Math 201-003 Fall 2019 (Darij Grinberg): final exam

1. Reminders

1.1. Matrix basics

- For any two matrices A and B, if the product AB is well-defined, then

$$(AB)_{i,j} = A_{i,1}B_{1,j} + A_{i,2}B_{2,j} + \cdots + A_{i,m}B_{m,j} = \operatorname{row}_i A \cdot \operatorname{col}_j B$$

for all indices i and j.

- The transpose A^T of an $n \times m$-matrix A is the $m \times n$-matrix whose entries are $\left(A^T\right)_{i,j} = A_{j,i}$.

1.2. Operations and RREFs

Definition 1.1. The following operations on a matrix A are called **elementary row operations** (for short **EROs**, or just **row operations**):

- **ERO1:** Exchange two rows.
- **ERO2:** Scale a row by a nonzero constant.
- **ERO3:** Add a multiple of one row to another row. (That is, add $\lambda \operatorname{row}_i A$ to $\operatorname{row}_j A$ for some $\lambda \in \mathbb{R}$ and $i \neq j$.)

Definition 1.2. A matrix is **in RREF** if and only if it satisfies the following four conditions:

- **RREF0:** Any zero row (= row full of zeros) is below any nonzero row (= row with at least some nonzero entries).
- **RREF1:** In any nonzero row, the first nonzero entry is equal to 1. This entry is called the *pivot* of the row.
- **RREF2:** The pivot of any nonzero row must be further to the right than the pivot of the previous nonzero row.
- **RREF3:** If a **column** contains a pivot, then all other entries in the column are zero.

The **RREF of a matrix** A is the unique matrix B that is in RREF and can be obtained from A by a sequence of EROs.

Definition 1.3. Let A be an $n \times n$-matrix. Then, the **determinant** $\det A$ of A is defined to be the sum

$$\sum_{\sigma \text{ is a permutation of } [n]} \operatorname{sign}(\sigma) \cdot A_{1,\sigma(1)} A_{2,\sigma(2)} \cdots A_{n,\sigma(n)}.$$

Here, $[n]$ means $\{1, 2, \ldots, n\}$.

Theorem 1.4 (Laplace expansion). Let A be an $n \times n$-matrix. For each $p, q \in [n]$, we let $M_{p,q}$ be the $(n-1) \times (n-1)$-matrix obtained from A by removing row p and column q. Then:

(a) For each $p \in [n]$, we have

$$\det A = \sum_{q=1}^{n} (-1)^{p+q} A_{p,q} \det \left(M_{p,q} \right).$$

(This is called **Laplace expansion along the p-th row**.)

(b) For each $q \in [n]$, we have

$$\det A = \sum_{p=1}^{n} (-1)^{p+q} A_{p,q} \det \left(M_{p,q} \right).$$

(This is called **Laplace expansion along the q-th column**.)

Definition 1.5. Let A be an $n \times n$-matrix. Let λ be a scalar (i.e., a real number).

(a) A **λ-eigenvector of** A means a nonzero vector $v \in \mathbb{R}^n$ such that $Av = \lambda v$.

(b) We say that λ is an **eigenvalue** of A if and only if there exists a λ-eigenvector of A.

(c) The **characteristic polynomial** of A is the polynomial

$$\chi_A(t) = \det(A - tI_n).$$

2. Matrix basics

Exercise 1. Let A be a 3×2-matrix. Which of the following matrices are well-defined, and (if they are) what are their dimensions? (For example, A^T is a well-defined 2×3-matrix, i.e., has dimensions 2×3. But A^2 is not well-defined.)

No justifications are required in this exercise. Just write the dimensions (or "N" for "not well-defined") into the respective box!

(a) $A + A^T$: [ ]

(b) AA^T: [ ]

(c) $A^T A$: [ ]

(d) $AA^T A$: [ ]

(e) $AA^T AA^T$: [ ]

Solution to Exercise 1. Here are the answers:

(a) Not well-defined (since we cannot add a 3×2-matrix with a 2×3-matrix).

(b) Well-defined and has dimensions 3×3.

(c) Well-defined and has dimensions 2×2.

(d) Well-defined and has dimensions 3×2.

(e) Well-defined and has dimensions 3×3.

