PROBLEMS FOR MAS201 (LINEAR MATHEMATICS FOR APPLICATIONS)

1. Lecture 1

Exercise 1. Calculate AB, where

$$A = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 3 & 2 & 0 & 0 \\ 3 & 6 & 2 & 0 \\ 3 & 6 & 3 & 2 \end{bmatrix} \qquad B = \begin{bmatrix} 4 & 2 & 2 & 2 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 4 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Solution:

$$AB = \begin{bmatrix} 8 & 4 & 4 & 4 \\ 12 & 8 & 4 & 4 \\ 12 & 12 & 8 & 4 \\ 12 & 12 & 12 & 8 \end{bmatrix}.$$

□

Exercise 2. Consider the following matrices:

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 10 \\ 100 & 1000 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 0 \\ 11 & 0 \\ 111 & 0 \end{bmatrix}.$$

For each of the following products, either evaluate the product or explain why it is undefined:

$$A^2 \qquad AB \qquad AC \qquad BA \qquad B^2 \qquad BC \qquad CA \qquad CB \qquad C^2$$

Solution: The products that are defined are as follows:

$$BA = \begin{bmatrix} 41 & 32 & 23 & 14 \\ 4100 & 3200 & 2300 & 1400 \end{bmatrix}$$

$$B^2 = \begin{bmatrix} 1001 & 10010 \\ 100100 & 1001000 \end{bmatrix}$$

$$CA = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 11 & 22 & 33 & 44 \\ 111 & 222 & 333 & 444 \end{bmatrix}$$

$$CB = \begin{bmatrix} 1 & 10 \\ 11 & 110 \\ 111 & 1110 \end{bmatrix}.$$

The other products are undefined. For example, A is a 2×4 matrix (with 4 columns) and B is a 2×2 matrix (with 2 rows). As the numer of columns in A is different from the number of rows in B, we cannot define the product AB. □

Exercise 3. Find examples as follows.

(a) Matrices A and B such that AB is defined but BA is not.
(b) Matrices C and D such that CD and DC are both defined but have different sizes.
(c) Matrices E and F such that EF and FE are both defined and have the same size but are not equal.
(d) Matrices G and H such that GH and HG are both defined and have the same size and are equal.

Solution: In each case there are many possible examples. We will give a selection.

(a) Here A must be an $m \times n$ matrix and B must be an $n \times p$ matrix where m and p are different. We could take $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ (a 2×2 matrix) and $B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$ (a 2×3 matrix). The entries in these matrices are not really relevant, only the shape matters. We could therefore simplify things by taking $A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. For an even more minimalist example, we could take $A = \begin{bmatrix} 0 \end{bmatrix}$ (a 1×1 matrix) and $B = \begin{bmatrix} 0 & 0 \end{bmatrix}$ (a 1×2 matrix).

(b) Here C must be an $m \times n$ matrix and D must be an $n \times m$ matrix for some integers m and n with $m \neq n$. For a realistic example, we can take

$$C = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \end{bmatrix} \qquad D = \begin{bmatrix} 3 & 4 \\ 3 & 4 \\ 3 & 4 \end{bmatrix}$$

giving

$$CD = \begin{bmatrix} 9 & 12 \\ 18 & 24 \end{bmatrix} \qquad DC = \begin{bmatrix} 11 & 11 & 11 \\ 11 & 11 & 11 \\ 11 & 11 & 11 \end{bmatrix}.$$

For a minimalist example we can take

$$C = \begin{bmatrix} 0 & 0 \end{bmatrix} \qquad D = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad CD = \begin{bmatrix} 0 \end{bmatrix} \qquad DC = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

(c) Here E and F must be square matrices of shape $n \times n$ for some $n > 1$. If we choose a pair of 2×2 matrices at random then it will usually work. For example, we could have

$$E = \begin{bmatrix} 1 & 5 \\ 3 & 2 \end{bmatrix} \qquad F = \begin{bmatrix} 3 & 1 \\ 4 & 6 \end{bmatrix} \qquad EF = \begin{bmatrix} 23 & 31 \\ 17 & 15 \end{bmatrix} \qquad FE = \begin{bmatrix} 6 & 17 \\ 22 & 32 \end{bmatrix}.$$

For a minimal example, we have

$$E = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \qquad F = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad EF = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad FE = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

(d) Here G and H must be square matrices of the same size, say $n \times n$. We can take G to be the zero matrix and H to be any $n \times n$ matrix, and then we have $GH = 0 = HG$, so this gives an example. Alternatively, we can take G to be the identity matrix I_n and H to be any $n \times n$ matrix, and then we have $GH = H = HG$, so this gives another example. Yet another possibility is to let H be any $n \times n$ matrix and then take $G = H$, so that $GH = HG = H^2$. For a minimal example, we can take $n = 1$ and $G = H = \begin{bmatrix} 0 \end{bmatrix}$.

□

Exercise 4. Find a nonzero matrix A such that A^2 is defined and is zero.

Solution: We could take $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ or $A = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$. □

Exercise 5. The *trace* of a square matrix is the sum of the diagonal entries. Show that if $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $B = \begin{bmatrix} p & q \\ r & s \end{bmatrix}$ then the trace of $AB - BA$ is zero.

Solution:

$$\begin{aligned}
AB &= \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} p & q \\ r & s \end{bmatrix} = \begin{bmatrix} ap + br & aq + bs \\ cp + dr & cq + ds \end{bmatrix} \\
BA &= \begin{bmatrix} p & q \\ r & s \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} ap + cq & bp + dq \\ ar + cs & br + ds \end{bmatrix} \\
AB - BA &= \begin{bmatrix} ap + br & aq + bs \\ cp + dr & cq + ds \end{bmatrix} - \begin{bmatrix} ap + cq & bp + dq \\ ar + cs & br + ds \end{bmatrix} \\
&= \begin{bmatrix} br - cq & aq + bs - bp - dq \\ cp + dr - ar - cs & cq - br \end{bmatrix}
\end{aligned}$$

$$\text{trace}(AB - BA) = (br - cq) + (cq - br) = 0.$$

□

2. Lecture 2

Exercise 6. Which of the following matrices are in reduced row-echelon form?

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 3 & 0 & 2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$D = \begin{bmatrix} 3 & 1 & 0 & 2 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad E = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Solution:

- A is not in RREF because the row of zeros occurs at the top, instead of the bottom.

- B is not in RREF because the pivot in the second row is to the left of the pivot in the first row, not to the right.
- C is in RREF.
- D is not in RREF because the first nonzero entry in the first row is equal to 3, not 1. Similarly, the first nonzero entry in the second row is not equal to 1.
- E is not in RREF because the last column contains a nonzero entry above the pivot in the third row.

□

Exercise 7. Give an example of a 4×7 matrix in RREF with pivots in columns 2, 5 and 7 (and no other columns) and with precisely six nonzero entries.

Solution: Every 4×7 matrix with pivots in the specified columns has the form

$$A = \begin{bmatrix} 0 & 1 & a & b & 0 & c & 0 \\ 0 & 0 & 0 & 0 & 1 & d & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

for some scalars a, b, c and d. If all of these scalars are nonzero then (together with the three pivots) we would have seven nonzero entries in the matrix. We want to have only six nonzero entries, so we can choose $a = b = c = 42$ and $d = 0$ (for example) giving

$$A = \begin{bmatrix} 0 & 1 & 42 & 42 & 0 & 42 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

□

Exercise 8. Use the augmented matrix method to solve the following system of linear equations, or prove that there is no solution.

$$\begin{aligned} 10a &= 10b + c \\ 10c + b &= 10a - 9 \\ a + 100c &= 100b + 11. \end{aligned}$$

Solution: We can tidy up the equations as follows:

$$\begin{array}{rrrl} 10a & -10b & -c & = 0 \\ 10a & -b & -10c & = 9 \\ a & -100b & +100c & = 11. \end{array}$$

Using this we can write down the augmented matrix and row-reduce it as follows:

$$\left[\begin{array}{ccc|c} 10 & -10 & -1 & 0 \\ 10 & -1 & -10 & 9 \\ 1 & -100 & 100 & 11 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & -1 & -0.1 & 0 \\ 10 & -1 & -10 & 9 \\ 1 & -100 & 100 & 11 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & -1 & -0.1 & 0 \\ 0 & 9 & -9 & 9 \\ 0 & -99 & 100.1 & 11 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|c} 1 & -1 & -0.1 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & -99 & 100.1 & 11 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & -1.1 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1.1 & 110 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & -1.1 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 100 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & 111 \\ 0 & 1 & 0 & 101 \\ 0 & 0 & 1 & 100 \end{array}\right]$$

We conclude that there is a unique solution, namely $a = 111$ and $b = 101$ and $c = 100$. □

Exercise 9. Use the augmented matrix method to solve the following system of linear equations, or prove that there is no solution.

$$\begin{aligned} 2w - x - y - 2z &= 1 \\ 3w - 2x - 2y - 3z &= -1 \\ 5w - 3x - 3y - 5z &= 0. \end{aligned}$$

Solution: We can write down the augmented matrix and row-reduce it as follows:

$$\left[\begin{array}{cccc|c} 2 & -1 & -1 & -2 & 1 \\ 3 & -2 & -2 & -3 & -1 \\ 5 & -3 & -3 & -5 & 0 \end{array}\right] \to \left[\begin{array}{cccc|c} 2 & -1 & -1 & -2 & 1 \\ -1 & 0 & 0 & 1 & -3 \\ -1 & 0 & 0 & 1 & -3 \end{array}\right] \to$$

$$\left[\begin{array}{cccc|c} 0 & -1 & -1 & 0 & -5 \\ 1 & 0 & 0 & -1 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 3 \\ 0 & 1 & 1 & 0 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

The final matrix corresponds to the system

$$\begin{aligned} w - z &= 3 \\ x + y &= 5 \\ 0 &= 0. \end{aligned}$$

There are pivots in columns 1 and 2, corresponding to the dependent variables w and x. After rearranging the equations to give the dependent variables in terms of the independent variables, we get $w = z + 3$ and $x = 5 - y$ with y and z arbitrary. Thus, we have an infinite family of solutions. □

Exercise 10. Use the augmented matrix method to solve the following system of linear equations, or prove that there is no solution.

$$\begin{aligned} p + q + r &= 30 \\ p + q - r &= 16 \\ p - q + r &= 24 \\ p - q - r &= 11 \end{aligned}$$

Solution: We can write down the augmented matrix and row-reduce it as follows:

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 30 \\ 1 & 1 & -1 & 16 \\ 1 & -1 & 1 & 24 \\ 1 & -1 & -1 & 12 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & 1 & 30 \\ 0 & 0 & -2 & -14 \\ 0 & -2 & 0 & -6 \\ 0 & -2 & -2 & -18 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & 1 & 30 \\ 0 & 0 & 1 & 7 \\ 0 & 1 & 0 & 3 \\ 0 & 1 & 1 & 9 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|c} 1 & 0 & 0 & 20 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & 2 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

The final matrix has a pivot in the last column, which means that the original system of equations has no solution. □

3. Lecture 3

Exercise 11. Put

$$p_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \qquad p_2 = \begin{bmatrix} 3 \\ 6 \end{bmatrix} \qquad p_3 = \begin{bmatrix} 2 \\ 4 \end{bmatrix}.$$

Describe geometrically which vectors in $\mathbb{R}^2$ can be expressed as a linear combination of p_1, p_2 and p_3. Give an example of a vector that cannot be described as such a linear combination.

Solution: Any linear combination of the vectors p_i has the form

$$\lambda_1 p_1 + \lambda_2 p_2 + \lambda_3 p_3 = \begin{bmatrix} \lambda_1 + 3\lambda_2 + 2\lambda_3 \\ 2\lambda_1 + 6\lambda_2 + 4\lambda_3 \end{bmatrix} = (\lambda_1 + 3\lambda_2 + 2\lambda_3) \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

Thus, these linear combinations are just the multiples of the vector $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$, so they form the line with equation $y = 2x$. This means that any vector $\begin{bmatrix} x \\ y \end{bmatrix}$ with $y \neq 2x$ cannot be expressed as a linear combination of the vectors p_i. For example, the vector $e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ cannot be expressed as a linear combination of the vectors p_i. □

Exercise 12. Put

$$u_1 = \begin{bmatrix} 1 \\ 1 \\ 7 \end{bmatrix} \qquad u_2 = \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix} \qquad u_3 = \begin{bmatrix} 3 \\ 3 \\ 1 \end{bmatrix} \qquad u_4 = \begin{bmatrix} 4 \\ 4 \\ 5 \end{bmatrix} \qquad u_5 = \begin{bmatrix} 5 \\ 5 \\ 2 \end{bmatrix}$$

Give an example of a vector $v \in \mathbb{R}^3$ that cannot be expressed as a linear combination of $u_1, \dots, u_5$.

Solution: Each of the vectors u_i has the first two components the same, so every linear combination of $u_1, \dots, u_5$ will also have the first two components the same. Thus, if we choose any vector v whose first two components are not the same, then it will not be a linear combination of $u_1, \dots, u_5$. The simplest example is to take $v = e_1 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^T$. □

Exercise 13. Consider the vectors

$$a_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \qquad a_2 = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 2 \end{bmatrix} \qquad a_3 = \begin{bmatrix} 2 \\ 2 \\ 1 \\ 1 \end{bmatrix} \qquad a_4 = \begin{bmatrix} 1 \\ 2 \\ 2 \\ 1 \end{bmatrix} \qquad b = \begin{bmatrix} 3 \\ -2 \\ 0 \\ 5 \end{bmatrix}.$$

You may assume the row-reduction

$$\left[\begin{array}{cccc|c} 1 & 1 & 2 & 1 & 3 \\ 1 & 1 & 2 & 2 & -2 \\ 1 & 2 & 1 & 2 & 0 \\ 1 & 2 & 1 & 1 & 5 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 0 & 3 & 0 & 6 \\ 0 & 1 & -1 & 0 & 2 \\ 0 & 0 & 0 & 1 & -5 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

Use this to give a formula expressing b as a linear combination of $a_1, \dots, a_4$.

Solution: The left hand matrix is $[a_1|a_2|a_3|a_4|b]$, so the row-reduction tells us that the equation $\lambda_1 a_1 + \dots + \lambda_4 a_4 = b$ is equivalent to the system of equations corresponding to the right hand matrix, namely

$$\begin{aligned} \lambda_1 + 3\lambda_3 &= 6 \\ \lambda_2 - \lambda_3 &= 2 \\ \lambda_4 &= -5. \end{aligned}$$

Here λ_3 is independent so it can take arbitrary values. We can choose $\lambda_3 = 0$, giving $\lambda_1 = 6$ and $\lambda_2 = 2$ and $\lambda_4 = -5$. This means that we have

$$b = \sum_i \lambda_i a_i = 6a_1 + 2a_2 - 5a_4.$$

□

Exercise 14. Consider the vectors

$$u_1 = \begin{bmatrix} 1 & 2 & -1 & 0 \end{bmatrix}^T \qquad u_2 = \begin{bmatrix} 3 & -1 & 4 & -2 \end{bmatrix}^T \qquad u_3 = \begin{bmatrix} -1 & 5 & -6 & 2 \end{bmatrix}^T$$
$$v = \begin{bmatrix} 5 & -4 & 9 & -4 \end{bmatrix}^T \qquad w = \begin{bmatrix} 4 & -2 & 3 & 1 \end{bmatrix}^T$$

and the matrix

$$A = \left[\begin{array}{c|c|c|c|c} u_1 & u_2 & u_3 & v & w \end{array}\right].$$

(a) Row-reduce A.
(b) Is v a linear combination of u_1, u_2 and u_3?
(c) Is w a linear combination of u_1, u_2 and u_3?

(Note that you do not need any additional row-reductions for parts (b) and (c). Remark 6.7 in the notes is relevant here.)

Solution:

(a) We have

$$A = \left[\begin{array}{c|c|c|c|c} u_1 & u_2 & u_3 & v & w \end{array}\right] = \begin{bmatrix} 1 & 3 & -1 & 5 & 4 \\ 2 & -1 & 5 & -4 & -2 \\ -1 & 4 & -6 & 9 & 3 \\ 0 & -2 & 2 & -4 & 1 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 3 & -1 & 5 & 4 \\ 0 & -7 & 7 & -14 & -10 \\ 0 & 7 & -7 & 14 & 7 \\ 0 & -2 & 2 & -4 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & -1 & 5 & 4 \\ 0 & 1 & -1 & 2 & 1 \\ 0 & -7 & 7 & -14 & -10 \\ 0 & -2 & 2 & -4 & 1 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 3 & -1 & 5 & 4 \\ 0 & 1 & -1 & 2 & 1 \\ 0 & 0 & 0 & 0 & -3 \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 2 & -1 & 1 \\ 0 & 1 & -1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 0 & 2 & -1 & 0 \\ 0 & 1 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(b) As in Remark 6.7 we can delete the last column and we still have a valid row-reduction

$$\left[\begin{array}{c|c|c|c} u_1 & u_2 & u_3 & v \end{array}\right] = \begin{bmatrix} 1 & 3 & -1 & 5 \\ 2 & -1 & 5 & -4 \\ -1 & 4 & -6 & 9 \\ 0 & -2 & 2 & -4 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 2 & -1 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The matrix on the right is in RREF with no pivot in the last column, which means (by Method 7.6) that v is indeed a linear combination of u_1, u_2 and u_3. More specifically, we see that the equation $\lambda_1 u_1 + \lambda_2 u_2 + \lambda_3 u_3 = v$ is equivalent to the system of equations corresponding to the above matrix, namely

$$\begin{aligned} \lambda_1 + 2\lambda_3 &= -1 \\ \lambda_2 - \lambda_3 &= 2 \\ 0 &= 0 \\ 0 &= 0. \end{aligned}$$

The solution is $\lambda_1 = -1 - 2\lambda_3$ and $\lambda_2 = 2 + \lambda_3$ with λ_3 arbitrary. We can take $\lambda_3 = 0$ giving $\lambda_1 = -1$ and $\lambda_2 = 2$, which means that $v = -u_1 + 2u_2$.

(b) As in Remark 6.7 we can delete the fourth column and we still have a valid row-reduction

$$\left[\begin{array}{c|c|c|c} u_1 & u_2 & u_3 & w \end{array}\right] = \begin{bmatrix} 1 & 3 & -1 & 4 \\ 2 & -1 & 5 & -2 \\ -1 & 4 & -6 & 3 \\ 0 & -2 & 2 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Here we have a pivot in the last column, indicating that w cannot be expressed as a linear combination of u_1, u_2 and u_3.

□

Exercise 15. Let u_1 and u_2 be vectors in $\mathbb{R}^n$, and put $v_1 = u_1 + u_2$ and $v_2 = u_1 - u_2$.

(a) Show that if a vector w can be expressed as a linear combination of v_1 and v_2, then it can also be expressed as a linear combination of u_1 and u_2.
(b) Give a formula for u_1 in terms of v_1 and v_2, and also a formula for u_2 in terms of v_1 and v_2.
(c) As a converse to (a), show that if a vector w can be expressed as a linear combination of u_1 and u_2, then it can also be expressed as a linear combination of v_1 and v_2.

Solution:

(a) Suppose that w can be expressed as a linear combination of v_1 and v_2. This means that $w = \lambda_1 v_1 + \lambda_2 v_2$ for some scalars λ_1 and λ_2. After substituting in the definition of v_1 and v_2, we get

$$w = \lambda_1(u_1 + u_2) + \lambda_2(u_1 - u_2) = (\lambda_1 + \lambda_2)u_1 + (\lambda_1 - \lambda_2)u_2.$$

Thus, if we define scalars μ_i by $\mu_1 = \lambda_1 + \lambda_2$ and $\mu_2 = \lambda_1 - \lambda_2$, we have $w = \mu_1 u_1 + \mu_2 u_2$. This expresses w as a linear combination of u_1 and u_2, as required.

(b) By adding the equations $v_1 = u_1 + u_2$ and $v_2 = u_1 - u_2$ we get $2u_1 = v_1 + v_2$ and so $u_1 = v_1/2 + v_2/2$. Similarly, we have $u_2 = v_1/2 - v_2/2$.

