambda_2 \\ -1 = -1\lambda_1 + 0\lambda_2 \end{cases}$). In other words, we want to know whether the system $\begin{cases} 0 = 1\lambda_1 + 1\lambda_2 \\ 1 = 2\lambda_1 + 1\lambda_2 \\ -1 = -1\lambda_1 + 0\lambda_2 \end{cases}$ of linear equations (in the unknowns λ_1, λ_2) has a solution. But this question is easy to answer (e.g., by Gaussian elimination), and the answer is "yes". Thus, our entry $(0,1,-1)^T$ is a linear combination of the entries before it[2]. Thus, we remove the entry from **a**, and start from scratch with the new (shorter) **a**.

- We scan the new list $\mathbf{a} = \left((1,2,-1)^T, (1,1,0)^T, (1,1,1)^T\right)$ (the result of removing $(0,1,-1)^T$ from the old list **a**) from left to right. Thus, we begin at its first entry, which is $(1,2,-1)^T$.

- Is this first entry $(1,2,-1)^T$ a linear combination of the entries before it? We have already answered this question during our previous scan of the list (since the segment of our list **a** up to its first entry has not changed when

[2]Namely, $(0,1,-1)^T = 1(1,2,-1)^T + (-1)(1,1,0)^T$. But we don't need to know these specifics.

we removed $(0,1,-1)^T$), and thus we already know that the answer is "no". Hence, we proceed to the second entry.

- Is this second entry $(1,1,0)^T$ a linear combination of the entries before it? Again, this is a question we have already answered during our previous scan of the list (since the segment of our list **a** up to its second entry has not changed when we removed $(0,1,-1)^T$), and thus we already know that the answer is "no". Hence, we proceed to the third entry.

- Is this third entry $(1,1,1)^T$ a linear combination of entries before it? The entries before it are $(1,2,-1)^T$ and $(1,1,0)^T$. Hence, we are asking whether $(1,1,1)^T$ is a linear combination of the vectors $(1,2,-1)^T$ and $(1,1,0)^T$. In other words, we are asking whether $(1,1,1)^T = \lambda_1 (1,2,-1)^T + \lambda_2 (1,1,0)^T$ for some $\lambda_1 \in \mathbb{R}$. Equivalently, we want to know whether $\begin{cases} 1 = 1\lambda_1 + 1\lambda_2 \\ 1 = 2\lambda_1 + 1\lambda_2 \\ 1 = -1\lambda_1 + 0\lambda_2 \end{cases}$ for some $\lambda_1, \lambda_2 \in \mathbb{R}$ (because the equation $(1,1,1)^T = \lambda_1 (1,2,-1)^T + \lambda_2 (1,1,0)^T$ is equivalent to the system of equations $\begin{cases} 1 = 1\lambda_1 + 1\lambda_2 \\ 1 = 2\lambda_1 + 1\lambda_2 \\ 1 = -1\lambda_1 + 0\lambda_2 \end{cases}$). In other words, we want to know whether the system $\begin{cases} 1 = 1\lambda_1 + 1\lambda_2 \\ 1 = 2\lambda_1 + 1\lambda_2 \\ 1 = -1\lambda_1 + 0\lambda_2 \end{cases}$ of linear equations (in the unknowns λ_1, λ_2) has a solution. But this question is easy to answer (e.g., by Gaussian elimination), and the answer is "no". Thus, our entry $(1,1,1)^T$ is not a linear combination of the entries before it. Thus, we have arrived at the end of the list.

We have thus ended up with the list $\left((1,2,-1)^T, (1,1,0)^T, (1,1,1)^T\right)$. This list is therefore a basis of $\mathbb{R}^3$ obtained by shrinking our (old) list **a**.

(b) There are various ways to do this. One particularly simple way is the following[3]: We scan the list **a** from left to right. Each time we read an entry of **a**, we check if this entry is a linear combination of the (current) entries of **b**. If it isn't, then we append this entry to **b**. In either case, we proceed to the next entry. By the time we have scanned all entries of **a**, the list **b** has become a basis of $\mathbb{R}^3$. (This is easy to prove[4].)

[3] In this algorithm, we treat **b** as a mutable variable.

[4] *Proof.* Consider the following:

- Every time we append an entry of **a** to the list **b**, the list **b** remains linearly independent (because we append an entry to **b** only if this entry is not a linear combination of the existing entries of **b**; but this guarantees that the linear independence of the list **b** is preserved).

- By the time we have scanned all entries of **a**, the list **b** has the property that each entry of

In order to simplify our life, we use not the original list

$$\mathbf{a} = \left((1,2,-1)^T, (1,1,0)^T, (0,1,-1)^T, (1,1,1)^T \right),$$

but the shorter list

$$\mathbf{a} = \left((1,2,-1)^T, (1,1,0)^T, (1,1,1)^T \right)$$

obtained at the end of the shrinking process in part **(a)** of the problem. Indeed, this shorter list works just as well (it is a basis of $\mathbb{R}^3$ and thus spans $\mathbb{R}^3$), and clearly its elements are elements of the original list $\mathbf{a}$ as well.