[All four parts **(b)**, **(c)**, **(d)** and **(e)** follow from the same principle: The product XY of a $p \times q$-matrix X with an $r \times s$-matrix Y is well-defined if and only if $q = r$; furthermore, in this case, it is a $p \times s$-matrix. Parts **(c)** and **(d)** follow directly from this principle; parts **(d)** and **(e)** require applying it twice or thrice.] □

3. RREF

Exercise 2. **(a)** Let A_3 be the 3×3-matrix $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 1 \end{pmatrix}$. Find its RREF.

(b) Let A_5 be the 5×5-matrix $\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 2 & 1 \\ 1 & 2 & 3 & 2 & 1 \\ 1 & 2 & 2 & 2 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}$. Find its RREF.

[**Note:** The pattern in A_3 and in A_5 is exactly what it looks like: 1's along the border of the matrix, 2's one step further inward, etc.]

Solution to Exercise 2. **(a)** Let us bring A_3 into RREF using [Strickland, Method 6.4] (with the standard allowance for not needlessly freezing rows when the matrix is already in RREF):

$$A_3 = \begin{pmatrix} \boxed{1} & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 1 \end{pmatrix} \xrightarrow{\text{add } (-1)\cdot\text{row } 1 \text{ to row } 2} \begin{pmatrix} \boxed{1} & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$
$$\xrightarrow{\text{add } (-1)\cdot\text{row } 1 \text{ to row } 3} \begin{pmatrix} \boxed{1} & 1 & 1 \\ 0 & \boxed{1} & 0 \\ 0 & 0 & 0 \end{pmatrix} \xrightarrow{\text{add } (-1)\cdot\text{row } 2 \text{ to row } 1} \begin{pmatrix} \boxed{1} & 0 & 1 \\ 0 & \boxed{1} & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The resulting matrix is the RREF of A_3.

(b) Let us bring A_5 into RREF using [Strickland, Method 6.4] (with the standard allowance for not needlessly freezing rows when the matrix is already in RREF):

$$A_5 = \begin{pmatrix} \boxed{1} & 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 2 & 1 \\ 1 & 2 & 3 & 2 & 1 \\ 1 & 2 & 2 & 2 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix} \xrightarrow{\text{add } (-1)\cdot\text{row } 1 \text{ to row } 2} \begin{pmatrix} \boxed{1} & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 \\ 1 & 2 & 3 & 2 & 1 \\ 1 & 2 & 2 & 2 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$
$$\xrightarrow{\text{add } (-1)\cdot\text{row } 1 \text{ to row } 3} \begin{pmatrix} \boxed{1} & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 \\ 1 & 2 & 2 & 2 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix} \xrightarrow{\text{add } (-1)\cdot\text{row } 1 \text{ to row } 4} \begin{pmatrix} \boxed{1} & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

$$\overset{\text{add } (-1)\cdot\text{row 1 to row 5}}{\longrightarrow} \begin{pmatrix} \boxed{1} & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \overset{\text{freeze row 1}}{\longrightarrow} \begin{pmatrix} \boxed{1} & 1 & 1 & 1 & 1 & \leftarrow \text{frozen} \\ 0 & \boxed{1} & 1 & 1 & 0 & \\ 0 & 1 & 2 & 1 & 0 & \\ 0 & 1 & 1 & 1 & 0 & \\ 0 & 0 & 0 & 0 & 0 & \end{pmatrix}$$

$$\overset{\text{add } (-1)\cdot\text{row 1 to row 2}}{\longrightarrow} \begin{pmatrix} \boxed{1} & 1 & 1 & 1 & 1 & \leftarrow \text{frozen} \\ 0 & \boxed{1} & 1 & 1 & 0 & \\ 0 & 0 & 1 & 0 & 0 & \\ 0 & 1 & 1 & 1 & 0 & \\ 0 & 0 & 0 & 0 & 0 & \end{pmatrix}$$

$$\overset{\text{add } (-1)\cdot\text{row 1 to row 3}}{\longrightarrow} \begin{pmatrix} \boxed{1} & 1 & 1 & 1 & 1 & \leftarrow \text{frozen} \\ 0 & \boxed{1} & 1 & 1 & 0 & \\ 0 & 0 & \boxed{1} & 0 & 0 & \\ 0 & 0 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 0 & 0 & \end{pmatrix}$$