(c) Suppose that w can be expressed as a linear combination of u_1 and u_2. This means that $w = \lambda_1 u_1 + \lambda_2 u_2$ for some scalars λ_1 and λ_2. After substituting in the equations from (b) we get

$$w = \lambda_1(v_1/2 + v_2/2) + \lambda_2(v_1/2 - v_2/2) = (\lambda_1/2 + \lambda_2/2)v_1 + (\lambda_1/2 - \lambda_2/2)v_2.$$

Thus, if we define scalars μ_i by $\mu_1 = \lambda_1/2+\lambda_2/2$ and $\mu_2 = \lambda_1/2-\lambda_2/2$, we have $w = \mu_1 v_1+\mu_2 v_2$. This expresses w as a linear combination of v_1 and v_2, as required.

□

Exercise 16. Decide whether the following lists are linearly dependent.

(a) $a_1 = \begin{bmatrix}1\\4\end{bmatrix}, \quad a_2 = \begin{bmatrix}5\\3\end{bmatrix}, \quad a_3 = \begin{bmatrix}4\\2\end{bmatrix}, \quad a_4 = \begin{bmatrix}6\\6\end{bmatrix}.$

(b) $b_1 = \begin{bmatrix}5\\0\\0\\3\end{bmatrix}, \quad b_2 = \begin{bmatrix}6\\4\\0\\0\end{bmatrix}, \quad b_3 = \begin{bmatrix}7\\0\\5\\0\end{bmatrix}$

(c) $c_1 = \begin{bmatrix}5\\4\\3\end{bmatrix}, \quad c_2 = \begin{bmatrix}4\\5\\4\end{bmatrix}, \quad c_3 = \begin{bmatrix}5\\3\\2\end{bmatrix}$

Solution:

(a) Here we have a list of 4 vectors in $\mathbb{R}^2$, and any such list is automatically linearly dependent. (In general, any linearly independent list in $\mathbb{R}^n$ has length at most n, so any list of length greater than n must be dependent.) As an example of a nontrivial linear relation, we have

$$4a_1 + 14a_2 - 8a_3 - 7a_4 = 0.$$

However, we do not need this in order to answer the question as asked.

(b) The list b_1, b_2, b_3 is easily seen to be linearly independent. Indeed, any linear relation $\lambda_1 b_1 + \lambda_2 b_2 + \lambda_3 b_3 = 0$ can be expanded as

$$\begin{bmatrix}5\lambda_1 + 6\lambda_2 + 7\lambda_3\\4\lambda_2\\5\lambda_3\\3\lambda_1\end{bmatrix} = \begin{bmatrix}0\\0\\0\\0\end{bmatrix}.$$

By looking at the fourth entry we see that $3\lambda_1 = 0$ so $\lambda_1 = 0$. Similarly, the second and third entries give $\lambda_2 = \lambda_3 = 0$, so all the λ_i are zero, so our linear relation is the trivial one. As there is only the trivial linear relation, the list is independent.

We can reach the same conclusion by row-reducing the matrix $[b_1|b_2|b_3]$:

$$\begin{bmatrix}5&6&7\\0&4&0\\0&0&5\\3&0&0\end{bmatrix} \to \begin{bmatrix}5&6&7\\0&1&0\\0&0&1\\1&0&0\end{bmatrix} \to \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\\5&6&7\end{bmatrix} \to \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\\0&0&0\end{bmatrix}$$

At the end we have a pivot in every column, so the original list is independent.

(c) Here there is no obvious shortcut so we just row-reduce the matrix $[c_1|c_2|c_3]$:

$$\begin{bmatrix}5&4&5\\4&5&3\\3&4&2\end{bmatrix} \to \begin{bmatrix}5&4&5\\1&1&1\\3&4&2\end{bmatrix} \to \begin{bmatrix}0&-1&0\\1&1&1\\0&1&-1\end{bmatrix} \to$$

$$\begin{bmatrix}0&1&0\\1&0&1\\0&0&-1\end{bmatrix} \to \begin{bmatrix}1&0&1\\0&1&0\\0&0&-1\end{bmatrix} \to \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$$

Again, we have a pivot in every column, so the list c_1, c_2, c_3 is independent.

□

Exercise 17. Consider the vectors $u = \begin{bmatrix}1\\2\\3\end{bmatrix}$ and $v = \begin{bmatrix}3\\2\\1\end{bmatrix}$. Give an example of a nonzero vector w such that the list u, w is independent and the list v, w is independent but the list u, v, w is dependent.

Solution: The simplest example is to put $w = u + v = \begin{bmatrix}4\\4\\4\end{bmatrix}$. To see that this works, recall that a list of two nonzero vectors is independent iff the two vectors are not multiples of each other. As w is not a multiple of u, we see that the list u, w is independent. Similarly, as w is not a multiple of v we see that the list v, w is independent. However, we have a nontrivial linear relation $u + v - w = 0$, which proves that the list u, v, w is dependent. □

4. Lecture 4

Exercise 18. Find examples as follows. All your vectors should be nonzero, and all your lists should have length at least 2 and not contain the same vector twice.

(a) A list of vectors in $\mathbb{R}^3$ that is linearly dependent and does not span $\mathbb{R}^3$.
(b) A list of vectors in $\mathbb{R}^3$ that is linearly dependent and spans $\mathbb{R}^3$.
(c) A list of vectors in $\mathbb{R}^3$ that is linearly independent and does not span $\mathbb{R}^3$.
(d) A list of vectors in $\mathbb{R}^3$ that is linearly independent and does not span $\mathbb{R}^3$.

Solution: There are many possible correct solutions. Here is one.

(a) Put $a_1 = e_1 = \begin{bmatrix}1\\0\\0\end{bmatrix}$ and $a_2 = -a_1$. Then the list a_1, a_2 is linearly dependent (because we have a nontrivial linear relation $a_1 + a_2 = 0$) and does not span (because e_2 cannot be written as a linear combination of a_1 and a_2).
(b) Put $b_1 = e_1$ and $b_2 = e_2$ and $b_3 = e_3$ and $b_4 = -e_3$. The list $b_1, \dots, b_4$ is linearly dependent, because we have the nontrivial linear relation $0b_1 + 0b_2 + b_3 + b_4 = 0$. It spans $\mathbb{R}^3$, because any vector $v = \begin{bmatrix}x & y & z\end{bmatrix}^T \in \mathbb{R}^3$ can be written as $v = xb_1 + yb_2 + zb_3 + 0b_4$, which expresses v as a linear combination of $b_1, \dots, b_4$.
(c) Put $c_1 = e_1$ and $c_2 = e_2$. The list c_1, c_2 is clearly linearly independent: a linear relation $\lambda_1 c_1 + \lambda_2 c_2 = 0$ expands to give $\begin{bmatrix}\lambda_1\\\lambda_2\\0\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$, so $\lambda_1 = \lambda_2 = 0$, so the linear relation is trivial. However, e_3 cannot be expressed as a linear combination of c_1 and c_2, so the list c_1, c_2 does not span.
(d) The list e_1, e_2, e_3 is linearly independent and spans.

□

Exercise 19. Decide whether the following statements are true or false. Justify your answers, and give explicit counterexamples for any statements that are false.

(a) Every list of 4 vectors in $\mathbb{R}^3$ spans $\mathbb{R}^3$.
(b) Every list of 4 vectors in $\mathbb{R}^3$ is linearly independent.
(c) If $\mathcal{A}$ is a list that spans $\mathbb{R}^4$ and $\mathcal{B}$ is a linearly independent list in $\mathbb{R}^4$ then $\mathcal{A}$ is at least as long as $\mathcal{B}$.
(d) There is a linearly independent list of length 5 in $\mathbb{R}^6$.

Solution:

(a) This is false. For example, the list e_1, e_1, e_1, e_1 is a list of four vectors in $\mathbb{R}^4$ that does not span.
(b) This is also false, and in fact is the opposite of the truth: every list of 4 vectors in $\mathbb{R}^3$ is linearly dependent, not linearly independent.
(c) This is true. As $\mathcal{A}$ spans $\mathbb{R}^4$ it must contain at least 4 vectors, and as $\mathcal{B}$ is linearly independent in $\mathbb{R}^4$ it must contain at most 4 vectors. Thus $\text{length}(\mathcal{B}) \leq 4 \leq \text{length}(\mathcal{A})$.
(d) This is true. The list e_1, e_2, e_3, e_4, e_5 is the most obvious example.

□

Exercise 20. Consider the list

$$u_1 = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}, \quad u_2 = \begin{bmatrix} 0 \\ 3 \\ -2 \end{bmatrix}, \quad u_3 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}, \quad u_4 = \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix}.$$

Does this span $\mathbb{R}^3$?

Solution: We use Method 9.7, which tells us to perform the following row-reduction:

$$\begin{bmatrix} u_1^T \\ \hline u_2^T \\ \hline u_3^T \\ \hline u_4^T \end{bmatrix} = \begin{bmatrix} 2 & -1 & 0 \\ 0 & 3 & -2 \\ 1 & -2 & 1 \\ 3 & 0 & -1 \end{bmatrix} \to \begin{bmatrix} 0 & 3 & -2 \\ 0 & 3 & -2 \\ 1 & -2 & 1 \\ 0 & 6 & -4 \end{bmatrix} \to$$

$$\begin{bmatrix} 1 & -2 & 1 \\ 0 & 3 & -2 \\ 0 & 3 & -2 \\ 0 & 6 & -4 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & -2/3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1/3 \\ 0 & 1 & -2/3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

In the final matrix we do not have a pivot in every column, so the specified list does not span $\mathbb{R}^3$. □

Exercise 21. Put $a = \begin{bmatrix} 1 & 3 & 5 & 7 \end{bmatrix} \in \mathbb{R}^4$.

(a) Suppose we have vectors $u_1, \dots, u_4 \in \mathbb{R}^4$ with $a.u_1 = a.u_2 = a.u_3 = a.u_4 = 0$. Prove that the list $u_1, \dots, u_4$ does not span $\mathbb{R}^4$.
(b) Give an example of a list $v_1, \dots, v_4$ that satisfies $a.v_1 = a.v_2 = a.v_3 = a.v_4 = 1$ and also spans $\mathbb{R}^4$.
(c) Give an example of a list $w_1, \dots, w_4$ that satisfies $a.w_1 = a.w_2 = a.w_3 = a.w_4 = 1$ and does not span $\mathbb{R}^4$.

Solution:

(a) If x is a linear combination of the vectors u_i, we have $x = \lambda_1 u_1 + \cdots + \lambda_4 u_4$ for some scalars $\lambda_1, \dots, \lambda_4$, so

$$a.x = a.(\lambda_1 u_1 + \cdots + \lambda_4 u_4) = \lambda_1(a.u_1) + \cdots + \lambda_4(a.u_4),$$

but $a.u_1 = a.u_2 = a.u_3 = a.u_4 = 0$ so $a.x = 0$. On the other hand, we have $a.e_1 = 1 \neq 0$, so e_1 cannot be a linear combination of the vectors u_i. This means that the u_i do not span $\mathbb{R}^4$.

(b) The obvious example is

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad v_2 = \begin{bmatrix} 0 \\ 1/3 \\ 0 \\ 0 \end{bmatrix} \qquad v_3 = \begin{bmatrix} 0 \\ 0 \\ 1/5 \\ 0 \end{bmatrix} \qquad v_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1/7 \end{bmatrix}.$$

To see that this spans, note that an arbitrary vector $x = \begin{bmatrix} a & b & c & d \end{bmatrix}^T$ in $\mathbb{R}^4$ can be expressed as

$$x = av_1 + 3bv_2 + 5cv_3 + 7dv_4,$$

which is a linear combination of the list $v_1, \dots, v_4$.

(c) The most obvious solution is to take $w_1 = w_2 = w_3 = w_4 = e_1$. If we prefer to avoid repetitions, we can instead use

$$w_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad w_2 = \begin{bmatrix} 4 \\ -1 \\ 0 \\ 0 \end{bmatrix} \qquad w_3 = \begin{bmatrix} 7 \\ -2 \\ 0 \\ 0 \end{bmatrix} \qquad w_4 = \begin{bmatrix} 10 \\ -3 \\ 0 \\ 0 \end{bmatrix}.$$

It is clear that any linear combination of $w_1, \dots, w_4$ has zeros in the third and fourth places. In particular, the standard vector e_4 is not a linear combination of the list $w_1, \dots, w_4$, so the list does not span $\mathbb{R}^4$.

□

Exercise 22. The vectors

$$u_1 = \begin{bmatrix}1\\0\\0\\0\end{bmatrix} \quad u_2 = \begin{bmatrix}1\\1\\0\\0\end{bmatrix} \quad u_3 = \begin{bmatrix}1\\1\\1\\0\end{bmatrix} \quad u_4 = \begin{bmatrix}1\\1\\1\\1\end{bmatrix} \quad u_5 = \begin{bmatrix}0\\1\\1\\1\end{bmatrix} \quad u_6 = \begin{bmatrix}0\\0\\1\\1\end{bmatrix} \quad u_7 = \begin{bmatrix}0\\0\\0\\1\end{bmatrix}$$

span $\mathbb{R}^4$, because an arbitrary vector $x = \begin{bmatrix}a & b & c & d\end{bmatrix}^T$ can be expressed as a linear combination of u_i by the formula

$$x = (a-b)u_1 + bu_2 + cu_6 + (d-c)u_7,$$

or alternatively by the formula

$$x = -bu_1 + bu_2 - du_3 + (a+d)u_4 - au_5 + cu_6 - cu_7.$$

(a) Check the above formulae.
(b) Give a similar explicit formula to prove that the following vectors span $\mathbb{R}^4$:

$$v_1 = \begin{bmatrix}1\\1\\1\\0\end{bmatrix} \quad v_2 = \begin{bmatrix}1\\1\\0\\1\end{bmatrix} \quad v_3 = \begin{bmatrix}1\\1\\1\\1\end{bmatrix} \quad v_4 = \begin{bmatrix}1\\0\\1\\1\end{bmatrix} \quad v_5 = \begin{bmatrix}0\\1\\1\\1\end{bmatrix}$$

(c) Use the row-reduction method to show again that the vectors v_i span $\mathbb{R}^4$.

Solution:

(a) For the first formula we have

$$\begin{aligned}&(a-b)u_1 + bu_2 + cu_6 + (d-c)u_7\\ &= \begin{bmatrix}a-b\\0\\0\\0\end{bmatrix} + \begin{bmatrix}b\\b\\0\\0\end{bmatrix} + \begin{bmatrix}0\\0\\c\\c\end{bmatrix} + \begin{bmatrix}0\\0\\0\\d-c\end{bmatrix} = \begin{bmatrix}a\\b\\c\\d\end{bmatrix}.\end{aligned}$$

For the second, we have

$$\begin{aligned}&-bu_1 + bu_2 - du_3 + (a+d)u_4 - au_5 + cu_6 - cu_7\\ &= \begin{bmatrix}-b\\0\\0\\0\end{bmatrix} + \begin{bmatrix}b\\b\\0\\0\end{bmatrix} + \begin{bmatrix}-d\\-d\\-d\\0\end{bmatrix} + \begin{bmatrix}a+d\\a+d\\a+d\\a+d\end{bmatrix} + \begin{bmatrix}0\\-a\\-a\\-a\end{bmatrix} + \begin{bmatrix}0\\0\\c\\c\end{bmatrix} + \begin{bmatrix}0\\0\\0\\-c\end{bmatrix} = \begin{bmatrix}a\\b\\c\\d\end{bmatrix}.\end{aligned}$$

(b) One possible formula is as follows: if $x = \begin{bmatrix}a & b & c & d\end{bmatrix}^T$, then

$$x = -dv_1 - cv_2 + (a+b+c+d)v_3 - bv_4 - av_5.$$

This can be found as follows: we note that

$$e_1 = v_3 - v_5 \qquad e_2 = v_3 - v_4 \qquad e_3 = v_3 - v_2 \qquad e_4 = v_3 - v_1,$$

and it follows that

$$\begin{aligned}x &= ae_1 + be_2 + ce_3 + de_4\\ &= a(v_3 - v_5) + b(v_3 - v_4) + c(v_3 - v_2) + d(v_3 - v_1)\\ &= -dv_1 - cv_2 + (a+b+c+d)v_3 - bv_4 - av_5.\end{aligned}$$

(c) The general method for these kinds of questions is to construct a matrix A whose rows are the vectors v_i^T, and then row-reduce it:

$$A = \begin{bmatrix}1&1&1&0\\1&1&0&1\\1&1&1&1\\1&0&1&1\\0&1&1&1\end{bmatrix} \to \begin{bmatrix}1&1&1&0\\0&0&-1&1\\0&0&0&1\\0&-1&0&1\\0&1&1&1\end{bmatrix} \to \begin{bmatrix}1&1&1&0\\0&0&1&-1\\0&0&0&1\\0&0&1&2\\0&1&1&1\end{bmatrix} \to$$

$$\begin{bmatrix}1&1&1&0\\0&0&1&-1\\0&0&0&1\\0&0&0&3\\0&1&1&1\end{bmatrix} \to \begin{bmatrix}1&1&1&0\\0&1&1&1\\0&0&1&-1\\0&0&0&1\\0&0&0&3\end{bmatrix} \to \begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\\0&0&0&0\end{bmatrix}$$

The final matrix has a pivot in every column, so the vectors v_i span $\mathbb{R}^4$.

□

5. Lecture 5

Exercise 23. (a) Is the list $a_1 = \begin{bmatrix}3\\5\end{bmatrix}, a_2 = \begin{bmatrix}2\\7\end{bmatrix}, a_3 = \begin{bmatrix}4\\4\end{bmatrix}$ a basis for $\mathbb{R}^2$?

(b) Is the list $b_1 = \begin{bmatrix}9\\8\\7\end{bmatrix}, b_2 = \begin{bmatrix}8\\7\\6\end{bmatrix}, b_3 = \begin{bmatrix}3\\2\\1\end{bmatrix}$ a basis for $\mathbb{R}^3$?

(c) Is the list $c_1 = \begin{bmatrix}1\\8\\5\\4\end{bmatrix}, c_2 = \begin{bmatrix}7\\3\\9\\5\end{bmatrix}, c_3 = \begin{bmatrix}5\\1\\9\\9\end{bmatrix}$ a basis for $\mathbb{R}^4$?

Solution: Any basis for $\mathbb{R}^n$ must contain exactly n vectors. In particular, a basis for $\mathbb{R}^2$ must contain precisely 2 vectors, so a_1, a_2, a_3 cannot be a basis for $\mathbb{R}^2$. (In fact, there is a linear relation $-20a_1 + 8a_2 + 11a_3 = 0$, showing that the list is linearly dependent and so cannot form a basis. However, it is not strictly necessary to work this out.) Similarly, as the list c_1, c_2, c_3 does not have length 4, it cannot form a basis for $\mathbb{R}^4$. This just leaves part (b). Here we can observe that

$$\begin{aligned} b_1 - b_2 &= \begin{bmatrix}1 & 1 & 1\end{bmatrix}^T \\ b_2 - b_3 &= \begin{bmatrix}5 & 5 & 5\end{bmatrix}^T = 5(b_1 - b_2), \end{aligned}$$

and this rearranges to give a nontrivial linear relation $6b_1 - 5b_2 + b_3 = 0$. This proves that the list b_1, b_2, b_3 is linearly dependent, so again we do not have a basis. This can also be seen by row-reducing the matrix $[b_1|b_2|b_3]$:

$$\begin{bmatrix}9&8&3\\8&7&2\\7&6&1\end{bmatrix} \to \begin{bmatrix}1&8/9&1/3\\8&7&2\\7&6&1\end{bmatrix} \to \begin{bmatrix}1&8/9&1/3\\0&-1/9&-2/3\\0&-2/9&-4/3\end{bmatrix} \to \begin{bmatrix}1&8/9&1/3\\0&1&6\\0&-2/9&-4/3\end{bmatrix} \to \begin{bmatrix}1&0&-5\\0&1&6\\0&0&0\end{bmatrix}$$

As the final result is not the identity matrix, we see that the list b_1, b_2, b_3 is not a basis. □

Exercise 24. Consider the list

$$a_1 = \begin{bmatrix}1\\1\\1\\2\end{bmatrix}, \quad a_2 = \begin{bmatrix}1\\1\\3\\4\end{bmatrix}, \quad a_3 = \begin{bmatrix}1\\4\\5\\6\end{bmatrix}, \quad a_4 = \begin{bmatrix}7\\8\\9\\10\end{bmatrix}.$$

Is this a basis for $\mathbb{R}^4$?