Let us now execute our algorithm step by step:

- We scan the list $\mathbf{a}$ from left to right. Thus, we begin at its first entry, which is $(1,2,-1)^T$.

- Is this first entry $(1,2,-1)^T$ a linear combination of the entries of $\mathbf{b}$? The entries of $\mathbf{b}$ are $(-1,0,1)^T$ and $(2,3,4)^T$. Hence, we are asking whether $(1,2,-1)^T$ is a linear combination of the vectors $(-1,0,1)^T$ and $(2,3,4)^T$. In other words, we are asking whether $(1,2,-1)^T = \lambda_1 (-1,0,1)^T + \lambda_2 (2,3,4)^T$ for some $\lambda_1, \lambda_2 \in \mathbb{R}$. Equivalently, we want to know whether $\begin{cases} 1 = (-1)\lambda_1 + 2\lambda_2 \\ 2 = 0\lambda_1 + 3\lambda_2 \\ -1 = 1\lambda_1 + 4\lambda_2 \end{cases}$ for some $\lambda_1, \lambda_2 \in \mathbb{R}$ (because the equation $(1,2,-1)^T = \lambda_1 (-1,0,1)^T + \lambda_2 (2,3,4)^T$ is equivalent to the system of equations $\begin{cases} 1 = (-1)\lambda_1 + 2\lambda_2 \\ 2 = 0\lambda_1 + 3\lambda_2 \\ -1 = 1\lambda_1 + 4\lambda_2 \end{cases}$). In other words, we want to know whether the system $\begin{cases} 1 = (-1)\lambda_1 + 2\lambda_2 \\ 2 = 0\lambda_1 + 3\lambda_2 \\ -1 = 1\lambda_1 + 4\lambda_2 \end{cases}$ of linear equations (in the unknowns λ_1, λ_2) has a solution. But this question is easy to answer (e.g., by Gaussian elimination), and the answer is "no". Thus, our entry $(1,2,-1)^T$ is not a linear combination of the entries of $\mathbf{b}$. Thus, we append this entry to $\mathbf{b}$, so that $\mathbf{b}$ becomes $\left((-1,0,1)^T, (2,3,4)^T, (1,2,-1)^T \right)$. We now proceed to the second entry of $\mathbf{a}$.

$\mathbf{a}$ is a linear combination of the entries of $\mathbf{b}$ (because when we scanned this entry, we have ensured that it became such a linear combination by appending it to $\mathbf{b}$, if it wasn't already one). In other words, every entry of $\mathbf{a}$ belongs to $\operatorname{span}(\mathbf{b})$. Thus, $\operatorname{span}(\mathbf{a}) \subseteq \operatorname{span}(\mathbf{b})$. But since $\mathbf{a}$ spans $\mathbb{R}^3$, we have $\operatorname{span}(\mathbf{a}) = \mathbb{R}^3$, so that $\mathbb{R}^3 = \operatorname{span}(\mathbf{a}) \subseteq \operatorname{span}(\mathbf{b})$ and thus $\operatorname{span}(\mathbf{b}) = \mathbb{R}^3$.

Hence, by the time we have scanned all entries of $\mathbf{a}$, the list $\mathbf{b}$ is linearly independent and satisfies $\operatorname{span}(\mathbf{b}) = \mathbb{R}^3$. In other words, this list $\mathbf{b}$ has become a basis of $\mathbb{R}^3$.

- Is this second entry $(1,1,0)^T$ a linear combination of the entries of $\mathbf{b}$? The entries of $\mathbf{b}$ are $(-1,0,1)^T$, $(2,3,4)^T$ and $(1,2,-1)^T$ (keep in mind that $\mathbf{b}$ has changed in the previous step!). Hence, we are asking whether $(1,1,0)^T$ is a linear combination of the vectors $(-1,0,1)^T$, $(2,3,4)^T$ and $(1,2,-1)^T$. By now, we have seen often enough how to answer such questions (of course, we now have to solve a system of equations in three unknowns $\lambda_1, \lambda_2, \lambda_3$). The answer is "yes". Thus, our entry $(1,1,0)^T$ is a linear combination of the entries of $\mathbf{b}$. Hence, we proceed to the third entry of $\mathbf{a}$ (without adding anything to $\mathbf{b}$). Recall that we have used the shorter list $\mathbf{a} = \left((1,2,-1)^T, (1,1,0)^T, (1,1,1)^T\right)$ as our $\mathbf{a}$, so this third entry is $(1,1,1)^T$.
- Is this third entry $(1,1,1)^T$ a linear combination of the entries of $\mathbf{b}$? The answer is "yes" (found in the same way as many times before). Hence, we arrive at the end of $\mathbf{a}$ (without adding anything to $\mathbf{b}$).