$$\overset{\text{add } (-1)\cdot\text{row 2 to row 1}}{\longrightarrow} \begin{pmatrix} \boxed{1} & 1 & 1 & 1 & 1 & \leftarrow \text{frozen} \\ 0 & \boxed{1} & 0 & 1 & 0 & \\ 0 & 0 & \boxed{1} & 0 & 0 & \\ 0 & 0 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 0 & 0 & \end{pmatrix}$$

$$\overset{\text{unfreeze row 1}}{\longrightarrow} \begin{pmatrix} \boxed{1} & 1 & 1 & 1 & 1 \\ 0 & \boxed{1} & 0 & 1 & 0 \\ 0 & 0 & \boxed{1} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \overset{\text{add } (-1)\cdot\text{row 2 to row 1}}{\longrightarrow} \begin{pmatrix} \boxed{1} & 0 & 1 & 0 & 1 \\ 0 & \boxed{1} & 0 & 1 & 0 \\ 0 & 0 & \boxed{1} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\overset{\text{add } (-1)\cdot\text{row 3 to row 1}}{\longrightarrow} \begin{pmatrix} \boxed{1} & 0 & 0 & 0 & 1 \\ 0 & \boxed{1} & 0 & 1 & 0 \\ 0 & 0 & \boxed{1} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

The resulting matrix is the RREF of A_5.

[*Remark:* Similarly, for any $n \in \mathbb{N}$, you can define the $n \times n$-matrix A_n whose (i,j)-th entry is "the distance from cell (i,j) to the boundary of the matrix" (i.e., the number $\min\{i, j, n+1-i, n+1-j\}$). Its RREF looks exactly like the RREFs of A_3 and A_5 we found above: The entries in cells (i,i) and $(i, n+1-i)$ for all $i \in \{1, 2, \ldots, \lceil n/2 \rceil\}$ equal 1 (where $\lceil n/2 \rceil$ denotes the smallest integer that is $n/2$), and all remaining entries equal 0.] $\square$

4. Linear independence

Exercise 3. Consider the four matrices

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}, \qquad C = \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}, \qquad D = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}.$$

(a) Are these matrices A, B, C, D linearly independent?
(b) Do these matrices A, B, C, D span the vector space $\mathbb{R}^{2\times 2}$?

Solution to Exercise 3. **(a)** Yes.

Proof. Let $aA + bB + cC + dD = 0_{2\times 2}$ be a relation between A, B, C, D. We shall prove that this relation is trivial, i.e., that we have $a = b = c = d = 0$.

Indeed, the definitions of A, B, C, D yield

$$\begin{aligned} aA + bB + cC + dD &= a\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} + b\begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix} + c\begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix} + d\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \\ &= \begin{pmatrix} 2a+b+c+d & a+2b+c+d \\ a+b+2c+d & a+b+c+2d \end{pmatrix}. \end{aligned}$$

Comparing this with $aA + bB + cC + dD = 0_{2\times 2}$, we find

$$\begin{pmatrix} 2a+b+c+d & a+2b+c+d \\ a+b+2c+d & a+b+c+2d \end{pmatrix} = 0_{2\times 2} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

In other words, the four equalities

$$\begin{cases} 2a+b+c+d = 0 \\ a+2b+c+d = 0 \\ a+b+2c+d = 0 \\ a+b+c+2d = 0 \end{cases}$$

all hold. We can treat them as a system of linear equations in a, b, c, d and solve it using Gaussian elimination, but we can also proceed in a simpler way: Subtracting the equality $a + 2b + c + d = 0$ from the equality $2a + b + c + d = 0$, we obtain $(2a + b + c + d) - (a + 2b + c + d) = 0$. This rewrites as $a - b = 0$. In other words, $b = a$. Similarly, $c = a$ and $d = a$. Now, the equality $2a + b + c + d = 0$ yields $0 = 2a + \underbrace{b}_{=a} + \underbrace{c}_{=a} + \underbrace{d}_{=a} = 2a + a + a + a = 5a$, so that $5a = 0$ and therefore $a = 0$. Similarly, $b = 0$ and $c = 0$ and $d = 0$. Hence, $a = b = c = d = 0$. Thus, the relation $aA + bB + cC + dD = 0_{2\times 2}$ is trivial.

We thus have shown that each any relation $aA + bB + cC + dD = 0_{2\times 2}$ between A, B, C, D must be trivial. In other words, A, B, C, D are linearly independent.