Solution: We can check this by row-reducing the matrix $[a_1|a_2|a_3|a_4]$:

$$\begin{bmatrix}1&1&1&7\\1&1&4&8\\1&4&5&6\\7&8&9&10\end{bmatrix} \to \begin{bmatrix}1&1&1&7\\0&0&3&1\\0&3&4&-1\\0&1&2&-39\end{bmatrix} \to \begin{bmatrix}1&0&-1&46\\0&0&3&1\\0&0&-2&116\\0&1&2&-39\end{bmatrix} \to \begin{bmatrix}1&0&-1&46\\0&0&3&1\\0&0&1&-58\\0&1&2&-39\end{bmatrix} \to$$

$$\begin{bmatrix}1&0&0&-12\\0&0&0&175\\0&0&1&-58\\0&1&0&77\end{bmatrix} \to \begin{bmatrix}1&0&0&-12\\0&0&0&1\\0&0&1&-58\\0&1&0&77\end{bmatrix} \to \begin{bmatrix}1&0&0&0\\0&0&0&1\\0&0&1&0\\0&1&0&0\end{bmatrix} \to \begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}$$

As we end up with the identity matrix, the original list is a basis. □

Exercise 25. Put $u_1 = \begin{bmatrix}1\\2\\3\end{bmatrix}$ and $u_2 = \begin{bmatrix}2\\3\\4\end{bmatrix}$. Find a vector u_3 such that the list u_1, u_2, u_3 is a basis for $\mathbb{R}^3$.

Solution: Any vector will do provided that it does not lie in the plane spanned by u_1 and u_2, so if you choose u_3 randomly then it will probably work. The simplest choice is to take $u_3 = e_1 = \begin{bmatrix}1\\0\\0\end{bmatrix}$. To check that u_1, u_2, u_3 is a basis we can row-reduce the matrix $U = [u_1|u_2|u_3]$ and check that we get the identity:

$$\begin{bmatrix}1&2&1\\2&3&0\\3&4&0\end{bmatrix} \to \begin{bmatrix}1&2&1\\0&-1&-2\\0&-2&-3\end{bmatrix} \to \begin{bmatrix}1&0&-3\\0&1&2\\0&0&1\end{bmatrix} \to \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}.$$

□

Exercise 26. Suppose that the list a_1, a_2, a_3, a_4, a_5 is a basis for $\mathbb{R}^5$. Show that the list a_1, a_3, a_5 is linearly independent.

Solution: Suppose we have a linear relation $\lambda a_1 + \mu a_3 + \nu a_5 = 0$. This gives a linear relation

$$\lambda a_1 + 0a_2 + \mu a_3 + 0a_4 + \nu a_5 = 0$$

on the whole list. However, the whole list is a basis for $\mathbb{R}^5$, so in particular it is linearly independent. Thus, the above linear relation must be the trivial one, so the coefficients $\lambda, 0, \mu, 0, \nu, 0$ must all be zero. As λ, μ and ν are zero, we see that the original relation on the list a_1, a_3, a_5 is the trivial relation. This means that the list a_1, a_3, a_5 is linearly independent, as claimed. □

6. Lecture 6

Exercise 27. Find the inverse of the matrix

$$A = \begin{bmatrix}0&0&0&1\\0&0&1&1\\0&1&1&1\\1&1&1&1\end{bmatrix}.$$

Solution: We row-reduce the matrix $[A|I_4]$:

$$\left[\begin{array}{cccc|cccc}0&0&0&1&1&0&0&0\\0&0&1&1&0&1&0&0\\0&1&1&1&0&0&1&0\\1&1&1&1&0&0&0&1\end{array}\right] \xrightarrow{1} \left[\begin{array}{cccc|cccc}0&0&0&1&1&0&0&0\\0&0&1&1&0&1&0&0\\0&1&1&1&0&0&1&0\\1&0&0&0&0&0&-1&1\end{array}\right] \xrightarrow{2} \left[\begin{array}{cccc|cccc}0&0&0&1&1&0&0&0\\0&0&1&1&0&1&0&0\\0&1&0&0&0&-1&1&0\\1&0&0&0&0&0&-1&1\end{array}\right]$$

$$\xrightarrow{3} \left[\begin{array}{cccc|cccc}0&0&0&1&1&0&0&0\\0&0&1&0&-1&1&0&0\\0&1&0&0&0&-1&1&0\\1&0&0&0&0&0&-1&1\end{array}\right] \xrightarrow{4} \left[\begin{array}{cccc|cccc}1&0&0&0&0&0&-1&1\\0&1&0&0&0&-1&1&0\\0&0&1&0&-1&1&0&0\\0&0&0&1&1&0&0&0\end{array}\right].$$

The conclusion is that

$$A^{-1} = \begin{bmatrix}0&0&-1&1\\0&-1&1&0\\-1&1&0&0\\1&0&0&0\end{bmatrix}.$$

□

Exercise 28. Consider the matrix

$$A_0 = \begin{bmatrix}0&10&100&-1&10\\0&11&110&-1&21\\0&-1&-10&0&-11\end{bmatrix}$$

(a) Find a row reduction

$$A_0 \to A_1 \to A_2 \to A_3 \to A_4 \to A_5 \to A_6$$

where each step uses only a single row-operation and A_6 is in RREF.

(b) Find elementary matrices $U_1, \ldots, U_6$ such that $A_i = U_i A_{i-1}$.

(c) Hence find an invertible matrix U such that $A_6 = UA_0$. (Be careful about the order of multiplication.)

Solution: The relevant matrices are as follows:

$$A_1 = \begin{bmatrix} 0 & 10 & 100 & -1 & 10 \\ 0 & 11 & 110 & -1 & 21 \\ 0 & 1 & 10 & 0 & 11 \end{bmatrix} \qquad U_1 = D_3(-1) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} 0 & 0 & 0 & -1 & -100 \\ 0 & 11 & 110 & -1 & 21 \\ 0 & 1 & 10 & 0 & 11 \end{bmatrix} \qquad U_2 = E_{13}(-10) = \begin{bmatrix} 1 & 0 & -10 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A_3 = \begin{bmatrix} 0 & 0 & 0 & -1 & -100 \\ 0 & 0 & 0 & -1 & -100 \\ 0 & 1 & 10 & 0 & 11 \end{bmatrix} \qquad U_3 = E_{23}(-11) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -11 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A_4 = \begin{bmatrix} 0 & 1 & 10 & 0 & 11 \\ 0 & 0 & 0 & -1 & -100 \\ 0 & 0 & 0 & -1 & -100 \end{bmatrix} \qquad U_4 = F_{13} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$A_5 = \begin{bmatrix} 0 & 1 & 10 & 0 & 11 \\ 0 & 0 & 0 & 1 & 100 \\ 0 & 0 & 0 & -1 & -100 \end{bmatrix} \qquad U_5 = D_2(-1) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A_6 = \begin{bmatrix} 0 & 1 & 10 & 0 & 11 \\ 0 & 0 & 0 & 1 & 100 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad U_6 = E_{32}(1) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}.$$

Indeed, the reduction steps are as follows:

(1) Multiply row 3 by -1.
(2) Add -10 times row 3 to row 1.
(3) Add -11 times row 3 ro row 2.
(4) Swap rows 1 and 3.
(5) Multiply row 3 by -1.
(6) Add row 2 to row 3.

The matrices U_i correspond to these row operations as in Proposition 11.3. It follows that

$$\begin{aligned} A_1 &= U_1 A_0 \\ A_2 &= U_2 A_1 = U_2 U_1 A_0 \\ A_3 &= U_3 A_2 = U_3 U_2 U_1 A_0 \end{aligned}$$

and so on, so $A_6 = UA_0$ where $U = U_6U_5U_4U_3U_2U_1$. Here

$$U_6U_5U_4 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & -1 & 0 \end{bmatrix}$$

$$U_3U_2U_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -11 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -10 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 10 \\ 0 & 1 & 11 \\ 0 & 0 & -1 \end{bmatrix}$$

$$U = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 10 \\ 0 & 1 & 11 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -1 \\ 0 & -1 & -11 \\ 1 & -1 & -1 \end{bmatrix}.$$

As a check, we can verify directly that

$$UA_0 = \begin{bmatrix} 0 & 0 & -1 \\ 0 & -1 & -11 \\ 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} 0 & 10 & 100 & -1 & 10 \\ 0 & 11 & 110 & -1 & 21 \\ 0 & -1 & -10 & 0 & -11 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 10 & 0 & 11 \\ 0 & 0 & 0 & 1 & 100 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = A_6.$$

□

Exercise 29. Which of the following matrices are invertible? Justify your answers.

$$A = \begin{bmatrix} 1 & 2 & 2 & 1 \\ 2 & 3 & 3 & 2 \\ 2 & 3 & 3 & 2 \\ 1 & 2 & 2 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 2 & 2 & 2 & 2 \\ 1 & 2 & 2 & 2 \\ 1 & 1 & 2 & 2 \\ 1 & 1 & 1 & 2 \end{bmatrix} \quad C = \begin{bmatrix} 2 & 1 & 1 & 1 \\ 2 & 2 & 1 & 1 \\ 2 & 2 & 2 & 1 \\ 2 & 2 & 2 & 2 \end{bmatrix} \quad D = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 5 & 6 \\ 0 & 0 & 7 & 8 \\ 0 & 0 & 0 & 9 \end{bmatrix} \quad E = \begin{bmatrix} 1 & 1 \\ 10 & 11 \\ 100 & 111 \\ 1000 & 1111 \end{bmatrix}$$

Solution:

(a) The matrix A is not invertible. Indeed, the first and last rows are the same, as are the middle two rows. Thus, we can perform row operations on A to get a matrix A' with two rows of zeros. It follows that A cannot row-reduce to the identity. Alternatively, we can say that there are only two distinct columns, which means that the columns cannot possibly form a basis for $\mathbb{R}^4$, which again means that the matrix is not invertible.

(b) We can start row-reducing B as follows:

$$B = \begin{bmatrix} 2 & 2 & 2 & 2 \\ 1 & 2 & 2 & 2 \\ 1 & 1 & 2 & 2 \\ 1 & 1 & 1 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 2 \\ 1 & 1 & 2 & 2 \\ 1 & 1 & 1 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} = B'.$$

As B' is upper-triangular with 1s on the diagonal we have $\det(B') = 1$, and it follows that $\det(B) \neq 0$, so B is invertible. More specifically, only the first of our row operations (where we multiplied row 1 by 1/2) affects the determinant, so $\det(B) = \det(B')/(1/2) = 2$. Alternatively, we can just carry out a few more row operations to see that $B' \to I_4$.

(c) We have $C = B^T$ and it is clear from Theorem 11.5 that the transpose of any invertible matrix is invertible, so C is invertible.

(d) As D is upper triangular, the determinant is the product of the diagonal entries, which is zero because $D_{22} = 0$. It follows that D is not invertible. This can also be seen from the row-reduction

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 5 & 6 \\ 0 & 0 & 7 & 8 \\ 0 & 0 & 0 & 9 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 6/5 \\ 0 & 0 & 1 & 8/7 \\ 0 & 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

(e) The matrix E is not invertible, just because invertibility only makes sense for square matrices.

□

Exercise 30. Find the inverse of the following matrix, either by creative experimentation or by row-reduction.

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & a & b \\ 0 & 1 & c & d \end{bmatrix}$$

Solution: The answer is

$$A^{-1} = \begin{bmatrix} -a & -b & 1 & 0 \\ -c & -d & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}.$$

There are various ways to see this. Perhaps the most conceptual is as follows. We can put $B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and divide A into 2×2 blocks. We then have $A = \left[\begin{array}{c|c} 0 & I \\ \hline I & B \end{array}\right]$, and the claim is that $A^{-1} = \left[\begin{array}{c|c} -B & I \\ \hline I & 0 \end{array}\right]$. To check this we just need the equation

$$\left[\begin{array}{c|c} 0 & I \\ \hline I & B \end{array}\right] \left[\begin{array}{c|c} -B & I \\ \hline I & 0 \end{array}\right] = \left[\begin{array}{c|c} I & 0 \\ \hline 0 & I \end{array}\right].$$

This is clear provided that we believe that we can treat the 2×2 blocks as though they were just numbers when we perform the above matrix product. This is not completely obvious, but it can be justified.

For a more pedestrian approach, we row-reduce the matrix $[A|I_4]$:

$$\left[\begin{array}{cccc|cccc} 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & a & b & 0 & 0 & 1 & 0 \\ 0 & 1 & c & d & 0 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{cccc|cccc} 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & b & -a & 0 & 1 & 0 \\ 0 & 1 & 0 & d & -c & 0 & 0 & 1 \end{array}\right] \to$$

$$\left[\begin{array}{cccc|cccc} 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & -a & -b & 1 & 0 \\ 0 & 1 & 0 & 0 & -c & -d & 0 & 1 \end{array}\right] \to \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & -a & -b & 1 & 0 \\ 0 & 1 & 0 & 0 & -c & -d & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \end{array}\right]$$

(Subtract multiples of row 1 from rows 3 and 4; subtract multiples of row 2 from rows 3 and 4; swap rows 1 and 3, and also swap rows 2 and 4.) The matrix A^{-1} appears as the right hand half of the final result. □

7. Lecture 7

Exercise 31. Calculate the determinant of the matrix

$$A = \begin{bmatrix} a & 0 & b & c \\ d & 0 & 0 & 0 \\ e & f & g & h \\ i & 0 & 0 & j \end{bmatrix}$$

Solution: The most obvious approach is to expand along the top row. This gives

$$\det(A) = a\det(B_1) - 0\det(B_2) + b\det(B_3) - c\det(B_4),$$

where

$$B_1 = \begin{bmatrix} 0 & 0 & 0 \\ f & g & h \\ 0 & 0 & j \end{bmatrix} \qquad B_2 = \begin{bmatrix} d & 0 & 0 \\ e & g & h \\ i & 0 & j \end{bmatrix} \qquad B_3 = \begin{bmatrix} d & 0 & 0 \\ e & f & h \\ i & 0 & j \end{bmatrix} \qquad B_4 = \begin{bmatrix} d & 0 & 0 \\ e & f & g \\ i & 0 & 0 \end{bmatrix}$$

As B_1 has a row of zeros we have $\det(B_1) = 0$. As $\det(B_2)$ gets multiplied by zero, we need not evaluate it. Straightforward expansion gives $\det(B_3) = dfj$ and $\det(B_4) = 0$. Putting this together, we get $\det(A) = bdfj$.

Alternatively, we can expand $\det(A)$ down the second column, and then along the second row, giving

$$\det(A) = (-1)^{3+2} f \det \begin{bmatrix} a & b & c \\ d & 0 & 0 \\ i & 0 & j \end{bmatrix} = (-1)^{3+2}(-1)^{2+1} fd \det \begin{bmatrix} b & c \\ 0 & j \end{bmatrix} = fdbj = bdfj.$$

□

Exercise 32. Consider the matrix

$$A = \begin{bmatrix} a & b & c & d \\ e & 0 & 0 & f \\ g & 0 & 0 & h \\ i & j & k & l \end{bmatrix}.$$

Prove that $\det(A) = \det \begin{bmatrix} e & f \\ g & h \end{bmatrix} \det \begin{bmatrix} b & c \\ j & k \end{bmatrix}$. (You can reduce the work involved if you choose carefully how to expand the determinant.)

Solution: We expand along the second row. Note that e occurs in the $(2,1)$ position and so comes with a sign $(-1)^{2+1} = -1$, whereas f occurs in the $(2,4)$ position with a sign $(-1)^{2+4} = +1$. We thus have

$$\det(A) = -e \det \begin{bmatrix} b & c & d \\ 0 & 0 & h \\ j & k & l \end{bmatrix} + f \det \begin{bmatrix} a & b & c \\ g & 0 & 0 \\ i & j & k \end{bmatrix}.$$

We now expand out these two 3×3 determinants along the middle row. Note that h is in the $(2,3)$ position of the first 3×3 matrix and so comes with a sign -1, and g is in the $(2,1)$ position of the second

3×3 matrix and so also comes with a sign -1. This gives

$$\det \begin{bmatrix} b & c & d \\ 0 & 0 & h \\ j & k & l \end{bmatrix} = -h \det \begin{bmatrix} b & c \\ j & k \end{bmatrix} = -h(bk - cj)$$

$$\det \begin{bmatrix} a & b & c \\ e & 0 & 0 \\ i & j & k \end{bmatrix} = -g \det \begin{bmatrix} b & c \\ j & k \end{bmatrix} = -g(bk - cj).$$

Putting this together we get

$$\det(A) = (-e)(-h)(bk - cj) + f(-g)(bk - cj) = (eh - fg)(bk - cj) = \det \begin{bmatrix} e & f \\ g & h \end{bmatrix} \det \begin{bmatrix} b & c \\ j & k \end{bmatrix}.$$

□

Exercise 33. Calculate the determinant of the matrix

$$A = \begin{bmatrix} a & a & a & a \\ a & b & b & b \\ a & b & c & c \\ a & b & c & d \end{bmatrix}.$$

(The easiest method is to start with some carefully chosen row operations as in Method 12.9.)

Solution: We subtract the third row from the fourth row, the second row from the third row, and the first row from the second row to get a new matrix B:

$$A = \begin{bmatrix} a & a & a & a \\ a & b & b & b \\ a & b & c & c \\ a & b & c & d \end{bmatrix} \to \begin{bmatrix} a & a & a & a \\ 0 & b-a & b-a & b-a \\ 0 & 0 & c-b & c-b \\ 0 & 0 & 0 & d-c \end{bmatrix} = B.$$

As we have not swapped any rows or multiplied any rows by a constant, there are no correcting factors and Method 12.9 just tells us that $\det(A) = \det(B)$. As B is upper triangular, the determinant is just the product of the diagonal entries, giving

$$\det(A) = a(b - a)(c - b)(d - c).$$

□

Exercise 34. Find the adjugate, determinant and inverse of the matrix $C = \begin{bmatrix} a & b & c \\ b & c & a \\ c & a & b \end{bmatrix}$.

(Note that the intermediate calculations that you need for $\det(C)$ are a subset of those that you need for $\operatorname{adj}(C)$. Try not to repeat work unnecessarily.)

Solution: The minors are

$$m_{11} = \det \begin{bmatrix} c & a \\ a & b \end{bmatrix} = bc - a^2 \qquad m_{12} = \det \begin{bmatrix} b & a \\ c & b \end{bmatrix} = b^2 - ac \qquad m_{13} = \det \begin{bmatrix} b & c \\ c & a \end{bmatrix} = ab - c^2$$

$$m_{21} = \det \begin{bmatrix} b & c \\ a & b \end{bmatrix} = b^2 - ac \qquad m_{22} = \det \begin{bmatrix} a & c \\ c & b \end{bmatrix} = ab - c^2 \qquad m_{23} = \det \begin{bmatrix} a & b \\ c & a \end{bmatrix} = a^2 - bc$$

$$m_{31} = \det \begin{bmatrix} b & c \\ c & a \end{bmatrix} = ab - c^2 \qquad m_{32} = \det \begin{bmatrix} a & c \\ b & a \end{bmatrix} = a^2 - bc \qquad m_{33} = \det \begin{bmatrix} a & b \\ b & c \end{bmatrix} = ac - b^2.$$

This gives

$$\operatorname{adj}(C) = \begin{bmatrix} m_{11} & -m_{21} & m_{31} \\ -m_{12} & m_{22} & -m_{32} \\ m_{13} & -m_{23} & m_{33} \end{bmatrix} = \begin{bmatrix} bc - a^2 & ac - b^2 & ab - c^2 \\ ac - b^2 & ab - c^2 & bc - a^2 \\ ab - c^2 & bc - a^2 & ac - b^2 \end{bmatrix}$$

$$\det(C) = C_{11}m_{11} - C_{12}m_{12} + C_{13}m_{13} = a(bc - a^2) - b(b^2 - ac) + c(ab - c^2)$$
$$= 3abc - a^3 - b^3 - c^3$$

$$C^{-1} = \frac{\operatorname{adj}(C)}{\det(C)} = \frac{1}{3abc - a^3 - b^3 - c^3} \begin{bmatrix} bc - a^2 & ac - b^2 & ab - c^2 \\ ac - b^2 & ab - c^2 & bc - a^2 \\ ab - c^2 & bc - a^2 & ac - b^2 \end{bmatrix}$$

□

Exercise 35. Find the adjugate, determinant and inverse of the matrix $H = \begin{bmatrix} 1 & 1/2 & 1/3 \\ 1/2 & 1/3 & 1/4 \\ 1/3 & 1/4 & 1/5 \end{bmatrix}$. (Note again that the intermediate calculations that you need for $\det(H)$ are a subset of those that you need for $\operatorname{adj}(H)$.)