We have thus ended up with the list $\mathbf{b} = \left((-1,0,1)^T, (2,3,4)^T, (1,2,-1)^T\right)$. This list is therefore a basis of $\mathbb{R}^3$ obtained by appending some elements from $\mathbf{a}$ to the (old) list $\mathbf{b}$.

[*Remark:* We could have made our life much easier. In fact, we could have stopped our algorithm immediately after adding $(1,2,-1)^T$ to the list $\mathbf{b}$, because the list $\mathbf{b}$ had become a basis of $\mathbb{R}^3$ at that moment (being a linearly independent list of 3 vectors in $\mathbb{R}^3$).

There are other ways to solve this exercise, and some of them lead to different results. For example, $\left((-1,0,1)^T, (2,3,4)^T, (0,1,-1)^T\right)$ is an equally valid answer to part **(b)**.] □

Solution to Exercise 5. There are many ways to solve this. The most systematic one is to follow the algorithm sketched in Example 1.3.5 in the class notes from 2019-12-04; this algorithm proceeds in three phases:

Phase 1: first, transform A into RREF using row operations;

Phase 2: then, clear out the nonzero entries to the right of the pivots using column operations of the ECO3 kind;

Phase 3: finally, move all pivots as far left as possible using column operations of the ECO1 kind).

I will not show this procedure in detail, but let me remark that the RREF of A is $\begin{pmatrix} \boxed{1} & 0 & \frac{13}{7} & \frac{6}{7} \\ 0 & \boxed{1} & \frac{6}{7} & \frac{13}{7} \\ 0 & 0 & 0 & 0 \end{pmatrix}$ (where the pivots are boxed, as usual), and thus Phase 2 results in $\begin{pmatrix} \boxed{1} & 0 & 0 & 0 \\ 0 & \boxed{1} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$. The pivots here are already in the first two columns, so Phase 3 is not necessary, and we conclude that the rank normal form of A is $\begin{pmatrix} \boxed{1} & 0 & 0 & 0 \\ 0 & \boxed{1} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$. The rank of A is the number of pivots; thus, it equals 2.

[However, let me show one alternative way to find the rank normal form of A, without going through the RREF. Recall that we can do any row operations and any column operations, as long as the final result is in rank normal form. So we can make our job easier by picking the easiest operations to do:

$$\begin{pmatrix} 1 & 6 & 7 & 12 \\ 2 & 5 & 8 & 11 \\ 3 & 4 & 9 & 10 \end{pmatrix} \xrightarrow[\text{to column 3}]{\text{add } -1\cdot\text{column 1}} \begin{pmatrix} 1 & 6 & 6 & 12 \\ 2 & 5 & 6 & 11 \\ 3 & 4 & 6 & 10 \end{pmatrix} \xrightarrow[\text{to column 4}]{\text{add } -1\cdot\text{column 2}} \begin{pmatrix} 1 & 6 & 6 & 6 \\ 2 & 5 & 6 & 6 \\ 3 & 4 & 6 & 6 \end{pmatrix}$$

$$\xrightarrow[\text{to column 4}]{\text{add } -1\cdot\text{column 3}} \begin{pmatrix} 1 & 6 & 6 & 0 \\ 2 & 5 & 6 & 0 \\ 3 & 4 & 6 & 0 \end{pmatrix} \xrightarrow[\text{by } 1/6]{\text{scale column 3}} \begin{pmatrix} 1 & 6 & 1 & 0 \\ 2 & 5 & 1 & 0 \\ 3 & 4 & 1 & 0 \end{pmatrix} \xrightarrow[\text{to column 2}]{\text{add } 1\cdot\text{column 1}} \begin{pmatrix} 1 & 7 & 1 & 0 \\ 2 & 7 & 1 & 0 \\ 3 & 7 & 1 & 0 \end{pmatrix}$$

$$\xrightarrow[\text{by } 1/7]{\text{scale column 2}} \begin{pmatrix} 1 & 1 & 1 & 0 \\ 2 & 1 & 1 & 0 \\ 3 & 1 & 1 & 0 \end{pmatrix} \xrightarrow[\text{to column 3}]{\text{add } -1\cdot\text{column 2}} \begin{pmatrix} 1 & 1 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 3 & 1 & 0 & 0 \end{pmatrix} \xrightarrow[\text{to row 2}]{\text{add } -2\cdot\text{row 1}} \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 3 & 1 & 0 & 0 \end{pmatrix}$$

$$\xrightarrow[\text{to row 3}]{\text{add } -3\cdot\text{row 1}} \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & -2 & 0 & 0 \end{pmatrix} \xrightarrow[\text{to row 3}]{\text{add } -2\cdot\text{row 2}} \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow[\text{to row 1}]{\text{add } 1\cdot\text{row 2}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\xrightarrow[\text{by } -1]{\text{scale row 1}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

The matrix we have obtained is clearly in rank normal form, so it must be the rank normal form of A. Moreover, it has 2 pivots, so the rank of A must be 2.] $\square$