(b) Yes.

We shall give two proofs of this fact:

First proof. The list (A, B, C, D) is $\mathbb{R}$-linearly independent (by part **(a)** of this exercise), i.e., is an independent list of $\mathbb{R}^{2\times 2}$ (using the terminology from our class notes from 2019-12-04).

The $\mathbb{R}$-vector space $\mathbb{R}^{2\times 2}$ has dimension $2 \cdot 2 = 4$ (since the list

$$\left(\underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}}_{\substack{\text{the four matrices, each of which has a 1 in} \\ \text{some position, and zeroes everywhere else}}} \right)$$

is a basis of this space). In other words, $\dim\left(\mathbb{R}^{2\times 2}\right) = 4$. Hence, Theorem 1.1.7 **(h)** from our class notes from 2019-12-04 (applied to $\mathbb{K} = \mathbb{R}$ and $V = \mathbb{R}^{2\times 2}$) shows that any independent list of $\mathbb{R}^{2\times 2}$ that has length 4 must be a basis of $\mathbb{R}^{2\times 2}$. Applying this to the independent list (A, B, C, D), we conclude that (A, B, C, D) must be a basis of $\mathbb{R}^{2\times 2}$ (since (A, B, C, D) is an independent list of $\mathbb{R}^{2\times 2}$). Thus, in particular, the list (A, B, C, D) spans $\mathbb{R}^{2\times 2}$. In other words, the matrices A, B, C, D span the vector space $\mathbb{R}^{2\times 2}$.

Second proof. Any matrix in $\mathbb{R}^{2\times 2}$ can be written as a linear combination of A, B, C, D: Namely, if the matrix is $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then it can be written as

$$\frac{4a - b - c - d}{5} A + \frac{4b - c - d - a}{5} B + \frac{4c - d - a - b}{5} C + \frac{4d - a - b - c}{5} D.$$

(You can find this by solving the equation $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = xA + yB + zC + wD$ for x, y, z, w.) $\square$

5. Inverses

Exercise 4. **(a)** Is the matrix $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ invertible?

(b) Is the matrix $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 1 \end{pmatrix}$ invertible?

Solution to Exercise 4. **(a) Yes.**

Proof. One way to see this is by constructing its inverse:

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}^{-1} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & -1 & 0 \end{pmatrix}.$$

Another way is to observe that its determinant is $-1 \neq 0$, and thus it is invertible (by the implication **(l)** $\Longrightarrow$ **(k)** in Theorem 2.1.3 in our class notes from 2019-11-04).

(b) No.

Proof. The columns of this matrix are linearly dependent (since its first and third columns are identical). Hence, the matrix is not invertible (by the implication **(b')** $\Longrightarrow$ **(k')** in Theorem 2.1.4 in our class notes from 2019-11-04). □

6. Determinants

Exercise 5. **(a)** Compute $\det \begin{pmatrix} a & 0 & 0 & 1 \\ 0 & b & 0 & 1 \\ 0 & 0 & c & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$, where a, b, c, d are reals.

[**Hint:** The result will have the form $x - y - z - w$, where x, y, z, w are four simple expressions.]

(b) Compute $\det \begin{pmatrix} a & b & c & d \\ b & b & c & d \\ c & c & c & d \\ d & d & d & d \end{pmatrix}$, where a, b, c, d are reals. Write the result as a product of four linear expressions in a, b, c, d.

[**Hint:** In part **(b)**, don't waste your time expanding the determinant by its definition; the result will not be easy to factor. Instead, try simplifying the determinant by appropriate transformations.]

[**Note:** The matrix in **(b)** has an a in its top-left cell; b's in the three cells bordering it; c's in the five cells bordering them; d's everywhere else.]