Solution: The minors are

$$m_{11} = \det\begin{bmatrix} 1/3 & 1/4 \\ 1/4 & 1/5 \end{bmatrix} = \frac{1}{240} \qquad m_{12} = \det\begin{bmatrix} 1/2 & 1/4 \\ 1/3 & 1/5 \end{bmatrix} = \frac{1}{60} \qquad m_{13} = \det\begin{bmatrix} 1/2 & 1/3 \\ 1/3 & 1/4 \end{bmatrix} = \frac{1}{72}$$

$$m_{21} = \det\begin{bmatrix} 1/2 & 1/3 \\ 1/4 & 1/5 \end{bmatrix} = \frac{1}{60} \qquad m_{22} = \det\begin{bmatrix} 1 & 1/3 \\ 1/3 & 1/5 \end{bmatrix} = \frac{4}{45} \qquad m_{23} = \det\begin{bmatrix} 1 & 1/2 \\ 1/3 & 1/4 \end{bmatrix} = \frac{1}{12}$$

$$m_{31} = \det\begin{bmatrix} 1/2 & 1/3 \\ 1/3 & 1/4 \end{bmatrix} = \frac{1}{72} \qquad m_{32} = \det\begin{bmatrix} 1 & 1/3 \\ 1/2 & 1/4 \end{bmatrix} = \frac{1}{12} \qquad m_{33} = \det\begin{bmatrix} 1 & 1/2 \\ 1/2 & 1/3 \end{bmatrix} = \frac{1}{12}.$$

This gives

$$\operatorname{adj}(H) = \begin{bmatrix} m_{11} & -m_{21} & m_{31} \\ -m_{12} & m_{22} & -m_{32} \\ m_{13} & -m_{23} & m_{33} \end{bmatrix} = \begin{bmatrix} \frac{1}{240} & -\frac{1}{60} & \frac{1}{72} \\ -\frac{1}{60} & \frac{4}{45} & -\frac{1}{12} \\ \frac{1}{72} & -\frac{1}{12} & \frac{1}{12} \end{bmatrix}$$

$$\det(H) = H_{11}m_{11} - H_{12}m_{12} + H_{13}m_{13} = \frac{1}{240} - \frac{1}{2} \times \frac{1}{60} + \frac{1}{3} \times \frac{1}{72}$$

$$= \frac{1}{2160}$$

$$H^{-1} = \operatorname{adj}(H)/\det(H) = \begin{bmatrix} \frac{2160}{240} & -\frac{2160}{60} & \frac{2160}{72} \\ -\frac{2160}{60} & \frac{4\times 2160}{45} & -\frac{2160}{12} \\ \frac{2160}{72} & -\frac{2160}{12} & \frac{2160}{12} \end{bmatrix}$$

$$= \begin{bmatrix} 9 & -36 & 30 \\ -36 & 192 & -180 \\ 30 & -180 & 180 \end{bmatrix}.$$

□

8. Lecture 8

Exercise 36. Find the characteristic polynomial of the matrix

$$A = \begin{bmatrix} 0 & 0 & 0 & -d \\ 1 & 0 & 0 & -c \\ 0 & 1 & 0 & -b \\ 0 & 0 & 1 & -a \end{bmatrix}$$

Solution: The characteristic polynomial is the determinant of the matrix

$$A - tI = \begin{bmatrix} -t & 0 & 0 & -d \\ 1 & -t & 0 & -c \\ 0 & 1 & -t & -b \\ 0 & 0 & 1 & -a-t \end{bmatrix}.$$

Expanding along the top row, we get

$$\det(A - tI) = -t\det\begin{bmatrix} -t & 0 & -c \\ 1 & -t & -b \\ 0 & 1 & -a-t \end{bmatrix} + d\det\begin{bmatrix} 1 & -t & 0 \\ 0 & 1 & -t \\ 0 & 0 & 1 \end{bmatrix}.$$

The second matrix above is upper triangular and so the determinant is easily seen to be 1. For the first matrix we have

$$\det\begin{bmatrix} -t & 0 & -c \\ 1 & -t & -b \\ 0 & 1 & -a-t \end{bmatrix} = -t\det\begin{bmatrix} -t & -b \\ 1 & -a-t \end{bmatrix} - c\det\begin{bmatrix} 1 & -t \\ 0 & 1 \end{bmatrix} = -t(t^2+at+b)-c = -(t^3+at^2+bt+c).$$

Putting this together, we get

$$\det(A - tI) = t(t^3 + at^2 + bt + c) + d = t^4 + at^3 + bt^2 + ct + d.$$

□

Exercise 37. Find the characteristic polynomial, eigenvalues and all the corresponding eigenvectors of the matrix

$$A = \begin{bmatrix} 1 & 4 & 6 \\ 0 & 2 & 5 \\ 0 & 0 & 3 \end{bmatrix}$$

Solution: The characteristic polynomial is

$$\det(A - tI_3) = \det \begin{bmatrix} 1-t & 4 & 6 \\ 0 & 2-t & 5 \\ 0 & 0 & 3-t \end{bmatrix} = -(t-1)(t-2)(t-3).$$

(Recall that the determinant of an upper triangular 3×3 matrix is the product of the diagonal entries.) Hence the eigenvalues of A are 1, 2 and 3.

To find the eigenvectors corresponding to the eigenvalue 1, we solve the system of linear equations $(A - 1I_3)\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$. The augmented matrix of this system

$$\big(A - I_3|0\big) = \begin{bmatrix} 0 & 4 & 6 & 0 \\ 0 & 1 & 5 & 0 \\ 0 & 0 & 2 & 0 \end{bmatrix} \sim \begin{bmatrix} 0 & 2 & 3 & 0 \\ 0 & 1 & 5 & 0 \\ 0 & 0 & 2 & 0 \end{bmatrix}$$
$$\sim \begin{bmatrix} 0 & 2 & 3 & 0 \\ 0 & 0 & \frac{7}{2} & 0 \\ 0 & 0 & 2 & 0 \end{bmatrix} \sim \begin{bmatrix} 0 & 2 & 3 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

in row echelon form. We therefore see by back substitution that the general solution of the system is given by $z = 0$, $y = 0$, $x = \mu$ where μ can be any number; therefore the set of eigenvectors of A corresponding to the eigenvalue 1 is

$$\left\{ \mu \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} : 0 \neq \mu \in \mathbb{R} \right\}.$$

To find the eigenvectors corresponding to the eigenvalue 2, we solve the system of linear equations $(A - 2I_3)\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$. The augmented matrix of this system

$$\big(A - 2I_3|0\big) = \begin{bmatrix} -1 & 4 & 6 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} -1 & 4 & 6 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

in row echelon form. We therefore see by back substitution that the general solution of the system is given by $z = 0$, $y = \mu$, $x = 4\mu$ where μ can be any number; therefore the set of eigenvectors of A corresponding to the eigenvalue 2 is

$$\left\{ \mu \begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix} : 0 \neq \mu \in \mathbb{R} \right\}.$$

To find the eigenvectors corresponding to the eigenvalue 3, we solve the system of linear equations $(A - 3I_3)\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$. The augmented matrix of this system

$$\big(A - 3I_3|0\big) = \begin{bmatrix} -2 & 4 & 6 & 0 \\ 0 & -1 & 5 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

is in row echelon form. We therefore see by back substitution that the general solution of the system is given by $z = \mu$, $y = 5\mu$, $x = \frac{1}{2}(4(5\mu) + 6\mu) = 13\mu$, where μ can be any number; therefore the set of eigenvectors of A corresponding to the eigenvalue 3 is

$$\left\{ \mu \begin{bmatrix} 13 \\ 5 \\ 1 \end{bmatrix} : 0 \neq \mu \in \mathbb{R} \right\}.$$

□

Exercise 38. Find the characteristic polynomial, eigenvalues and all the corresponding eigenvectors of the matrix

$$B = \begin{bmatrix} 3 & 2 & 1 \\ 0 & 1 & 2 \\ 0 & 1 & -1 \end{bmatrix}$$

Solution: The characteristic polynomial is

$$\det(B - tI_3) = \det \begin{bmatrix} 3-t & 2 & 1 \\ 0 & 1-t & 2 \\ 0 & 1 & -1-t \end{bmatrix} = -(t-3)((1-t)(-1-t) - 2)$$
$$= -(t-3)(t^2 - 3) = -(t-3)(t-\sqrt{3})(t+\sqrt{3}).$$

Hence the eigenvalues of B are 3, $\sqrt{3}$ and $-\sqrt{3}$.

To find the eigenvectors corresponding to the eigenvalue 3, we solve the system of linear equations $(B - 3I_3)\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$. The augmented matrix of this system

$$\big(B - 3I_3|0\big) = \begin{bmatrix} 0 & 2 & 1 & 0 \\ 0 & -2 & 2 & 0 \\ 0 & 1 & -4 & 0 \end{bmatrix} \sim \begin{bmatrix} 0 & 2 & 1 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & -\frac{9}{2} & 0 \end{bmatrix} \sim \begin{bmatrix} 0 & 2 & 1 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

in row echelon form. We therefore see by back substitution that the general solution of the system is given by $z = 0$, $y = 0$, $x = \mu$ where μ can be any number; therefore the set of eigenvectors of B corresponding to the eigenvalue 1 is

$$\left\{ \mu \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} : 0 \neq \mu \in \mathbb{R} \right\}.$$

To find the eigenvectors corresponding to the eigenvalue $\sqrt{3}$, we solve the system of linear equations $(B - \sqrt{3}I_3)\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$. The augmented matrix of this system

$$\big(B - \sqrt{3}I_3|0\big) = \begin{bmatrix} 3-\sqrt{3} & 2 & 1 & 0 \\ 0 & 1-\sqrt{3} & 2 & 0 \\ 0 & 1 & -1-\sqrt{3} & 0 \end{bmatrix} \sim \begin{bmatrix} 3-\sqrt{3} & 2 & 1 & 0 \\ 0 & 1 & -1-\sqrt{3} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

in row echelon form. We therefore see by back substitution that the general solution of the system is given by $z = \mu$, $y = (1+\sqrt{3})\mu$, and

$$x = \left[\tfrac{(-2(1+\sqrt{3})-1)}{(3-\sqrt{3})}\right]\mu = \left[\tfrac{(-3-2\sqrt{3})(3+\sqrt{3})}{(3-\sqrt{3})(3+\sqrt{3})}\right]\mu = \left[\tfrac{-15-9\sqrt{3}}{6}\right]\mu = \left[\tfrac{-5-3\sqrt{3}}{2}\right]\mu,$$

where μ can be any number; therefore the set of eigenvectors of B corresponding to the eigenvalue $\sqrt{3}$ is

$$\left\{ \mu \begin{bmatrix} \frac{-5-3\sqrt{3}}{2} \\ 1+\sqrt{3} \\ 1 \end{bmatrix} : 0 \neq \mu \in \mathbb{R} \right\}.$$

To find the eigenvectors corresponding to the eigenvalue $-\sqrt{3}$, we solve the system of linear equations $(B+\sqrt{3}I_3)\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$. The augmented matrix of this system

$$\left(B+\sqrt{3}I_3|0\right) = \begin{bmatrix} 3+\sqrt{3} & 2 & 1 & 0 \\ 0 & 1+\sqrt{3} & 2 & 0 \\ 0 & 1 & -1+\sqrt{3} & 0 \end{bmatrix} \sim \begin{bmatrix} 3+\sqrt{3} & 2 & 1 & 0 \\ 0 & 1 & -1+\sqrt{3} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

in row echelon form. We therefore see by back substitution that the general solution of the system is given by $z = \mu$, $y = (1-\sqrt{3})\mu$, and

$$x = \left[\tfrac{(-2(1-\sqrt{3})-1)}{(3+\sqrt{3})}\right]\mu = \left[\tfrac{(-3+2\sqrt{3})(3-\sqrt{3})}{(3+\sqrt{3})(3-\sqrt{3})}\right]\mu = \left[\tfrac{-15+9\sqrt{3}}{6}\right]\mu = \left[\tfrac{-5+3\sqrt{3}}{2}\right]\mu,$$

where μ can be any number; therefore the set of eigenvectors of B corresponding to the eigenvalue $-\sqrt{3}$ is

$$\left\{ \mu \begin{bmatrix} \frac{-5+3\sqrt{3}}{2} \\ 1-\sqrt{3} \\ 1 \end{bmatrix} : 0 \neq \mu \in \mathbb{R} \right\}.$$

□

Exercise 39. Show, directly from the definition of eigenvalue, that 0 is an eigenvalue of the matrix

$$N := \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Show, also directly from the definition of eigenvalue, that an arbitrary non-zero number k is not an eigenvalue of N. Find all the eigenvectors of N.

Solution: A real number k is an eigenvalue of N if and only if the system of linear equations

$$(\dagger_k) \qquad (N-kI_4)\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = 0$$

has a non-trivial solution.

When $k=0$, the augmented matrix of $(\dagger_0)$ is

$$\left(N|0\right) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

and this is already in row echelon form. Thus we see by back substitution that the complete solution of $(\dagger_0)$ is $w=0$, $z=0$, $y=0$, $x=\mu$, where μ can be any number. Thus there is a non-trivial solution to $(\dagger_0)$, and so 0 is an eigenvalue of N. Also the set of eigenvectors of N corresponding to the eigenvalue 0 is

$$\left\{ \mu \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} : 0 \neq \mu \in \mathbb{R} \right\}.$$

Now consider the case where $k \neq 0$. Then the augmented matrix of $(\dagger_k)$ is

$$\left(N-kI_4|0\right) = \begin{bmatrix} k & 1 & 0 & 0 & 0 \\ 0 & k & 1 & 0 & 0 \\ 0 & 0 & k & 1 & 0 \\ 0 & 0 & 0 & k & 0 \end{bmatrix}$$

and this is already in row echelon form. We see, by back substitution, that (because $k \neq 0$) the complete solution of $(\dagger_k)$ is $w=0$, $z=0$, $y=0$, $x=0$. Thus the only solution of $(\dagger_k)$ is the trivial one, and therefore k is not an eigenvalue of N. □

Exercise 40. Find the characteristic polynomial, eigenvalues and eigenvectors of the matrix

$$A = \begin{bmatrix} 3 & -5 & 5 \\ 2 & -4 & 5 \\ 2 & -2 & 3 \end{bmatrix}.$$

Solution: The characteristic polynomial is

$$\det(A - tI_3) = \det \begin{bmatrix} 3-t & -5 & 5 \\ 2 & -4-t & 5 \\ 2 & -2 & 3-t \end{bmatrix} = \begin{vmatrix} 1-t & t-1 & 0 \\ 2 & -4-t & 5 \\ 0 & t+2 & -2-t \end{vmatrix}$$

(on subtracting the middle row from each of the other two rows)

$$= (1-t)(2+t) \begin{vmatrix} 1 & -1 & 0 \\ 2 & -4-t & 5 \\ 0 & 1 & -1 \end{vmatrix} = (1-t)(2+t) \begin{vmatrix} 1 & 0 & 0 \\ 2 & -2-t & 5 \\ 0 & 1 & -1 \end{vmatrix}$$

(on adding the first column to the second column)

$$= (1-t)(2+t)(2+t-5) = -(t-1)(t+2)(t-3).$$

Hence the eigenvalues of A are 1, -2 and 3.

To find the eigenvectors corresponding to the eigenvalue 1, we solve the system of linear equations $(A - 1I_3) \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$. The augmented matrix of this system

$$\big(A - I_3 | 0\big) = \begin{bmatrix} 2 & -5 & 5 & 0 \\ 2 & -5 & 5 & 0 \\ 2 & -2 & 2 & 0 \end{bmatrix} \sim \begin{bmatrix} 2 & -5 & 5 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 3 & -3 & 0 \end{bmatrix} \sim \begin{bmatrix} 2 & -5 & 5 & 0 \\ 0 & 3 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

in row echelon form. We therefore see by back substitution that the general solution of the system is given by $z = \mu$, $y = \mu$, $x = \frac{1}{2}(5\mu - 5\mu) = 0$ where μ can be any number; therefore the set of eigenvectors of A corresponding to the eigenvalue 1 is

$$\left\{ \mu \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} : 0 \neq \mu \in \mathbb{R} \right\}.$$

To find the eigenvectors corresponding to the eigenvalue -2, we solve the system of linear equations $(A + 2I_3) \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$. The augmented matrix of this system

$$\big(A + 2I_3 | 0\big) = \begin{bmatrix} 5 & -5 & 5 & 0 \\ 2 & -2 & 5 & 0 \\ 2 & -2 & 5 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

in row echelon form. We therefore see by back substitution that the general solution of the system is given by $z = 0$, $y = \mu$, $x = \mu$ where μ can be any number; therefore the set of eigenvectors of A corresponding to the eigenvalue -2 is

$$\left\{ \mu \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} : 0 \neq \mu \in \mathbb{R} \right\}.$$

To find the eigenvectors corresponding to the eigenvalue 3, we solve the system of linear equations $(A - 3I_3) \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$. The augmented matrix of this system

$$\big(A - 3I_3 | 0\big) = \begin{bmatrix} 0 & -5 & 5 & 0 \\ 2 & -7 & 5 & 0 \\ 2 & -2 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 2 & -2 & 0 & 0 \\ 2 & -7 & 5 & 0 \\ 0 & -5 & 5 & 0 \end{bmatrix} \sim \begin{bmatrix} 2 & -2 & 0 & 0 \\ 0 & -5 & 5 & 0 \\ 0 & -5 & 5 & 0 \end{bmatrix} \sim \begin{bmatrix} 2 & -2 & 0 & 0 \\ 0 & -5 & 5 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

in row echelon form. We therefore see by back substitution that the general solution of the system is given by $z = \mu$, $y = \mu$, $x = \mu$, where μ can be any number; therefore the set of eigenvectors of A

corresponding to the eigenvalue 3 is

$$\left\{ \mu \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} : 0 \neq \mu \in \mathbb{R} \right\}.$$

□

Exercise 41. Let A be an $n \times n$ matrix, and let $\lambda_1, \ldots, \lambda_h$ be h distinct eigenvalues of A. For each $i = 1, \ldots, h$, let the vectors $v_{i,1}, \ldots, v_{i,t_i}$ be linearly independent eigenvectors of A all corresponding to the eigenvalue λ_i. We collect these lists together into a single list

$$v_{1,1}, \ldots, v_{1,t_1}, v_{2,1}, \ldots, v_{2,t_2}, \ldots, v_{h,1}, \ldots, v_{h,t_h}.$$

Prove (as was stated in lectures) that this list is linearly independent.

Solution: For each $i = 1, \ldots, h$, the vectors $v_{i,1}, \ldots, v_{i,t_i}$ are linearly independent eigenvectors of A all corresponding to the eigenvalue λ_i. We show that

$$v_{1,1}, \ldots, v_{1,t_1}, v_{2,1}, \ldots, v_{2,t_2}, \ldots, v_{h,1}, \ldots, v_{h,t_h}$$

(taken all together) are linearly independent by induction on h.

When $h = 1$, there is nothing to prove, because we are given that $v_{1,1}, \ldots, v_{1,t_1}$ are linearly independent.

Assume now that $h > 1$ and that the claim is true for $h - 1$ distinct eigenvalues of A.

Let

$$a_{1,1}, \ldots, a_{1,t_1}, a_{2,1}, \ldots, a_{2,t_2}, \ldots, a_{h,1}, \ldots, a_{h,t_h}$$

be scalars such that

$$\sum_{i=1}^{h} \sum_{j=1}^{t_i} a_{i,j} v_{i,j} = 0. \tag{1}$$

Then

$$0 = A0 = A \left[\sum_{i=1}^{h} \sum_{j=1}^{t_i} a_{i,j} v_{i,j} \right] = \sum_{i=1}^{h} \sum_{j=1}^{t_i} a_{i,j} A v_{i,j}$$

and so

$$\sum_{i=1}^{h} \sum_{j=1}^{t_i} a_{i,j} \lambda_i v_{i,j} = 0 \tag{2}$$

because $A v_{i,j} = \lambda_i v_{i,j}$ for all $j = 1, \ldots, t_i$ and $i = 1, \ldots, h$. If we now subtract λ_h times (1) from (2) we get

$$\sum_{i=1}^{h} \sum_{j=1}^{t_i} a_{i,j} (\lambda_i - \lambda_h) v_{i,j} = 0,$$

that is

$$\sum_{i=1}^{h-1} \sum_{j=1}^{t_i} a_{i,j} (\lambda_i - \lambda_h) v_{i,j} = 0$$

(since the addends for $i = h$ are zero).

By the induction hypothesis,

$$v_{1,1}, \ldots, v_{1,t_1}, v_{2,1}, \ldots, v_{2,t_2}, \ldots, v_{h-1,1}, \ldots, v_{h-1,t_{h-1}}$$

(taken all together) are linearly independent. Therefore

$$a_{i,j}(\lambda_i - \lambda_h) = 0 \quad \text{for all } j = 1, \ldots, t_i \text{ and } i = 1, \ldots, h-1.$$

Since $\lambda_i - \lambda_h \neq 0$ for all $i = 1, \ldots, h - 1$, it follows that

$$a_{i,j} = 0 \quad \text{for all } j = 1, \ldots, t_i \text{ and } i = 1, \ldots, h-1.$$

With this information, equation (1) now simplifies to

$$\sum_{j=1}^{t_h} a_{h,j} v_{h,j} = 0,$$

and so it follows from the fact that $v_{h,1}, \dots, v_{h,t_h}$ are linearly independent that $a_{h,1} = \cdots = a_{h,t_h} = 0$. Thus, $a_{i,j} = 0$ for all $j = 1, \dots, t_i$ and all $i = 1, \dots, h$.

We have now shown that

$$v_{1,1}, \dots, v_{1,t_1}, v_{2,1}, \dots, v_{2,t_2}, \dots, v_{h,1}, \dots, v_{h,t_h}$$

are linearly independent. This completes the inductive step. By the Principle of Mathematical Induction, the claim is proved.

□

9. Lecture 9

Exercise 42. Consider the matrix $A = \begin{bmatrix} 4 & 1 \\ -6 & 9 \end{bmatrix}$. Find an invertible matrix U and a diagonal matrix D such that $A = UDU^{-1}$. Check directly that the equation $A = UDU^{-1}$ holds.

Solution: The characteristic polynomial is

$$\chi_A(t) = \det \begin{bmatrix} 4-t & 1 \\ -6 & 9-t \end{bmatrix} = (4-t)(9-t) - (-6) = t^2 - 13t + 42 = (t-6)(t-7).$$

Thus, the eigenvalues are $\lambda_1 = 6$ and $\lambda_2 = 7$. To find the corresponding eigenvectors we use the following row-reductions:

$$A - \lambda_1 I = \begin{bmatrix} -2 & 1 \\ -6 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & -1/2 \\ -6 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & -1/2 \\ 0 & 0 \end{bmatrix} = B_1$$
$$A - \lambda_2 I = \begin{bmatrix} -3 & 1 \\ -6 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & -1/3 \\ -6 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & -1/3 \\ 0 & 0 \end{bmatrix} = B_2$$

The eigenvector u_1 must satisfy $B_1 u_1 = 0$, and it is clear that $u_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ will do. Similarly, the eigenvector u_2 must satisfy $B_2 u_2 = 0$, and it is clear that $u_2 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ will do. We now take

$$U = [u_1 | u_2] = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}$$
$$D = \operatorname{diag}(\lambda_1, \lambda_2) = \begin{bmatrix} 6 & 0 \\ 0 & 7 \end{bmatrix}.$$

The general method (Proposition 14.4) tells us that $A = UDU^{-1}$. To check this directly, we need to work out U^{-1}. The general formula

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

gives

$$U^{-1} = \frac{1}{3-2} \begin{bmatrix} 3 & -1 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ -2 & 1 \end{bmatrix}.$$

We thus have

$$UDU^{-1} = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 6 & 0 \\ 0 & 7 \end{bmatrix} \begin{bmatrix} 3 & -1 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 18 & -6 \\ -14 & 7 \end{bmatrix} = \begin{bmatrix} 4 & 1 \\ -6 & 9 \end{bmatrix}.$$

As expected, this is the same as A. □

Exercise 43. Show that the matrix $A = \begin{bmatrix} 4 & 1 \\ -1 & 2 \end{bmatrix}$ cannot be diagonalised.

Solution: The characteristic polynomial is

$$\det(A - tI) = \det \begin{bmatrix} 4-t & 1 \\ -1 & 2-t \end{bmatrix} = (4-t)(2-t) + 1 = 9 - 6t + t^2 = (t-3)^2.$$

This shows that the only eigenvalue is 3. The eigenvectors of eigenvalue 3 are the vectors $u = \begin{bmatrix} x \\ y \end{bmatrix}$ satisfying $(A - 3I)u = 0$. Here $A - 3I = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$, so $(A - 3I)u = \begin{bmatrix} x + y \\ -x - y \end{bmatrix}$. This means that

u is an eigenvector iff $x+y=0$, or in other words $u=\begin{bmatrix}x\\-x\end{bmatrix}=x\begin{bmatrix}1\\-1\end{bmatrix}$. As every eigenvector is a nonzero multiple of $\begin{bmatrix}1\\-1\end{bmatrix}$, we see that any two eigenvectors are multiples of each other and so are linearly dependent. Thus, there is no basis of eigenvectors. Proposition 14.4 therefore tells us that A cannot be diagonalised. □

Exercise 44. Consider the matrix

$$A=\begin{bmatrix}100 & 10 & 1\\ 100 & 10 & 1\\ 100 & 10 & 1\end{bmatrix}.$$

Find a basis for $\mathbb{R}^3$ consisting of eigenvectors for A. Using this, find a diagonalisation $A=UDU^{-1}$.

Solution: The characteristic polynomial is as follows.

$$\begin{aligned}\chi_A(t)&=\det\begin{bmatrix}100-t & 10 & 1\\ 100 & 10-t & 1\\ 100 & 10 & 1-t\end{bmatrix}\\ &=(100-t)\det\begin{bmatrix}10-t & 1\\ 10 & 1-t\end{bmatrix}-10\det\begin{bmatrix}100 & 1\\ 100 & 1-t\end{bmatrix}+\det\begin{bmatrix}100 & 10-t\\ 100 & 10\end{bmatrix}\\ &=(100-t)(t^2-11t)-10(-100t)+(100t)=-t^3+111t^2=-t^2(t-111).\end{aligned}$$

It follows that the eigenvalues are 0 and 111. The eigenvectors of eigenvalue 0 are the vectors $u=\begin{bmatrix}x & y & z\end{bmatrix}^T$ satisfying $Au=0$ or equivalently $100x+10y+z=0$. This gives $z=-100x-10y$, so

$$u=\begin{bmatrix}x\\ y\\ -100x-10y\end{bmatrix}=x\begin{bmatrix}1\\ 0\\ -100\end{bmatrix}+y\begin{bmatrix}0\\ 1\\ -10\end{bmatrix}.$$

Taking $x=1$ and $y=0$ gives $u_1=\begin{bmatrix}1 & 0 & -100\end{bmatrix}^T$. Taking $x=0$ and $y=1$ gives $u_2=\begin{bmatrix}0\\ 1\\ -10\end{bmatrix}$. These are two linearly independent eigenvectors of eigenvalue zero.

Next, to find an eigenvector of eigenvalue 111 we row-reduce the matrix $A-111I$. If we row-reduce in the obvious way we get the following sequence:

$$\begin{bmatrix}1 & -10/11 & -1/11\\ 100 & -101 & 1\\ 100 & 10 & -110\end{bmatrix}\to\begin{bmatrix}1 & -10/11 & -1/11\\ 0 & -111/11 & 111/11\\ 100 & 10 & -110\end{bmatrix}\to\begin{bmatrix}1 & -10/11 & -1/11\\ 0 & -111/11 & 111/11\\ 0 & 1110/11 & -1110/11\end{bmatrix}\to$$

$$\begin{bmatrix}1 & -10/11 & -1/11\\ 0 & 1 & -1\\ 0 & 1110/11 & -1110/11\end{bmatrix}\to\begin{bmatrix}1 & -10/11 & -1/11\\ 0 & 1 & -1\\ 0 & 0 & 0\end{bmatrix}\to\begin{bmatrix}1 & 0 & -1\\ 0 & 1 & -1\\ 0 & 0 & 0\end{bmatrix}$$

If we proceed in a more creative order we can avoid fractions:

$$\begin{bmatrix}-11 & 10 & 1\\ 100 & -101 & 1\\ 100 & 10 & -110\end{bmatrix}\to\begin{bmatrix}-11 & 10 & 1\\ 100 & -101 & 1\\ 0 & 111 & -111\end{bmatrix}\to\begin{bmatrix}-11 & 10 & 1\\ 100 & -101 & 1\\ 0 & 1 & -1\end{bmatrix}\to$$

$$\begin{bmatrix}-11 & 0 & 11\\ 100 & 0 & -100\\ 0 & 1 & -1\end{bmatrix}\to\begin{bmatrix}1 & 0 & -1\\ 0 & 0 & 0\\ 0 & 1 & -1\end{bmatrix}\to\begin{bmatrix}1 & 0 & -1\\ 0 & 1 & -1\\ 0 & 0 & 0\end{bmatrix}$$

Either way, we get the same final matrix B. An eigenvector $u=\begin{bmatrix}x & y & z\end{bmatrix}^T$ of eigenvalue 111 must satisfy $Bu=0$, which means that $x=z$ and $y=z$. Thus, we can take $u_3=\begin{bmatrix}1 & 1 & 1\end{bmatrix}^T$. In fact, if we were sufficiently alert we could have seen that this vector satisfies $Au_3=111u_3$ by inspection, and avoided the whole row-reduction process. We now put

$$U = \left[\begin{array}{c|c|c} u_1 & u_2 & u_3 \end{array}\right] = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ -100 & -10 & 1 \end{bmatrix}$$

$$D = \operatorname{diag}(\lambda_1, \lambda_2, \lambda_3) = \operatorname{diag}(0, 0, 111) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 111 \end{bmatrix}.$$

The general theory now tells us that $A = UDU^{-1}$. It would not be hard to check this directly, but the question does not ask us to do so. We just record the value of U^{-1} for any students who wish to check their work:

$$U^{-1} = \frac{1}{111} \begin{bmatrix} 11 & -10 & -1 \\ -100 & 101 & -1 \\ 100 & 10 & 1 \end{bmatrix}.$$

□

Exercise 45. Consider the matrix

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}.$$

Find a basis for $\mathbb{R}^4$ consisting of eigenvectors for A. Using this, find a diagonalisation $A = UDU^{-1}$. Ideally, you should do all this by inspection rather than using the characteristic polynomial and row-reduction.

Solution: In terms of the standard basis vectors e_i, we have

$$Ae_1 = e_3 \qquad Ae_2 = e_4 \qquad Ae_3 = e_1 \qquad Ae_4 = e_2.$$

It follows that if we put

$$u_1 = e_1 + e_3 \qquad u_2 = e_2 + e_4 \qquad u_3 = e_1 - e_3 \qquad u_4 = e_2 - e_4$$

then

$$Au_1 = u_1 \qquad Au_2 = u_2 \qquad Au_3 = u_3 \qquad Au_4 = u_4,$$

so the vectors u_i are eigenvectors, with eigenvalues $\lambda_1 = \lambda_2 = 1$ and $\lambda_3 = \lambda_4 = -1$. Thus, if we put

$$U = [u_1|u_2|u_3|u_4] = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix} \qquad D = \operatorname{diag}(\lambda_1, \lambda_2, \lambda_3, \lambda_4) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

then we have $A = UDU^{-1}$. Also, it is not hard to see that $U^2 = 2I_4$, so $U^{-1} = \frac{1}{2}U$. □

Exercise 46. Diagonalise the matrix $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$.

Hint: One of the eigenvalues, and the corresponding eigenvector, involves $\sqrt{3}$. You can find another eigenvalue and eigenvector by just changing $\sqrt{3}$ to $-\sqrt{3}$ everywhere. You may also find it useful to remember the rule

$$\frac{1}{a + b\sqrt{3}} = \frac{a - b\sqrt{3}}{(a - b\sqrt{3})(a + b\sqrt{3})} = \frac{a - b\sqrt{3}}{a^2 - 3b^2}.$$

Solution: The characteristic polynomial is

$$\begin{aligned}
\chi_A(t) &= \det \begin{bmatrix} 1-t & 1 & 1 \\ 1 & -t & 1 \\ 1 & 1 & 1-t \end{bmatrix} \\
&= (1-t)\det \begin{bmatrix} -t & 1 \\ 1 & 1-t \end{bmatrix} - \det \begin{bmatrix} 1 & 1 \\ 1 & 1-t \end{bmatrix} + \det \begin{bmatrix} 1 & -t \\ 1 & 1 \end{bmatrix} \\
&= (1-t)(t^2 - t - 1) - (-t) + (1+t) = t^2 - t - 1 - t^3 + t^2 + t + t + 1 + t = -t^3 + 2t^2 + 2t \\
&= -t(t^2 - 2t - 2).
\end{aligned}$$

The quadratic formula tells that the roots of $t^2 - 2t - 2$ are $1 \pm \sqrt{3}$. Thus, the eigenvalues are $\lambda_1 = 0$ and $\lambda_2 = 1 + \sqrt{3}$ and $\lambda_3 = 1 - \sqrt{3}$. By inspection, the vector $u_1 = \begin{bmatrix} 1 & 0 & -1 \end{bmatrix}^T$ satisfies $Au_1 = 0$, so it is an eigenvector of eigenvalue 0. To find an eigenvector of eigenvalue $\lambda_2 = 1 + \sqrt{3}$, we row-reduce the matrix $A - \lambda_2 I$:

$$\begin{bmatrix} -\sqrt{3} & 1 & 1 \\ 1 & -1-\sqrt{3} & 1 \\ 1 & 1 & -\sqrt{3} \end{bmatrix} \xrightarrow{1} \begin{bmatrix} -\sqrt{3} & 1 & 1 \\ 1 & -1-\sqrt{3} & 1 \\ 0 & 2+\sqrt{3} & -1-\sqrt{3} \end{bmatrix} \xrightarrow{2} \begin{bmatrix} 0 & -2-\sqrt{3} & 1+\sqrt{3} \\ 1 & -1-\sqrt{3} & 1 \\ 0 & 2+\sqrt{3} & -1-\sqrt{3} \end{bmatrix} \xrightarrow{3}$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 1 & -1-\sqrt{3} & 1 \\ 0 & 2+\sqrt{3} & -1-\sqrt{3} \end{bmatrix} \xrightarrow{4} \begin{bmatrix} 0 & 0 & 0 \\ 1 & -1-\sqrt{3} & 1 \\ 0 & 1 & 1-\sqrt{3} \end{bmatrix} \xrightarrow{5} \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 1-\sqrt{3} \end{bmatrix} \xrightarrow{6} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1-\sqrt{3} \\ 0 & 0 & 0 \end{bmatrix}$$

The steps are as follows:

(1) Subtract row 2 from row 3.
(2) Add $\sqrt{3}$ times row 2 to row 1.
(3) Add row 3 to row 1.
(4) We now want to divide row 3 by $2+\sqrt{3}$. Taking $a = 2$ and $b = 1$ in the equation for $1/(a+b\sqrt{3})$ we get $1/(2+\sqrt{3}) = 2-\sqrt{3}$. We therefore multiply row 3 by $2-\sqrt{3}$.
(5) Add $1+\sqrt{3}$ times row 3 to row 2.
(6) Reorder the rows.

We conclude that an eigenvector $u_2 = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ of eigenvalue $1+\sqrt{3}$ must satisfy $x - z = 0$ and $y + (1-\sqrt{3})z = 0$. Taking $z = 1$ we get $u_2 = \begin{bmatrix} 1 & -1+\sqrt{3} & 1 \end{bmatrix}^T$. Finally, following the hint we see that the final eigenvector u_3 is just $\begin{bmatrix} 1 & -1-\sqrt{3} & 1 \end{bmatrix}$ (obtained by changing the $\sqrt{3}$ in u_2 to $-\sqrt{3}$). We now have a diagonalisation $A = UDU^{-1}$, where

$$U = \left[\begin{array}{c|c|c} u_1 & u_2 & u_3 \end{array}\right] = \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1+\sqrt{3} & -1-\sqrt{3} \\ -1 & 1 & 1 \end{bmatrix}$$

$$D = \operatorname{diag}(\lambda_1, \lambda_2, \lambda_3) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1+\sqrt{3} & 0 \\ 0 & 0 & -1-\sqrt{3} \end{bmatrix}.$$

□

10. Lecture 10

Exercise 47. Let A be the 5×5 matrix in which every entry is 1.

(a) Show that $A^2 = 5A$.
(b) Suppose that λ is an eigenvalue of A, so there exists a nonzero vector u with $Au = \lambda u$. By considering $A^2 u$, show that $\lambda^2 = 5\lambda$, so $\lambda = 0$ or $\lambda = 5$. (You should not write out any matrices here, or attempt to calculate the characteristic polynomial; just use part (a).)
(c) Find an eigenvector v of eigenvalue 5, and a linearly independent list $w_1, \dots, w_4$ of eigenvectors of eigenvalue 0.
(d) Now put $B = \frac{1}{2} I_5 + \frac{1}{10} A$. Show that B is stochastic.
(e) Prove by induction on k that $B^k = 2^{-k} I_5 + (1 - 2^{-k}) A/5$ for all $k \geq 0$. (You should not write out any matrices here; just use part (a).) What happens when k is large?

Solution:

(a) One way to say this is to introduce the vector $v = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \end{bmatrix}^T$, so $v.v = 5$. We also have

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} = \left[\begin{array}{c} v^T \\ \hline v^T \\ \hline v^T \\ \hline v^T \\ \hline v^T \end{array}\right] = \left[\begin{array}{c|c|c|c|c} v & v & v & v & v \end{array}\right]$$

so

$$A^2 = \begin{bmatrix} v.v & v.v & v.v & v.v & v.v \\ v.v & v.v & v.v & v.v & v.v \\ v.v & v.v & v.v & v.v & v.v \\ v.v & v.v & v.v & v.v & v.v \\ v.v & v.v & v.v & v.v & v.v \end{bmatrix} = \begin{bmatrix} 5 & 5 & 5 & 5 & 5 \\ 5 & 5 & 5 & 5 & 5 \\ 5 & 5 & 5 & 5 & 5 \\ 5 & 5 & 5 & 5 & 5 \\ 5 & 5 & 5 & 5 & 5 \end{bmatrix} = 5A.$$

(b) Suppose we have an eigenvalue λ, and an associated eigenvector u (so $u \neq 0$ and $Au = \lambda u$). We then have

$$A^2u = A(Au) = A(\lambda u) = \lambda Au = \lambda^2 u.$$

On the other hand, we have $A^2 = 5A$, so

$$A^2u = 5Au = 5\lambda u.$$

By comparing these two equations, we see that $\lambda^2 u = 5\lambda u$, so $(\lambda^2 - 5\lambda)u = 0$ or $\lambda(\lambda - 5)u = 0$. As $u \neq 0$ it follows that $\lambda = 0$ or $\lambda = 5$.

(c) Put

$$v = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \qquad w_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad w_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} \qquad w_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \\ 0 \end{bmatrix} \qquad w_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}.$$

It is easy to see that $Av = 5v$ and $Aw_i = 0$ for all i, so v is an eigenvector of eigenvalue 5, and $w_1, \dots, w_4$ are eigenvectors of eigenvalue 0. It is also clear that the list $w_1, \dots, w_4$ is linearly independent. This is not the only possible answer. For example, the list

$$w_1' = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ -1 \end{bmatrix} \qquad w_2' = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ -1 \end{bmatrix} \qquad w_3' = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ -1 \end{bmatrix} \qquad w_4' = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}$$

would do equally well.