Solution to Exercise 5. **(a)** Let a, b, c, d be reals. Laplace expansion along the first row yields

$$\det \begin{pmatrix} a & 0 & 0 & 1 \\ 0 & b & 0 & 1 \\ 0 & 0 & c & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} = \underbrace{(-1)^{1+1}}_{=1} a \det \begin{pmatrix} b & 0 & 1 \\ 0 & c & 1 \\ 1 & 1 & 1 \end{pmatrix} + \underbrace{(-1)^{1+2} 0 \det \begin{pmatrix} 0 & 0 & 1 \\ 0 & c & 1 \\ 1 & 1 & 1 \end{pmatrix}}_{=0}$$

$$+ \underbrace{(-1)^{1+3} 0 \det \begin{pmatrix} 0 & b & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}}_{=0} + \underbrace{(-1)^{1+4}}_{=-1} 1 \det \begin{pmatrix} 0 & b & 0 \\ 0 & 0 & c \\ 1 & 1 & 1 \end{pmatrix}$$

$$= a \det \begin{pmatrix} b & 0 & 1 \\ 0 & c & 1 \\ 1 & 1 & 1 \end{pmatrix} - \det \begin{pmatrix} 0 & b & 0 \\ 0 & 0 & c \\ 1 & 1 & 1 \end{pmatrix}.$$

The two determinants on the right hand side of this equality are easily computed (either again by Laplace expansion along the first row, or by the fairly manageable[1] explicit definition of a determinant):

$$\det \begin{pmatrix} b & 0 & 1 \\ 0 & c & 1 \\ 1 & 1 & 1 \end{pmatrix} = bc - b - c; \qquad \det \begin{pmatrix} 0 & b & 0 \\ 0 & 0 & c \\ 1 & 1 & 1 \end{pmatrix} = bc.$$

[1] in the case of a 3×3-matrix

Thus, we can conclude our above computation as follows:

$$\det\begin{pmatrix} a & 0 & 0 & 1 \\ 0 & b & 0 & 1 \\ 0 & 0 & c & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} = a \underbrace{\det\begin{pmatrix} b & 0 & 1 \\ 0 & c & 1 \\ 1 & 1 & 1 \end{pmatrix}}_{=bc-b-c} - \underbrace{\det\begin{pmatrix} 0 & b & 0 \\ 0 & 0 & c \\ 1 & 1 & 1 \end{pmatrix}}_{=bc}$$
$$= a\left(bc-b-c\right) - bc = abc - ab - ac - bc.$$

[*Remark:* An alternative solution can be obtained just by applying the definition of a determinant and analyzing which of the 24 permutations of $[4]$ will give rise to nonzero terms. This is not as painful as it sounds; only 4 permutations give rise to nonzero terms. This method can be generalized to the case of an $n \times n$-matrix, resulting in the formula

$$\det\begin{pmatrix} a_1 & 0 & \cdots & 0 & 1 \\ 0 & a_2 & \cdots & 0 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & a_{n-1} & 1 \\ 1 & 1 & \cdots & 1 & 1 \end{pmatrix}$$
$$= a_1 a_2 \cdots a_{n-1}$$
$$- \left(\text{the sum of all products of } n-2 \text{ of the } n-1 \text{ numbers } a_1 a_2 \cdots a_{n-1}\right).$$

If $a_1, a_2, \ldots, a_{n-1}$ are nonzero, then this can be simplified to

$$\det\begin{pmatrix} a_1 & 0 & \cdots & 0 & 1 \\ 0 & a_2 & \cdots & 0 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & a_{n-1} & 1 \\ 1 & 1 & \cdots & 1 & 1 \end{pmatrix}$$
$$= a_1 a_2 \cdots a_{n-1} - \left(\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_{n-1}}\right) a_1 a_2 \cdots a_{n-1}.$$

For an even more general statement, see [18s-hw4s, Exercise 6 **(b)**].]

(b) Let a, b, c, d be reals. Let us recall some properties of determinants:

Property 1: Let A be an $n \times n$-matrix, and let p and q be two distinct elements of $[n]$. If we add $\lambda \cdot \operatorname{row}_p A$ to the q-th row of A, then $\det A$ does not change. (This is Corollary 1.2.5 in the class notes from 2019-10-30.)

Property 2: Let A be an $n \times n$-matrix, and let p and q be two distinct elements of $[n]$. If we subtract $\operatorname{row}_p A$ from the q-th row of A, then $\det A$ does not change. (This follows from Property 1 (applied to $\lambda = -1$), because adding $(-1) \cdot \operatorname{row}_p A$ to the q-th row of A is the same as subtracting $\operatorname{row}_p A$ from the q-th row of A.)

Property 3: If an $n \times n$-matrix A is triangular (i.e., upper-triangular or lower-triangular), then its determinant is the product of its diagonal elements:

$$\det A = A_{1,1} A_{2,2} \cdots A_{n,n}.$$

(This is Theorem 1.1.2 in the class notes from 2019-10-30.)

Now,

$$\begin{aligned}
&\det \begin{pmatrix} a & b & c & d \\ b & b & c & d \\ c & c & c & d \\ d & d & d & d \end{pmatrix} \\
&= \det \begin{pmatrix} a-b & 0 & 0 & 0 \\ b & b & c & d \\ c & c & c & d \\ d & d & d & d \end{pmatrix} \\
&\qquad \left(\begin{array}{c} \text{here, we have subtracted the 2-nd row of our matrix} \\ \text{from the 1-st row; this did not change the determinant} \\ \text{(by Property 2)} \end{array} \right) \\
&= \det \begin{pmatrix} a-b & 0 & 0 & 0 \\ b-c & b-c & 0 & 0 \\ c & c & c & d \\ d & d & d & d \end{pmatrix} \\
&\qquad \left(\begin{array}{c} \text{here, we have subtracted the 3-rd row of our matrix} \\ \text{from the 3-rd row; this did not change the determinant} \\ \text{(by Property 2)} \end{array} \right) \\
&= \det \begin{pmatrix} a-b & 0 & 0 & 0 \\ b-c & b-c & 0 & 0 \\ c-d & c-d & c-d & 0 \\ d & d & d & d \end{pmatrix} \\
&\qquad \left(\begin{array}{c} \text{here, we have subtracted the 4-th row of our matrix} \\ \text{from the 3-rd row; this did not change the determinant} \\ \text{(by Property 2)} \end{array} \right) \\
&= (a-b)(b-c)(c-d)\, d
\end{aligned}$$

(by Property 3, since our matrix is lower-triangular and its diagonal entries are $a-b, b-c, c-d, d$).

[*Remark:* See [Grinbe15, Remark 6.16] for a generalization of this result to $n \times n$-matrices (and for a different proof).] □

7. Eigenvalues and eigenvectors

Exercise 6. Diagonalize the matrix $A := \begin{pmatrix} 1 & 1 \\ 0 & 5 \end{pmatrix}$.

Solution to Exercise 6. The characteristic polynomial of A is

$$\chi_A(t) = \det(A - tI_2) = \det \begin{pmatrix} 1-t & 1 \\ 0 & 5-t \end{pmatrix} = (1-t)(5-t)$$

(since the determinant of an upper-triangular matrix equals the product of its diagonal entries). Hence, the roots of $\chi_A(t)$ are 1 and 5.

Recall that the eigenvalues of A are the roots of $\chi_A(t)$ (by Proposition 2.1.7 in the class notes from 2019-11-04). But the roots of $\chi_A(t)$ are 1 and 5. Hence, the eigenvalues of A are 1 and 5.

Next, we will find the eigenvectors. This is an easy matter of solving systems of linear equations:

- The 1-eigenvectors of A are the nonzero vectors $v \in \mathbb{R}^2$ satisfying $Av = 1v$. In other words, they are the nonzero vectors $\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2$ such that $A \begin{pmatrix} x \\ y \end{pmatrix} = 1 \begin{pmatrix} x \\ y \end{pmatrix}$. This is a system of 2 linear equations in the unknowns x, y; solving it by Gaussian elimination, we obtain $\{y = 0$ (where x is a free variable). Thus, they are the nonzero scalar multiples of the vector $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$.
- Likewise, the 5-eigenvectors of A are the nonzero scalar multiples of the vector $\begin{pmatrix} 1 \\ 4 \end{pmatrix}$. (Or, equivalently, of the vector $\begin{pmatrix} 1/4 \\ 1 \end{pmatrix}$, but I like my entries integer if possible.)

Finally, let us diagonalize A using the eigenvectors we found. We label our two eigenvalues as $\lambda_1 = 1$ and $\lambda_2 = 5$, and we label the corresponding eigenvectors as $u_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $u_2 = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$. Then, the pair (u_1, u_2) is a basis of $\mathbb{R}^2$ that consists of eigenvectors of A, and λ_1, λ_2 are the corresponding eigenvalues. Hence, we can find a diagonalization of A using Proposition 1.2.3 **(a)** in the class notes from 2019-11-11: We set

$$U = [u_1 \mid u_2] = \begin{pmatrix} 1 & 1 \\ 0 & 4 \end{pmatrix} \qquad \text{and}$$

$$D = \operatorname{diag}(\lambda_1, \lambda_2) = \operatorname{diag}(1, 5) = \begin{pmatrix} 1 & 0 \\ 0 & 5 \end{pmatrix}.$$

The pair U, D is a diagonalization of A (that is, U is invertible, and D is diagonal, and we have $A = UDU^{-1}$). $\square$

8. General

An $m \times n$-matrix A is said to be **left-invertible** if it has a left inverse (i.e., if there exists an $n \times m$-matrix B such that $BA = I_n$).