(d) In the metrix B, every entry away from the diagonal is $\frac{1}{10}$, and every entry on the diagonal is $\frac{1}{10} + \frac{1}{2} = \frac{6}{10}$. In particular, all entries are positive. Moreover, each column contains four entries equal to $\frac{1}{10}$ and one entry equal to $\frac{6}{10}$, adding up to $(4 \times 1 + 6)/10 = 1$. Thus, the matrix is stochastic.

(e) We claim that for all $k \geq 0$ we have $B^k = 2^{-k}I_5 + (1 - 2^{-k})A/5$. When $k = 0$ the left hand side is $B^0 = I_5$, whereas the right hand side is $2^0 I_5 + (1 - 2^0)A = I_5$, as required. When $k = 1$ the left hand side is $B^1 = B = \frac{1}{2}I_5 + \frac{1}{10}A$. We also have $2^{-1} = 1 - 2^{-1} = \frac{1}{2}$ so on the right hand side we have $\frac{1}{2}I_5 + \frac{1}{10}A$ again, as required.

Now suppose that the claim is true for a particular value of k. We can the equation $B = \frac{1}{2}I_5 + \frac{1}{10}A$ by the equation $B^k = 2^{-k}I_5 + (1 - 2^{-k})A/5$ and expand out to get

$$\begin{aligned} B^{k+1} &= (\tfrac{1}{2}I_5 + \tfrac{1}{10}A)(2^{-k}I_5 + (1 - 2^{-k})A/5) \\ &= \tfrac{1}{2}2^{-k}I_5 + \tfrac{1}{2}\tfrac{1}{5}(1 - 2^{-k})A + \tfrac{1}{10}2^{-k}A + \tfrac{1}{10}\tfrac{1}{5}(1 - 2^{-k})A^2. \end{aligned}$$

Using $A^2 = 5A$ this becomes

$$\begin{aligned} B^{k+1} &= \tfrac{1}{2}2^{-k}I_5 + \tfrac{1}{2}\tfrac{1}{5}(1 - 2^{-k})A + \tfrac{1}{10}2^{-k}A + \tfrac{1}{10}(1 - 2^{-k})A \\ &= 2^{-k-1}I_5 + \left(\tfrac{1}{2}(1 - 2^{-k}) + \tfrac{1}{2}2^{-k} + \tfrac{1}{2}(1 - 2^{-k})\right) A/5 \\ &= 2^{-k-1}I_5 + \left(\tfrac{1}{2} - 2^{-k-1} + 2^{-k-1} + \tfrac{1}{2} - 2^{-k-1})\right) A/5 \\ &= 2^{-k-1}I_5 + (1 - 2^{-k-1})A/5. \end{aligned}$$

This is the case $k + 1$ of our claim. It follows by induction that the claim holds for all k.

□

Exercise 48. Show that the matrix $A = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 2 & 1 \\ -1 & 0 & 3 \end{bmatrix}$ cannot be diagonalised.

Hint: the eigenvalues are small integers.

Solution: The characteristic polynomial is

$$\chi_A(t) = \det \begin{bmatrix} 1-t & 1 & 0 \\ -1 & 2-t & 1 \\ -1 & 0 & 3-t \end{bmatrix} = (1-t)\det \begin{bmatrix} 2-t & 1 \\ 0 & 3-t \end{bmatrix} - \det \begin{bmatrix} -1 & 1 \\ -1 & 3-t \end{bmatrix}$$
$$= (1-t)(2-t)(3-t) - (t-2) = (2-t)((1-t)(3-t)+1)$$
$$= (2-t)(4-4t+t^2) = (2-t)^3.$$

(If we had not spotted that $2-t$ was a common factor and had just expanded everything out, we would have found that $\chi_A(t) = -t^3 + 6t^2 - 12t + 8$. Using the hint we could have tried various small integers and found that $\chi_A(2) = 0$, then we could have divided $\chi_A(t)$ by $t-2$ to get $-t^2+4t-4$, then we could have used the quadratic formula to see that 2 is the only root.)

We now see that 2 is the only eigenvalue of A. To find the eigenvectors, we row-reduce $A - 2I$:

$$\begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ -1 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 0 \\ 0 & -1 & 1 \\ 0 & -1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

From this we see that the eigenvectors of eigenvalue 2 are just the nonzero vectors of the form $u = \begin{bmatrix} x & x & x \end{bmatrix}^T$. In particular, any two eigenvectors are multiples of each other, and so are linearly dependent. It follows that there is no basis of eigenvectors, so the matrix cannot be diagonalised. □

Exercise 49. Consider the matrix

$$A = \frac{1}{16}\begin{bmatrix} 10 & 2 & 2 \\ 3 & 11 & 7 \\ 3 & 3 & 7 \end{bmatrix}.$$

For this matrix it turns out that the powers A^n converge to a limit as $n \to \infty$. Use Maple to find a diagonalisation $A = UDU^{-1}$, then find the limit of D^n as $n \to \infty$, then find the limit of A^n.

Solution: We enter the definition of A and find the eigenvectors as follows:

```
with(LinearAlgebra):

A := <<10|2|2>,<3|11|7>,<3|3|7>>/16;

L,U := Eigenvectors(A);
```

Maple responds by printing

$$L, U := \begin{bmatrix} 1 \\ 1/4 \\ 1/2 \end{bmatrix}, \begin{bmatrix} 1 & 0 & -1 \\ 2 & -1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

This indicates that the eigenvalues are $\lambda_1 = 1$, $\lambda_2 = \frac{1}{4}$ and $\lambda_3 = \frac{1}{2}$, and the corresponding eigenvectors are the columns of the above matrix, namely

$$u_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \qquad u_2 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \qquad u_3 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}.$$

We therefore have a diagonalisation $A = UDU^{-1}$, where

$$U = \begin{bmatrix} 1 & 0 & -1 \\ 2 & -1 & 1 \\ 1 & 1 & 0 \end{bmatrix} \qquad D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/4 & 0 \\ 0 & 0 & 1/2 \end{bmatrix}.$$

We can calculate the inverse of U by entering `U^(-1)` in Maple; we find that

$$U^{-1} = \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 \\ -1 & -1 & 3 \\ -3 & 1 & 1 \end{bmatrix}$$

This gives

$$\lim_{n\to\infty} D^n = \lim_{n\to\infty} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/4^n & 0 \\ 0 & 0 & 1/2^n \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

We will call this matrix D^∞. As $A^n = UD^nU^{-1}$ we see that

$$\lim_{n\to\infty} A^n = UD^\infty U^{-1} = \frac{1}{4}\begin{bmatrix}1 & 0 & -1\\ 2 & -1 & 1\\ 1 & 1 & 0\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}1 & 1 & 1\\ -1 & -1 & 3\\ -3 & 1 & 1\end{bmatrix}$$
$$= \begin{bmatrix}0.25 & 0 & -0.25\\ 0.5 & -0.25 & 0.25\\ 0.25 & 0.25 & 0\end{bmatrix}\begin{bmatrix}1 & 1 & 1\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix} = \begin{bmatrix}0.25 & 0.25 & 0.25\\ 0.5 & 0.5 & 0.5\\ 0.25 & 0.25 & 0.25\end{bmatrix}.$$

As a check, we can enter `evalf(A^10)` in Maple to calculate a numerical approximation to A^{10}, which is

$$\begin{bmatrix}0.2507324219 & 0.2497558594 & 0.2497558594\\ 0.4992678165 & 0.5002443790 & 0.5002434254\\ 0.2499997616 & 0.2499997616 & 0.2500007153\end{bmatrix}.$$

This is already quite close to the limiting value. □

Exercise 50. Consider the matrix

$$A = \begin{bmatrix}1 & 1 & 1\\ 0 & 1 & 1\\ 0 & 0 & 1\end{bmatrix}.$$

You may assume that this matrix cannot be diagonalised. Nonetheless, the powers A^n follow a simple pattern. Calculate A^n for some small values of n, then see if you can find the general rule, then prove it by induction.

Solution: The first few powers are as follows:

$$A^0 = \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix} \qquad A^1 = \begin{bmatrix}1 & 1 & 1\\ 0 & 1 & 1\\ 0 & 0 & 1\end{bmatrix}$$
$$A^2 = \begin{bmatrix}1 & 2 & 3\\ 0 & 1 & 2\\ 0 & 0 & 1\end{bmatrix} \qquad A^3 = \begin{bmatrix}1 & 3 & 6\\ 0 & 1 & 3\\ 0 & 0 & 1\end{bmatrix}$$
$$A^4 = \begin{bmatrix}1 & 4 & 10\\ 0 & 1 & 4\\ 0 & 0 & 1\end{bmatrix} \qquad A^5 = \begin{bmatrix}1 & 5 & 15\\ 0 & 1 & 5\\ 0 & 0 & 1\end{bmatrix}.$$

From this it is at least clear that

$$A^n = \begin{bmatrix}1 & n & p_n\\ 0 & 1 & n\\ 0 & 0 & 1\end{bmatrix}$$

for some number p_n. The remaining problem is to find a formula for p_n. The first few cases are

$$p_0 = 0 \qquad p_1 = 1 \qquad p_2 = 3 \qquad p_3 = 6 \qquad p_4 = 10 \qquad p_5 = 15.$$

You might recognise these numbers as coming from Pascal's triangle, or you might notice that $p_n - p_{n-1} = n$ and work from there, or you might notice that p_n is approximately $n^2/2$ and so study $p_n - n^2/2$, or you might enter the above numbers in the Online Encyclopedia of Integer Sequences at `http://oeis.org` and see what it finds. By any of these means you can arrive at the formula

$$p_n = \binom{n+1}{2} = (n^2+n)/2.$$

We thus conclude that

$$A^n = \begin{bmatrix}1 & n & (n^2+n)/2\\ 0 & 1 & n\\ 0 & 0 & 1\end{bmatrix}.$$

We can prove this formally by induction. The claim is clearly true for $n = 0$. If it holds for a particular value of n, then we have

$$A^{n+1} = AA^n = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & n & (n^2+n)/2 \\ 0 & 1 & n \\ 0 & 0 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & n+1 & (n^2+n)/2+n+1 \\ 0 & 1 & n+1 \\ 0 & 0 & 1 \end{bmatrix}.$$

Here

$$(n^2+n)/2+n+1 = n^2/2+3n/2+1 = ((n+1)^2+(n+1))/2,$$

so we see that the claim also holds for $n+1$. Thus, by induction, it holds for all natural numbers n. □

11. Lecture 11

Exercise 51. Solve the following system of differential equations using the method in Section 15:

$$\dot{x}_1 = 0.2x_1 + 0.5x_2 + 0.3x_3$$
$$\dot{x}_2 = 0.6x_1 + 0.6x_2 + 0.7x_3$$
$$\dot{x}_3 = 0.1x_1 + 0.4x_2 + 0.8x_3,$$

with $x = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^T$ at $t = 0$. You should use Maple to calculate the relevant eigenvalues and eigenvectors. Unlike most examples in this course, this one has not been fine-tuned to work out with nice round numbers.

Solution: We have $\dot{x} = Ax$ and $x = c$ at $t = 0$, where

$$A = \begin{bmatrix} 0.2 & 0.5 & 0.3 \\ 0.6 & 0.6 & 0.7 \\ 0.1 & 0.4 & 0.8 \end{bmatrix} \qquad c = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

The general method is to diagonalise A as UDU^{-1} with $D = \text{diag}(\lambda_1, \lambda_2, \lambda_3)$ say, then $x = UEU^{-1}c$, where $E = \text{diag}(e^{\lambda_1 t}, e^{\lambda_2 t}, e^{\lambda_3 t})$. We can do this in Maple as follows:

```
with(LinearAlgebra):
unprotect('D'):

A := <<0.2|0.5|0.3>,<0.6|0.6|0.7>,<0.1|0.4|0.8>>;
L,U := Eigenvectors(A);
D := DiagonalMatrix(L);
E := DiagonalMatrix([exp(L[1]*t),exp(L[2]*t),exp(L[3]*t)]);
c := <1,0,0>;
x := U . E . U^(-1);
```

Maple responds with

$$x := \begin{bmatrix} 0.1471732926\,e^{1.442698079\,t} + 0.5641142246\,e^{-0.2096633632\,t} + 0.2887124828\,e^{0.3669652806\,t} \\ 0.2563257383\,e^{1.442698079\,t} - 0.5623411149\,e^{-0.2096633632\,t} + 0.3060153766\,e^{0.3669652806\,t} \\ 0.1824303322\,e^{1.442698079\,t} + 0.1669120914\,e^{-0.2096633632\,t} - 0.3493424236\,e^{0.3669652806\,t} \end{bmatrix}$$

which is the solution for x. Some comments on these commands:

- Maple usually uses the symbol D for differentiataion, so if we want to use D as the name of a matrix, we need to enter `unprotect('D')` first. The quotation marks are important here.
- The line `L,U := Eigenvectors(A)` sets L to be the vector $\begin{bmatrix} \lambda_1 & \lambda_2 & \lambda_3 \end{bmatrix}^T$, whose entries are the eigenvalues. It also sets U to be the usual matrix whose columns are the corresponding eigenvectors.

□

Exercise 52. Consider the matrix $A = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}$.

(a) Find the eigenvalues of A.

(b) For each eigenvalue, find a corresponding eigenvector of A.

(c) Define recursively a sequence of vectors $\begin{bmatrix} u_n \\ v_n \end{bmatrix}$ as follows: we have $u_0 = 1$ and $v_0 = 0$, and for all $n > 0$ we have

$$\begin{aligned} u_n &= u_{n-1} + v_{n-1} \\ v_n &= 2u_{n-1} + v_{n-1}. \end{aligned}$$

Use your eigenvectors of A to find expressions for u_n and v_n (for a general positive integer n).

Solution:

We have

$$\chi_A(t) = \begin{vmatrix} 1-t & 1 \\ 2 & 1-t \end{vmatrix} = (t-1)^2 - 2 = t^2 - 2t - 1 = \left[t - 1 - \sqrt{2}\right]\left[t - 1 + \sqrt{2}\right].$$

We thus see that the eigenvalues of A are $\lambda_1 = 1 + \sqrt{2}$ and $\lambda_2 = 1 - \sqrt{2}$. There are two distinct eigenvalues, and so it is possible to find a basis for $\mathbb{R}^2$ consisting of eigenvectors of A. Notice that $(1+\sqrt{2})(1-\sqrt{2}) = -1$ and $(1+\sqrt{2}) + (1-\sqrt{2}) = 2$.

To find an eigenvector corresponding to λ_1, we consider $(A - \lambda_1 I_2)\begin{bmatrix} x & y \end{bmatrix}^T = 0$:

$$\begin{aligned} \left(A - \lambda_1 I_2 | 0\right) &= \left[\begin{array}{cc|c} 1-(1+\sqrt{2}) & 1 & 0 \\ 2 & 1-(1+\sqrt{2}) & 0 \end{array}\right] = \left[\begin{array}{cc|c} -\sqrt{2} & 1 & 0 \\ 2 & -\sqrt{2} & 0 \end{array}\right] \\ &\to \left[\begin{array}{cc|c} -\sqrt{2} & 1 & 0 \\ 0 & 0 & 0 \end{array}\right], \end{aligned}$$

so that $w_1 := \begin{bmatrix} 1 \\ \sqrt{2} \end{bmatrix}$ is an eigenvector of A corresponding to λ_1.

To find an eigenvector corresponding to λ_2, we consider $(A - \lambda_2 I_2)\begin{bmatrix} x & y \end{bmatrix}^T = 0$:

$$\begin{aligned} \left(A - \lambda_2 I_2 | 0\right) &= \left[\begin{array}{cc|c} 1-(1-\sqrt{2}) & 1 & 0 \\ 2 & 1-(1-\sqrt{2}) & 0 \end{array}\right] = \left[\begin{array}{cc|c} \sqrt{2} & 1 & 0 \\ 2 & \sqrt{2} & 0 \end{array}\right] \\ &\to \left[\begin{array}{cc|c} \sqrt{2} & 1 & 0 \\ 0 & 0 & 0 \end{array}\right], \end{aligned}$$

so that $w_2 := \begin{bmatrix} 1 \\ -\sqrt{2} \end{bmatrix}$ is an eigenvector of A corresponding to λ_2.

We have

$$\begin{bmatrix} u_0 \\ v_0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} u_n \\ v_n \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} u_{n-1} \\ v_{n-1} \end{bmatrix} = A\begin{bmatrix} u_{n-1} \\ v_{n-1} \end{bmatrix} \quad \text{for } n > 0.$$

Therefore

$$\begin{bmatrix} u_n \\ v_n \end{bmatrix} = A^n \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{for } n > 0.$$

We calculate this by using the above eigenvectors w_1 and w_2 of A. Since w_1 and w_2 are eigenvectors of A corresponding to different eigenvalues, they are linearly independent, and so form a basis for $\mathbb{R}^2$. We express $\begin{bmatrix} 1 & 0 \end{bmatrix}^T$ as a linear combination of w_1 and w_2:

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \tfrac{1}{2}\begin{bmatrix} 1 \\ \sqrt{2} \end{bmatrix} + \tfrac{1}{2}\begin{bmatrix} 1 \\ -\sqrt{2} \end{bmatrix} = \tfrac{1}{2}w_1 + \tfrac{1}{2}w_2.$$

Therefore, for all $n > 0$,

$$\begin{aligned} \begin{bmatrix} u_n \\ v_n \end{bmatrix} &= A^n \begin{bmatrix} 1 \\ 0 \end{bmatrix} = A^n\left(\tfrac{1}{2}w_1 + \tfrac{1}{2}w_2\right) \\ &= \tfrac{1}{2}A^n w_1 + \tfrac{1}{2}A^n w_2 = \tfrac{1}{2}\lambda_1^n w_1 + \tfrac{1}{2}\lambda_2^n w_2 \\ &= \tfrac{1}{2}\left(1+\sqrt{2}\right)^n \begin{bmatrix} 1 \\ \sqrt{2} \end{bmatrix} + \tfrac{1}{2}\left(1-\sqrt{2}\right)^n \begin{bmatrix} 1 \\ -\sqrt{2} \end{bmatrix} \\ &= \begin{bmatrix} \tfrac{1}{2}\left(\left(1+\sqrt{2}\right)^n + \left(1-\sqrt{2}\right)^n\right) \\ \tfrac{\sqrt{2}}{2}\left(\left(1+\sqrt{2}\right)^n - \left(1-\sqrt{2}\right)^n\right) \end{bmatrix}. \end{aligned}$$

Thus

$$u_n = \tfrac{1}{2}\left((1+\sqrt{2})^n + (1-\sqrt{2})^n\right) \qquad \text{and} \qquad v_n = \tfrac{1}{\sqrt{2}}\left((1+\sqrt{2})^n - (1-\sqrt{2})^n\right)$$

for all $n > 0$. □

Exercise 53. The sequence $(a_n)_{n=0}^{\infty}$ is given by $a_0 = 1001001$, $a_1 = 1010100$, $a_2 = 1110000$ and

$$a_{n+3} = 111a_{n+2} - 1110a_{n+1} + 1000a_n \qquad (\text{ for } n > 2)$$

(a) Write down a matrix equation relating the vector u_n to u_{n+1}, where $u_n = \begin{bmatrix} a_{n+2} \\ a_{n+1} \\ a_n \end{bmatrix}$.
(b) Find the eigenvalues and eigenvectors of the matrix occuring in (a). (If you have done this correctly, the answers will be integers with a nice pattern.)
(c) Express u_0 as a linear combination of the eigenvectors in (b).
(d) Give a general formula for a_n.
(e) Check directly that your formula satisfies $a_{n+3} = 111a_{n+2} - 1110a_{n+1} + 1000a_n$ and that a_0, a_1 and a_2 are as they should be.

Solution:

(a) We have

$$u_{n+1} = \begin{bmatrix} a_{n+3} \\ a_{n+2} \\ a_{n+1} \end{bmatrix} = \begin{bmatrix} 111a_{n+2} - 1110a_{n+1} + 1000a_n \\ a_{n+2} \\ a_{n+1} \end{bmatrix} = \begin{bmatrix} 111 & -1110 & 1000 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} a_{n+2} \\ a_{n+1} \\ a_n \end{bmatrix}.$$

In other words, if we put

$$A = \begin{bmatrix} 111 & -1110 & 1000 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

then $u_{n+1} = Au_n$. It follows that for all $n \geq 0$ we have

$$u_n = A^n u_0 = A^n \begin{bmatrix} 1110000 \\ 1010100 \\ 1001001 \end{bmatrix}.$$

(b) The characteristic polynomial is

$$\chi_A(t) = \det \begin{bmatrix} 111-t & -1110 & 1000 \\ 1 & -t & 0 \\ 0 & 1 & -t \end{bmatrix} = (111-t)\det \begin{bmatrix} -t & 0 \\ 1 & -t \end{bmatrix} + 1110 \det \begin{bmatrix} 1 & 0 \\ 0 & -t \end{bmatrix} + 1000 \det \begin{bmatrix} 1 & -t \\ 0 & 1 \end{bmatrix}$$
$$= (111-t)t^2 - 1110t + 1000 = 1000 - 1110t + 111t^2 - t^3$$
$$= (1-t)(10-t)(100-t).$$

Thus, the eigenvalues are 1, 10 and 100. To find the corresponding eigenvectors, we perform the following row-reductions:

$$A - I = \begin{bmatrix} 110 & -1110 & 1000 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} =: B_1$$

$$A - 10I = \begin{bmatrix} 101 & -1110 & 1000 \\ 1 & -10 & 0 \\ 0 & 1 & -10 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -100 \\ 0 & 1 & -10 \\ 0 & 0 & 0 \end{bmatrix} =: B_2$$

$$A - 100I = \begin{bmatrix} 10 & -1110 & 1000 \\ 1 & -100 & 0 \\ 0 & 1 & -100 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -10000 \\ 0 & 1 & -100 \\ 0 & 0 & 0 \end{bmatrix} =: B_3.$$

To find an eigenvector $w_2 = \begin{bmatrix} x & y & z \end{bmatrix}^T$ of eigenvalue 10, we need to solve $(A - 10I)w_2 = 0$, or equivalently $B_2 w_2 = 0$, which just reduces to $x = 100z$ and $y = 10z$ with z arbitrary. Taking

$z = 1$, we see that $\begin{bmatrix}100 & 10 & 1\end{bmatrix}^T$ is an eigenvector of eigenvalue 10. Treating the other two eigenvalues in the same way, we find that the vectors

$$w_1 = \begin{bmatrix}1\\1\\1\end{bmatrix} \qquad w_2 = \begin{bmatrix}100\\10\\1\end{bmatrix} \qquad w_3 = \begin{bmatrix}10000\\100\\1\end{bmatrix}$$

are eigenvectors of eigenvalues 1, 10 and 100 respectively.

(c) By inspection we have

$$\begin{aligned} u_0 = \begin{bmatrix}1110000\\1010100\\1001001\end{bmatrix} &= \begin{bmatrix}1000000\\1000000\\1000000\end{bmatrix} + \begin{bmatrix}100000\\10000\\1000\end{bmatrix} + \begin{bmatrix}10000\\100\\1\end{bmatrix} \\ &= 1000000\begin{bmatrix}1\\1\\1\end{bmatrix} + 1000\begin{bmatrix}100\\10\\1\end{bmatrix} + \begin{bmatrix}10000\\100\\1\end{bmatrix} = 10^6 w_1 + 10^3 w_2 + w_3. \end{aligned}$$

(d) Recall that $Aw_1 = w_1$ and $Aw_2 = 10w_2$ and $Aw_3 = 100w_3$. It follows that for all $n \geq 0$ we have $A^n w_1 = w_1$ and $A^n w_2 = 10^n w_2$ and $A^n w_3 = 100^n w_3 = 10^{2n} w_3$. This gives

$$\begin{aligned} u_n = A^n u_0 &= A^n(10^6 w_1 + 10^3 w_2 + w_3) = 10^6 A^n w_1 + 10^3 A^n w_2 + A^n w_3 \\ &= 10^6 w_1 + 10^{n+3} w_2 + 10^{2n} w_3 = 10^6 \begin{bmatrix}1\\1\\1\end{bmatrix} + 10^{n+3}\begin{bmatrix}10^2\\10\\1\end{bmatrix} + 10^{2n}\begin{bmatrix}10^4\\10^2\\1\end{bmatrix} \\ &= \begin{bmatrix}10^6 + 10^{n+5} + 10^{2n+4}\\ 10^6 + 10^{n+4} + 10^{2n+2}\\ 10^6 + 10^{n+3} + 10^{2n}\end{bmatrix}. \end{aligned}$$

In particular, a_n is the bottom entry in u_n, which is

$$a_n = 10^6 + 10^{n+3} + 10^{2n}.$$

(e) Our formula gives

$$\begin{aligned} a_0 &= 10^6 + 10^3 + 10^0 = 1001001 \\ a_1 &= 10^6 + 10^4 + 10^2 = 1010100 \\ a_0 &= 10^6 + 10^5 + 10^4 = 1110000 \end{aligned}$$

as it should. We also have

$$\begin{aligned} &111a_{n+2} - 1110a_{n+1} + 1000a_n \\ =&111(10^6 + 10^{n+5} + 10^{2n+4}) - 1110(10^6 + 10^{n+4} + 10^{2n+2}) + 1000(10^6 + 10^{n+3} + 10^{2n}) \\ =&10^6(111 - 1110 + 1000) + 10^{n+3}(11100 - 11100 + 1000) + 10^{2n}(1110000 - 111000 + 1000) \\ =&10^6 + 1000 \times 10^{n+3} + 1000000 \times 10^{2n} = 10^6 + 10^{n+6} + 10^{2n+6} = a_{n+3}. \end{aligned}$$

□

Exercise 54. Let (a_n) be the sequence given by $a_0 = 2$ and $a_1 = 4$ and $a_{n+2} = 4a_{n+1} - a_n$ for $n \geq 0$. Give a general formula for a_n.

Solution: The vectors $v_n = \begin{bmatrix}a_n\\a_{n+1}\end{bmatrix}$ satisfy $v_0 = \begin{bmatrix}2\\4\end{bmatrix}$ and

$$v_{n+1} = \begin{bmatrix}a_{n+1}\\a_{n+2}\end{bmatrix} = \begin{bmatrix}a_{n+1}\\4a_{n+1} - a_n\end{bmatrix} = \begin{bmatrix}0 & 1\\-1 & 4\end{bmatrix}\begin{bmatrix}a_n\\a_{n+1}\end{bmatrix} = Av_n,$$

where $A = \begin{bmatrix}0 & 1\\-1 & 4\end{bmatrix}$. It follows that $v_k = A^k v_0$ for all $k \geq 0$. To understand this more explicitly, we need to find the eigenvalues and eigenvectors of A. The characteristic polynomial is

$$\chi_A(t) = \det\begin{bmatrix}-t & 1\\-1 & 4-t\end{bmatrix} = -t(4-t) - (-1) = t^2 - 4t + 1.$$

The eigenvalues of A are the roots of $\chi_A(t)$, which are $\lambda_1 = (4+\sqrt{16-4})/2 = 2+\sqrt{3}$ and $\lambda_2 = 2-\sqrt{3}$. We next want to find an eigenvector $u_1 = \begin{bmatrix} x \\ y \end{bmatrix}$ with $Au_1 = \lambda_1 u_1$, or in other words

$$\begin{bmatrix} 0 & 1 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \lambda_1 x \\ \lambda_1 y \end{bmatrix}$$

or $y = \lambda_1 x$ and $4y - x = \lambda_1 y$. If we substitute $y = \lambda_1 x$ then the equation $4y - x = \lambda_1 y$ becomes $4\lambda_1 x - x = \lambda_1^2 x$ or $(\lambda_1^2 - 4\lambda_1 + 1)x = 0$, which holds automatically because λ_1 is a root of $t^2 - 4t + 1 = 0$. It follows that we can take $u_1 = \begin{bmatrix} 1 \\ \lambda_1 \end{bmatrix}$. Similarly, the vector $u_2 = \begin{bmatrix} 1 \\ \lambda_2 \end{bmatrix}$ is an eigenvector of A with eigenvalue λ_2.

We next need to express the vector $v_0 = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$ as a linear combination of u_1 and u_2. Equivalently, we must find α_1 and α_2 such that

$$\begin{bmatrix} 2 \\ 4 \end{bmatrix} = \alpha_1 \begin{bmatrix} 1 \\ \lambda_1 \end{bmatrix} + \alpha_2 \begin{bmatrix} 1 \\ \lambda_2 \end{bmatrix}.$$

Looking on the top line gives $\alpha_2 = 2 - \alpha_1$, and then the second line gives $4 = \lambda_1\alpha_1 + \lambda_2(2-\alpha_1)$ and so $\alpha_1(\lambda_1 - \lambda_2) = 4 - 2\lambda_2$. Here $\lambda_1 - \lambda_2 = 2\sqrt{3}$ and $4 - 2\lambda_2 = 2\sqrt{3}$ as well so $\alpha_1 = 1$. It follows that $\alpha_2 = 2 - \alpha_1 = 1$, so $v_0 = u_1 + u_2$. (This can also be seen by inspection.)

We now have $A^n u_i = \lambda_i^n u_i$, so

$$\begin{aligned} v_n &= A^n v_0 = A^n(u_1 + u_2) = A^n u_1 + A^n u_2 = \lambda_1^n u_1 + \lambda_2^n u_2 \\ &= \begin{bmatrix} \lambda_1^n + \lambda_2^n \\ \lambda_1^{n+1} + \lambda_2^{n+1} \end{bmatrix}. \end{aligned}$$

On the other hand, we have $v_n = \begin{bmatrix} a_n \\ a_{n+1} \end{bmatrix}$, so we conclude that $a_n = \lambda_1^n + \lambda_2^n = (2+\sqrt{3})^n + (2-\sqrt{3})^n$. □

12. Lecture 12

Exercise 55. Over a period of 5 minutes, in a typical MAS201 lecture, 90% of students who are awake at the beginning of the 5-minute period will still be so at the end of it (but the other 10% will fall asleep) and 90% of students who are asleep at the beginning of the 5-minute period will still be so at the end of it (and the other 10% will wake up). If all the students are awake at the beginning of the lecture, what percentage will be awake at the end of the lecture, 50 minutes later?

Solution: For each $k = 0, \ldots, 10$, let a_k and b_k be the proportions of students who are awake and who are asleep after $5k$ minutes of the lecture, respectively, and set $v_k = \begin{bmatrix} a_k & b_k \end{bmatrix}^T$. We have

$$\begin{bmatrix} a_0 \\ b_0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} a_{k+1} \\ b_{k+1} \end{bmatrix} = \begin{bmatrix} \frac{9}{10} & \frac{1}{10} \\ \frac{1}{10} & \frac{9}{10} \end{bmatrix} \begin{bmatrix} a_k \\ b_k \end{bmatrix}$$

for $k = 0, \ldots, 9$. Set

$$A = \begin{bmatrix} \frac{9}{10} & \frac{1}{10} \\ \frac{1}{10} & \frac{9}{10} \end{bmatrix}.$$

We are thus considering the difference equation $v_{k+1} = Av_k$, so that $v_k = A^k v_0$ for $k = 0, \ldots, 10$, and we wish to find $v_{10} = A^{10} v_0$.

The matrix A is stochastic, and so has 1 as an eigenvalue. The characteristic polynomial of A is

$$\chi_A(t) = \begin{vmatrix} \frac{9}{10} - t & \frac{1}{10} \\ \frac{1}{10} & \frac{9}{10} - t \end{vmatrix} = \left(t - \tfrac{9}{10}\right)^2 - \left(\tfrac{1}{10}\right)^2 = (t-1)\left(t - \tfrac{8}{10}\right).$$

Thus the eigenvalues of A are 1 and $\frac{8}{10}$. Since A has 2 distinct eigenvalues, we can find a basis for $\mathbb{R}^2$ consisting of eigenvectors of A.

To find an eigenvector of A corresponding to the eigenvalue $\lambda_1 := 1$, we need to solve the system of linear equations $(A - I_2)\begin{bmatrix} x & y \end{bmatrix}^T = 0$. This has augmented matrix

$$[A - I_2|0] = \left[\begin{array}{cc|c} -\frac{1}{10} & \frac{1}{10} & 0 \\ \frac{1}{10} & -\frac{1}{10} & 0 \end{array}\right] \to \left[\begin{array}{cc|c} -\frac{1}{10} & \frac{1}{10} & 0 \\ 0 & 0 & 0 \end{array}\right],$$

and so $w_1 := \begin{bmatrix}1 & 1\end{bmatrix}^T$ is an eigenvector of A corresponding to the eigenvalue 1.

To find an eigenvector of A corresponding to the eigenvalue $\lambda_2 := \frac{8}{10}$, we need to solve the system of linear equations $\begin{bmatrix}A - \frac{8}{10}I_2\end{bmatrix}\begin{bmatrix}x & y\end{bmatrix}^T = 0$. This has augmented matrix

$$\begin{bmatrix}A - \tfrac{8}{10}I_2 | 0\end{bmatrix} = \left[\begin{array}{cc|c} \frac{1}{10} & \frac{1}{10} & 0 \\ \frac{1}{10} & \frac{1}{10} & 0 \end{array}\right] \to \left[\begin{array}{cc|c} \frac{1}{10} & \frac{1}{10} & 0 \\ 0 & 0 & 0 \end{array}\right],$$

and so $w_2 := \begin{bmatrix}1 & -1\end{bmatrix}^T$ is an eigenvector of A corresponding to the eigenvalue $\frac{8}{10}$.

Now, $w_1 = \begin{bmatrix}1 \\ 1\end{bmatrix}$ and $w_2 = \begin{bmatrix}1 \\ -1\end{bmatrix}$, being eigenvectors corresponding to distinct eigenvalues of A, form a basis for $\mathbb{R}^2$. We express v_0 as a linear combination of these two eigenvectors:

$$v_0 = \begin{bmatrix}1 \\ 0\end{bmatrix} = \tfrac{1}{2}\begin{bmatrix}1 \\ 1\end{bmatrix} + \tfrac{1}{2}\begin{bmatrix}1 \\ -1\end{bmatrix} = \tfrac{1}{2}w_1 + \tfrac{1}{2}w_2.$$

We have

$$v_k = A^k v_0 = A^k\left(\tfrac{1}{2}w_1 + \tfrac{1}{2}w_2\right) = \tfrac{1}{2}A^k w_1 + \tfrac{1}{2}A^k w_2 = \tfrac{1}{2}\lambda_1^k w_1 + \tfrac{1}{2}\lambda_2^k w_2 = \tfrac{1}{2}1^k w_1 + \tfrac{1}{2}(0.8)^k w_2.$$

In particular,

$$v_{10} = \begin{bmatrix}a_{10} \\ b_{10}\end{bmatrix} = \tfrac{1}{2}w_1 + \tfrac{1}{2}(0.8)^{10}w_2 = \tfrac{1}{2}\begin{bmatrix}1 \\ 1\end{bmatrix} + \tfrac{1}{2}(0.8)^{10}\begin{bmatrix}1 \\ -1\end{bmatrix}.$$

Since $(0.8)^{10} \approx 0.107374$, we conclude that approximately 55.37% of students are awake at the end of the lecture. □

Exercise 56. Put $d = \begin{bmatrix}1 & \cdots & 1\end{bmatrix}^T \in \mathbb{R}^n$.

(a) If $P \in M_n(\mathbb{R})$ is a stochastic matrix, show that $d^T P = d^T$.
(b) Deduce that if $q \in \mathbb{R}^n$ is a probability vector, then Pq is also a probability vector.
(c) Deduce that if $Q \in M_n(\mathbb{R})$ is another stochastic matrix, then PQ is also a stochastic matrix. (**Hint:** how are the columns of PQ related to the columns of Q?)

Solution:

(a) Let the columns of P be $v_1, \ldots, v_n$. As P is stochastic, we know that the sum of the entries in v_i is equal to 1, so $d.v_i = 1$. This means that

$$d^T P = \begin{bmatrix}1 & \cdots & 1\end{bmatrix}\left[\begin{array}{c|c|c} v_1 & \cdots & v_n\end{array}\right] = \begin{bmatrix}d.v_1 & \cdots & d.v_n\end{bmatrix} = \begin{bmatrix}1 & \cdots & 1\end{bmatrix} = d^T.$$

(b) Now let q be a probability vector. Then all entries in P and q are nonnegative, and the entries in Pq are sums of entries in P multiplied by entries in q, so they are again nonnegative. Moreover, the sum of the entries in Pq is $d.Pq = d^T Pq$, but $d^T P = d$, so this is the same as $d^T q = d.q$, which is 1 by assumption. This proves that Pq is a probability vector.

(c) Now let Q be another $n \times n$ stochastic matrix. Let $w_1, \ldots, w_n$ be the columns of Q, which are probability vectors. We then have

$$PQ = P\left[\begin{array}{c|c|c} w_1 & \cdots & w_n\end{array}\right] = \left[\begin{array}{c|c|c} Pw_1 & \cdots & Pw_n\end{array}\right].$$

The vectors $Pw_1, \ldots, Pw_n$ are probability vectors by part (b), and it follows that PQ is a stochastic matrix.

□

Exercise 57. Suppose that $0 < p < 1$ and $0 < q < 1$, and put $P = \begin{bmatrix}p & 1-q \\ 1-p & q\end{bmatrix}$ (so P is a stochastic matrix). Find the eigenvalues and eigenvectors of P in terms of p and q.

(**Hint:** a general theorem from lectures tells you one of the eigenvalues.)

Solution: The characteristic polynomial is

$$\chi_P(t) = \det\begin{bmatrix}p-t & 1-q \\ 1-p & q-t\end{bmatrix} = (p-t)(q-t) - (1-p)(1-q) = t^2 - (p+q)t + (p+q-1).$$

Every stochastic matrix has 1 as an eigenvalue, so one of the roots of $\chi_P(t)$ is at $t = 1$. We can divide $t^2 - (p+q)t + (p+q-1)$ by $t-1$ to obtain the factorisation $\chi_P(t) = (t-1)(t-(p+q-1))$, so the other eigenvalue is $r = p + q - 1$. To find an eigenvector $u_1 = \begin{bmatrix} x \\ y \end{bmatrix}$ of eigenvalue 1, we must solve

$$(P - I)u_1 = \begin{bmatrix} p-1 & 1-q \\ 1-p & q-1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

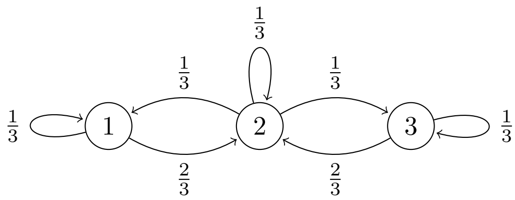
This reduces to $(1-p)x = (1-q)y$ so we can take $y = 1/(1-q)$ to get $x = 1/(1-p)$ and $u_1 = \begin{bmatrix} 1/(1-p) \\ 1/(1-q) \end{bmatrix}$.

Next, to find an eigenvector of eigenvalue r we note that

$$P - rI = \begin{bmatrix} p & 1-q \\ 1-p & q \end{bmatrix} - \begin{bmatrix} p+q-1 & 0 \\ 0 & p+q-1 \end{bmatrix} = \begin{bmatrix} 1-q & 1-q \\ 1-p & 1-p \end{bmatrix}.$$

It follows that the vector $u_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ satisfies $(P - rI)u_2 = 0$, so this is the required eigenvector. □

Exercise 58. Consider the following Markov chain:



Write down the transition matrix and find its eigenvalues and eigenvectors. What is the stationary distribution?