Exercise 7. True or false? **No justifications are required in this exercise. Just write Y or N into the respective box!**

(a) [ ] If a 3×3-matrix has at most 2 nonzero entries, then its determinant is 0.

(b) [ ] If all entries of an 3×3-matrix are nonzero, then its determinant is nonzero.

(c) [ ] The product of two invertible matrices is invertible (if this product is well-defined).

(d) [ ] The product of two left-invertible matrices is left-invertible (if this product is well-defined).

(e) [ ] If v and w are two eigenvectors of a 2×2-matrix A, then $v + w$ is an eigenvector of A as well.

(f) [ ] If A and B are two 2×2-matrices, then $\det(A + B) = \det A + \det B$.

(g) [ ] If A and B are two 2×2-matrices, then $\det(AB) = \det A \cdot \det B$.

(h) [ ] If two matrices A and B have the same determinant, then they have the same characteristic polynomial.

(i) [ ] The eigenvalues of a matrix A are also the eigenvalues of A^T.

(j) [ ] The eigenvectors of a matrix A are also the eigenvectors of A^T.

Solution to Exercise 7. **(a) YES.**

Proof. Let A be a 3×3-matrix that has at most 2 nonzero entries. We must prove that $\det A = 0$.

The matrix A has 3 rows. Thus, if each row of A had at least one nonzero entry, then A would have at least 3 nonzero entries, which would contradict our assumption that A has at most 2 nonzero entries. Hence, not every row of A has at least one nonzero entry. In other words, at least one row of A must have only zero entries. In other words, A has a zero row. Thus, Corollary 1.2.1 in the class notes from 2019-10-30 shows that $\det A = 0$.

(b) NO.

Proof. All entries of the 3×3-matrix $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$ are nonzero, yet this matrix has determinant 0.

(c) YES.

Proof. Let A_1 and A_2 be two invertible matrices such that the product A_1A_2 is well-defined. We must prove that A_1A_2 is invertible.

We have assumed that A_1 is invertible. In other words, A_1 has an inverse. Let B_1 be this inverse. Recalling the definition of "inverse", we see that this entails $A_1B_1 = I$ and $B_1A_1 = I$ (where the "I"s are identity matrices of appropriate sizes).

We have assumed that A_2 is invertible. In other words, A_2 has an inverse. Let B_2 be this inverse. Recalling the definition of "inverse", we see that this entails $A_2B_2 = I$ and $B_2A_2 = I$ (where the "I"s are identity matrices of appropriate sizes).

Now,

$$(A_1A_2)(B_2B_1) = A_1 \underbrace{A_2B_2}_{=I} B_1 = A_1 \underbrace{IB_1}_{=B_1} = A_1B_1 = I \qquad \text{and}$$
$$(B_2B_1)(A_1A_2) = B_2 \underbrace{B_1A_1}_{=I} A_2 = B_2 \underbrace{IA_2}_{=A_2} = B_2A_2 = I.$$

These two equalities show that B_2B_1 is an inverse of A_1A_2 (by the definition of "inverse"). Thus, the matrix A_1A_2 has an inverse, i.e., is invertible.

(d) YES.

Proof. Let A_1 and A_2 be two left-invertible matrices such that the product A_1A_2 is well-defined. We must prove that A_1A_2 is left-invertible.

We have assumed that A_1 is left-invertible. In other words, A_1 has a left inverse. Let B_1 be this left inverse. Recalling the definition of "left inverse", we see that this entails $B_1A_1 = I$ (where the "I" is an identity matrix of appropriate size).

We have assumed that A_2 is left-invertible. In other words, A_2 has a left inverse. Let B_2 be this left inverse. Recalling the definition of "left inverse", we see that this entails $B_2A_2 = I$ (where the "I" is an identity matrix of appropriate size).

Now,
$$(B_2B_1)(A_1A_2) = B_2 \underbrace{B_1A_1}_{=I} A_2 = B_2 \underbrace{IA_2}_{=A_2} = B_2A_2 = I.$$

This shows that B_2B_1 is a left inverse of A_1A_2 (by the definition of "left inverse"). Thus, the matrix A_1A_2 has a left inverse, i.e., is left-invertible.

[*Remark:* Clearly, an analogous argument can be used to prove the analogous fact about right-invertible matrices.]

(e) NO.

Proof. For a simple example, let us pick $A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$. Then, both $v := \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $w := \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ are eigenvectors of A (for eigenvalues 1 and 2, respectively), but $v + w$ is not.

[*Remark:* The claim can be salvaged: If v and w are two eigenvectors of A for **the same** eigenvalue, then $v + w$ is either the zero vector or an eigenvector of A for the same eigenvalue. In other words, if $\lambda \in \mathbb{R}$ and if $A \in \mathbb{R}^{n \times n}$, then the sum of any two λ-eigenvectors of A is either the zero vector or a λ-eigenvector of A again.]

(f) NO.

Proof. One of the simplest counterexamples is $A = I_2$ and $B = I_2$. These satisfy $\det(A+B) = 4$ but $\underbrace{\det A}_{=1} + \underbrace{\det B}_{=1} = 2$.

(g) YES.

Proof. This was Exercise 1 on homework set #3. It is also a particular case of the fact that $\det(AB) = \det A \cdot \det B$ whenever A and B are two square matrices of the same size. (This is Theorem 1.5.1 in the class notes from 2019-10-30, where I give two references to proofs of this fact.)

(h) NO.

Proof. For example, the matrices $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ and $\begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix}$ have the same determinant (namely, 4), but different characteristic polynomials (namely, $(t-2)^2$ versus $(t-4)(t-1)$).

(i) YES.

Proof. Let A be an $n \times n$-matrix. We must prove that the eigenvalues of A are the eigenvalues of A^T.

It suffices to show that the characteristic polynomial of A is the characteristic polynomial of A^T (since the eigenvalues of a matrix are the roots of its characteristic polynomial[2]).

[2] by Proposition 2.1.7 in the class notes from 2019-11-04

The matrix tI_n is a diagonal matrix (with t's on the diagonal), and thus equals its own transpose. In other words, $(tI_n)^T = tI_n$.

It is easy to see that $(C - D)^T = C^T - D^T$ for any two $n \times n$-matrices C and D. Thus, $(A - tI_n)^T = A^T - \underbrace{(tI_n)^T}_{=tI_n} = A^T - tI_n$. Taking determinants on both sides of this equality, we find $\det\left((A - tI_n)^T\right) = \det\left(A^T - tI_n\right)$.

But Theorem 1.3.1 in the class notes from 2019-10-23 shows that $\det\left(C^T\right) = \det C$ for any $n \times n$-matrix C. Applying this to $C = A - tI_n$, we obtain $\det\left((A - tI_n)^T\right) = \det(A - tI_n)$. Hence,

$$\det(A - tI_n) = \det\left((A - tI_n)^T\right) = \det\left(A^T - tI_n\right).$$

The left hand side of this equality is $\chi_A(t)$ (since this is how the characteristic polynomial χ_A is defined), whereas the right hand side is $\chi_{A^T}(t)$ (since this is how the characteristic polynomial χ_{A^T} is defined). Hence, this equality rewrites as $\chi_A(t) = \chi_{A^T}(t)$. In other words, the characteristic polynomial of A is the characteristic polynomial of A^T. This completes our proof.

(j) NO.

Proof. For example, $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is an eigenvector of the matrix $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, but not of its transpose. □

References

[18s-hw4s] Darij Grinberg, *UMN Fall 2018 Math 4707 homework set #4 with solutions*, `http://www.cip.ifi.lmu.de/~grinberg/t/18s/hw4s.pdf`

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The numbering of theorems and formulas in this link might shift when the project gets updated; for a "frozen" version whose numbering is guaranteed to match that in the citations above, see `https://github.com/darijgr/detnotes/releases/tag/2019-01-10` .

[Strickland] Neil Strickland, *MAS201 Linear Mathematics for Applications*, lecture notes, 28 September 2013.
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