Solution: The transition matrix is

$$P = \begin{bmatrix} p_{1\leftarrow 1} & p_{1\leftarrow 2} & p_{1\leftarrow 3} \\ p_{2\leftarrow 1} & p_{2\leftarrow 2} & p_{2\leftarrow 3} \\ p_{3\leftarrow 1} & p_{3\leftarrow 2} & p_{3\leftarrow 3} \end{bmatrix} = \begin{bmatrix} 1/3 & 1/3 & 0 \\ 2/3 & 1/3 & 2/3 \\ 0 & 1/3 & 1/3 \end{bmatrix}.$$

For the characteristic polynomial, we have

$$\begin{aligned}
\chi_P(t) &= \det \begin{bmatrix} 1/3-t & 1/3 & 0 \\ 2/3 & 1/3-t & 2/3 \\ 0 & 1/3 & 1/3-t \end{bmatrix} \\
&= (1/3-t)\det \begin{bmatrix} 1/3-t & 2/3 \\ 1/3 & 1/3-t \end{bmatrix} - (1/3)\det \begin{bmatrix} 2/3 & 2/3 \\ 0 & 1/3-t \end{bmatrix} \\
\det \begin{bmatrix} 1/3-t & 2/3 \\ 1/3 & 1/3-t \end{bmatrix} &= (1/3-t)^2 - 2/9 = t^2 - (2/3)t - 1/9 \\
\det \begin{bmatrix} 2/3 & 2/3 \\ 0 & 1/3-t \end{bmatrix} &= 2/9 - (2/3)t \\
\chi_P(t) &= (1/3-t)(t^2 - (2/3)t - 1/9) - (1/3)(2/9 - (2/3)t) \\
&= -1/9 + (1/9)t + t^2 - t^3 = (1-t)(t^2 - 1/9) = (1-t)(t-1/3)(t+1/3).
\end{aligned}$$

From this we see that the eigenvalues are $1/3$, $-1/3$ and 1. To find an eigenvector u_1 of eigenvalue $1/3$ we row-reduce $P - \frac{1}{3}I$:

$$\begin{bmatrix} 0 & 1/3 & 0 \\ 2/3 & 0 & 2/3 \\ 0 & 1/3 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

This means that if $u_1 = \begin{bmatrix} x & y & z \end{bmatrix}^T$ we must have

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

which gives $x = -z$ with $y = 0$. Taking $z = 1$ we get $u_1 = \begin{bmatrix} -1 & 0 & 1 \end{bmatrix}^T$. Next, to find an eigenvector u_2 of eigenvalue $-1/3$ we row-reduce $P + \frac{1}{3}I$:

$$\begin{bmatrix} 2/3 & 1/3 & 0 \\ 2/3 & 2/3 & 2/3 \\ 0 & 1/3 & 2/3 \end{bmatrix} \to \begin{bmatrix} 1 & 1/2 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}.$$

This means that if $u_2 = \begin{bmatrix} x & y & z \end{bmatrix}^T$ we must have

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

which gives $x = z$ and $y = -2z$. Taking $z = 1$ we get $u_2 = \begin{bmatrix} 1 & -2 & 1 \end{bmatrix}^T$. Finally, to find an eigenvector of eigenvalue 1 we row-reduce $P - I$:

$$\begin{bmatrix} -2/3 & 1/3 & 0 \\ 2/3 & -2/3 & 2/3 \\ 0 & 1/3 & -2/3 \end{bmatrix} \to \begin{bmatrix} 1 & -1/2 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}.$$

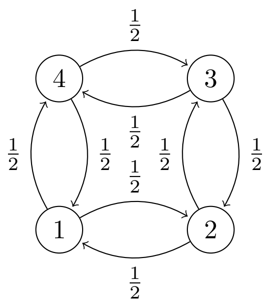
This means that if $u_3 = \begin{bmatrix} x & y & z \end{bmatrix}^T$ we must have

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

which gives $x = z$ and $y = 2z$. Taking $z = 1$, we get $u_3 = \begin{bmatrix} 1 & 2 & 1 \end{bmatrix}^T$.

We are also asked for a stationary distribution, which should be an eigenvector of eigenvalue 1 that is also a probability vector. To make u_3 into a probability vctor we need to divide it by 4, giving $\begin{bmatrix} \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \end{bmatrix}^T$ as the stationary distribution. □

Exercise 59. Consider the following Markov chain:



Write down the transition matrix P and check that $P^3 = P$. Deduce that $P^{2k+1} = P$ for all $k \geq 0$. If we start in state 1 at $t = 0$, what is the probability of being in state 3 at $t = 1111$?

Note: you do not need to calculate any eigenvalues or eigenvectors for this question.

Solution: The transition matrix is

$$P = \begin{bmatrix} p_{1\leftarrow 1} & p_{1\leftarrow 2} & p_{1\leftarrow 3} & p_{1\leftarrow 4} \\ p_{2\leftarrow 1} & p_{2\leftarrow 2} & p_{2\leftarrow 3} & p_{2\leftarrow 4} \\ p_{3\leftarrow 1} & p_{3\leftarrow 2} & p_{3\leftarrow 3} & p_{3\leftarrow 4} \\ p_{4\leftarrow 1} & p_{4\leftarrow 2} & p_{4\leftarrow 3} & p_{4\leftarrow 4} \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}.$$

This gives

$$P^2 = \frac{1}{4} \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

$$P^3 = \frac{1}{4} \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} = P.$$

We can now multiply both sides of the equation $P^3 = P$ by P^2 to get $P^5 = P^3$, but $P^3 = P$ so $P^5 = P$. We now multiply both sides by P^2 again to get $P^7 = P^3 = P$, and again to get $P^9 = P^3 = P$ and so on. Ths shows that $P^{2k+1} = P$ for all $k \geq 0$.

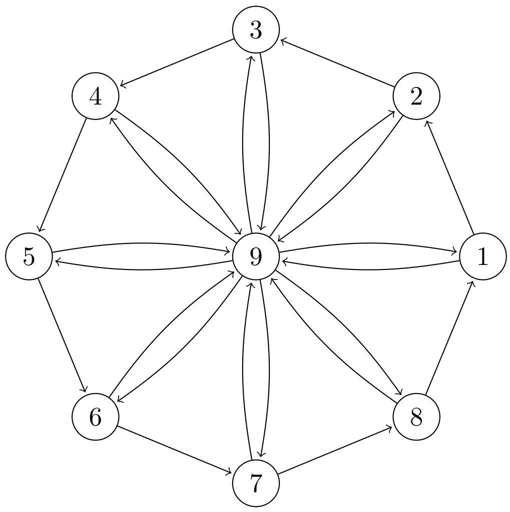
Now suppose we are definitely in state 1 at $t = 0$, so the distribution vector r_0 is $\begin{bmatrix}1 & 0 & 0 & 0\end{bmatrix}^T$. The distribution at $t = 1111$ is then $r_{1111} = P^{1111}r_0$, but we have just seen that $P^{1111} = P$, so

$$r_{1111} = Pr_0 = \frac{1}{2}\begin{bmatrix}0 & 1 & 0 & 1\\ 1 & 0 & 1 & 0\\ 0 & 1 & 0 & 1\\ 1 & 0 & 1 & 0\end{bmatrix}\begin{bmatrix}1\\0\\0\\0\end{bmatrix} = \begin{bmatrix}0\\1/2\\0\\1/2\end{bmatrix}.$$

By looking at the third entry, we see that the probability of being in state 3 at $t = 1111$ is zero. In fact, this can be seen even more directly. From the diagram we see that every second we switch from an odd-numbered state to an even-numbered state or *vice-versa*. We start in state 1 at $t = 0$, and at $t = 1111$ we have switched over an odd number of times, so we must be in an even-numbered state, and in particular we cannot be in state 3. □

13. Lecture 13

Exercise 60. Consider the following web of pages and links.



Let a be the PageRank of page 1, and let b be the PageRank of page 9. By symmetry, pages 2 to 8 must also have rank a. Use the consistency and normalisation conditions to find a and b (without writing down any 9×9 matrices).

Solution: First, the normalisation condition says that $\sum_{i=1}^{9} r_i = 1$. As $r_1 = \cdots = r_8 = a$ and $r_9 = b$, this means that $8a + b = 1$.

Next, note that the numbers of outgoing links are $N_1 = \cdots = N_8 = 2$ and $N_9 = 8$. As page 1 has links from pages 8 and 9, the consistency condition says that $r_1 = r_8/N_8 + r_9/N_9$, or in other words $a = a/2 + b/8$. By symmetry, pages 2 to 8 have the same consistency condition as page 1. On the other hand, page 9 has links from pages 1 to 8, so the consistency condition there is

$$b = r_9 = r_1/N_1 + \cdots + r_8/N_8 = a/2 + \cdots + a/2 = 4a.$$

Solving the equations $8a + b = 1$, $a = a/2 + b/8$ and $b = 4a$ gives $a = 1/12$ and $b = 1/3$. □

Exercise 61. Consider the following sets

$$P_0 = \{\begin{bmatrix} x & y \end{bmatrix}^T \in \mathbb{R}^2 \mid x^2 \geq 1\}$$
$$P_1 = \{\begin{bmatrix} x & y \end{bmatrix}^T \in \mathbb{R}^2 \mid xy \geq 0\}$$
$$P_2 = \{\begin{bmatrix} x & y \end{bmatrix}^T \in \mathbb{R}^2 \mid y \leq x^2\}$$
$$P_3 = \{\begin{bmatrix} x & y \end{bmatrix}^T \in \mathbb{R}^2 \mid x + y \text{ is an integer }\}$$
$$P_4 = \{\begin{bmatrix} x & y \end{bmatrix}^T \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}$$

The set P_0 is not closed under addition, because the vectors $u_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $u_1 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$ both lie in P_0, but the sum $u_0 + u_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ does not lie in P_0. Moreover, the set P_0 is not closed under scalar multiplication, because the vector $u_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ lies in P_0, but the product $0.5u_2 = \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}$ does not lie in P_0. Give similarly specific examples to show that

(a) P_1 is not closed under addition.
(b) P_2 is not closed under addition.
(c) P_2 is not closed under scalar multiplication.
(d) P_3 is not closed under scalar multiplication.
(e) P_4 is not closed under scalar multiplication.

Solution:

(a) P_1 contains the vectors $u_3 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and $u_4 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$ but not the sum $u_3 + u_4 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.
(b) P_2 contains the vectors $u_5 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $u_6 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ but not the sum $u_5 + u_6 = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$.
(c) P_2 contains the vector $u_7 = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$ but not the vector $(-1)u_7 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.
(d) P_3 contains the vector $u_8 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ but not the vector $0.5u_8 = \begin{bmatrix} 0.5 \\ 0 \end{bmatrix}$.
(e) P_4 contains the vector u_8 as above, but not the vector $2u_8 = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$.

□

Exercise 62. Which of the following sets is a subspace of $\mathbb{R}^4$?

(a) V_1 is the set of vectors of the form $\begin{bmatrix} s & t+s & t-s & t \end{bmatrix}^T$ (for some $s, t \in \mathbb{R}$).
(b) V_2 is the set of vectors of the form $\begin{bmatrix} t & t^2 & t^3 & t^4 \end{bmatrix}^T$ (for some $t \in \mathbb{R}$).
(c) V_3 is the set of vectors $v = \begin{bmatrix} w & x & y & z \end{bmatrix}^T$ that satisfy $w + 10x + 100y + 1000z = 1$.
(d) V_4 is the set of vectors $v = \begin{bmatrix} w & x & y & z \end{bmatrix}^T$ that satisfy $w - x + y - z = 0$.
(e) V_5 is the set of vectors $v = \begin{bmatrix} w & x & y & z \end{bmatrix}^T$ that satisfy $(w-x)^2 + (y-z)^2 = 0$.

Solution:

(a) The set V_1 is a subspace of $\mathbb{R}^4$. Indeed, if $v, v' \in V_1$ then we have $v = \begin{bmatrix} s & t+s & t-s & t \end{bmatrix}^T$ and $v' = \begin{bmatrix} s' & t'+s' & t'-s' & t' \end{bmatrix}^T$ for some $s, t, s', t' \in \mathbb{R}$. This means that

$$v + v' = \begin{bmatrix} s'' & t''+s'' & t''-s'' & t'' \end{bmatrix}^T,$$

where $s'' = s + s'$ and $t'' = t + t'$. It follows that $v + v' \in V_1$, so V_1 is closed under addition. Similarly, if a is any scalar, we have $av = \begin{bmatrix} s^* & t^*+s^* & t^*-s^* & t^* \end{bmatrix}^T$, where $s^* = as$ and $t^* = at$. This shows that $av \in V_1$, so V_1 is closed under scalar multiplication. Finally, by taking $s = t = 0$ we see that the zero vector lies in V_1.

(b) The set V_2 is not a subspace of $\mathbb{R}^4$. Indeed, by taking $t = 1$ we see that the vector $v = \begin{bmatrix}1 & 1 & 1 & 1\end{bmatrix}^T$ lies in V_2, but the vector $2v = \begin{bmatrix}2 & 2 & 2 & 2\end{bmatrix}^T$ does not lie in V_2, so V_2 is not closed under scalar multiplication.

(c) The set V_3 is not a subspace of $\mathbb{R}^4$, because the zero vector does not satisfy $w + 10x + 100y + 1000z = 1$ and so is not an element of V_3.

(d) The set V_4 is a subspace of $\mathbb{R}^4$. Indeed, the zero vector $\begin{bmatrix}w & x & y & z\end{bmatrix}^T = \begin{bmatrix}0 & 0 & 0 & 0\end{bmatrix}^T$ satisfies $w-x+y-z$ and so $0 \in V_4$. If we have elements $v = \begin{bmatrix}w & x & y & z\end{bmatrix}^T$ and $v' = \begin{bmatrix}w' & x' & y' & z'\end{bmatrix}^T$ in V_4 then the we have $w - x + y - z = 0$ and $w' - x' + y' - z' = 0$. By adding these equations we see that $(w + w') - (x + x') + (y + y') - (z + z') = 0$, which shows that the sum $v + v'$ is again an element of V_4, so V_4 is closed under addition. A similar argument shows that it is closed under scalar multiplication.

(e) The set V_5 is also a subspace of $\mathbb{R}^4$, although this fact is slightly disguised by the way that we have defined it. Because all squares are nonnegative, we see that the only way $(w - x)^2 + (y - z)^2$ can be zero is if $w = x$ and $y = z$. This means that V_5 is the set of vectors of the form $\begin{bmatrix}s & s & t & t\end{bmatrix}^T$, which is a subspace by the same method that we used in part (a).

□

Exercise 63. (a) Give an example of a subset $U_0 \subseteq \mathbb{R}^2$ that contains zero and is closed under addition but is not closed under scalar multiplication.

(b) Give an example of a subset $U_1 \subseteq \mathbb{R}^2$ that contains zero and is closed under scalar multiplication but is not closed under addition.

(c) Suppose that U_2 is a nonempty subset of $\mathbb{R}^2$ that is closed under addition and scalar multiplication. Show that U_2 contains the zero vector.

(d) Let U_3 be a subspace of $\mathbb{R}^1 = \mathbb{R}$. Show that U_3 is either $\{0\}$ or all of $\mathbb{R}$.

Solution:

(a) The simplest example is

$$U_0 = \left\{ \begin{bmatrix}x\\y\end{bmatrix} \in \mathbb{R}^2 \mid x, y \geq 0 \right\}.$$

This is not closed under scalar multiplication, because $\begin{bmatrix}1\\1\end{bmatrix} \in U_0$ but $(-1)\begin{bmatrix}1\\1\end{bmatrix} \notin U_0$.

(b) The simplest example is

$$U_1 = \left\{ \begin{bmatrix}x\\y\end{bmatrix} \in \mathbb{R}^2 \mid xy = 0 \right\}.$$

This is not closed under addition, because $\begin{bmatrix}1\\0\end{bmatrix} \in U_1$ and $\begin{bmatrix}0\\1\end{bmatrix} \in U_1$ but $\begin{bmatrix}1\\0\end{bmatrix} + \begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}1\\1\end{bmatrix} \notin U_1$.

(c) As U_2 is nonempty, we can choose a vector $u \in U_2$. As U_2 is closed under scalar multiplication, we can multiply the vector $u \in U_2$ by the scalar $0 \in \mathbb{R}$, and the result $0u$ will again be an element of U_2. Of course $0u$ is just the zero vector, so the zero vector is an element of U_2.

(d) Let U_3 be a subspace of $\mathbb{R}$. As it is a subspace, it must contain zero. If it does not contain anything else, then $U_3 = \{0\}$. Suppose instead that it does contain something else, so there is a nonzero element $u \in U_3$. Consider another element $v \in \mathbb{R}$. As we are working with elements of $\mathbb{R}^1$ which are just numbers, we can make sense of multiplication and division (which are not defined for vectors in $\mathbb{R}^2$ and beyond). We can thus express v as the product of the scalar v/u with the vector $u \in U_3$. (There is no problem with dividing by u, because we have assumed that $u \neq 0$.) As U_3 is closed under scalar multiplication, the product $(v/u)u$ lies in U_3, or in other words $v \in U_3$. This works for all vectors $v \in \mathbb{R}^1$, so we have $U_3 = \mathbb{R}^1$.

□

15. Lecture 15

Exercise 64. Let V be the set of vectors of the form

$$v = \begin{bmatrix}2p - q & q + r + s & 3p + 2s & r - s\end{bmatrix}$$

(where p, q, r and s are arbitrary real numbers). Find a list of vectors whose span is V.

Solution: This is similar to examples 19.16 and 19.17. The general form for elements of V is

$$v = \begin{bmatrix} 2p-q & q+r+s & 3p+2s & r-s \end{bmatrix} = p \begin{bmatrix} 2 \\ 0 \\ 3 \\ 0 \end{bmatrix} + q \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + r \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} + s \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}.$$

In other words, the elements of V are all the possible linear combinations of the four vectors occuring in the above formula. In other words, we have

$$V = \operatorname{span}\left(\begin{bmatrix} 2 \\ 0 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix} \right).$$

□

Exercise 65. Put

$$A = \begin{bmatrix} 1 & 6 & 8 \\ 7 & 2 & 3 \end{bmatrix}$$

and $V = \{v \in \mathbb{R}^3 \mid Av = 0\}$. Find a list of vectors whose annihilator is V.

Solution: This is an instance of Proposition 19.14: the space V is by definition the kernel of A, and that proposition tells us that the kernel is the annihilator of the transposed rows. Thus, if we put $a_1 = \begin{bmatrix} 1 & 6 & 8 \end{bmatrix}^T$ and $a_2 = \begin{bmatrix} 7 & 2 & 3 \end{bmatrix}^T$ then $V = \operatorname{ann}(a_1, a_2)$. This can also be seen quite easily without reference to Proposition 19.14. If $v = \begin{bmatrix} x & y & z \end{bmatrix}^T$ then

$$Av = \begin{bmatrix} 1 & 6 & 8 \\ 7 & 2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x+6y+8z \\ 7x+2y+3z \end{bmatrix} = \begin{bmatrix} a_1.v \\ a_2.v \end{bmatrix},$$

so v lies in V iff $Av = 0$ iff $a_1.v = a_2.v = 0$ iff v lies in $\operatorname{ann}(a_1, a_2)$; this means that $V = \operatorname{ann}(a_1, a_2)$ as before. □

Exercise 66. Put

$$a_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \qquad a_2 = \begin{bmatrix} 4 \\ 3 \\ 2 \\ 1 \end{bmatrix} \qquad u = \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix} \qquad v = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$

(a) Does u lie in $\operatorname{ann}(a_1, a_2)$?
(b) Does v lie in $\operatorname{ann}(a_1, a_2)$?
(c) Does u lie in $\operatorname{span}(a_1, a_2)$?
(d) Does v lie in $\operatorname{span}(a_1, a_2)$?

Solution:

(a) Yes, we have $u.a_1 = 1-2-3+4 = 0$ and $u.a_2 = 4-3-2+1 = 0$, so $u \in \operatorname{ann}(a_1, a_2)$.
(b) No, we have $v.a_1 = 1+2+3+4 = 10 \neq 0$, so $v \notin \operatorname{ann}(a_1, a_2)$. (We also have $v.a_2 \neq 0$, but the fact that $v.a_1 \neq 0$ is already enough to show that $v \notin \operatorname{ann}(a_1, a_2)$, so we do not really need to consider $v.a_2$.)
(c) No, u cannot be written as a linear combination of a_1 and a_2, so it does not lie in $\operatorname{span}(a_1, a_2)$. One way to check this is to use Method 7.6, which involves row-reducing the matrix $[a_1|a_2|u]$:

$$\begin{bmatrix} 1 & 4 & 1 \\ 2 & 3 & -1 \\ 3 & 2 & -1 \\ 4 & 1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 4 & 1 \\ 0 & -5 & -3 \\ 0 & -10 & -4 \\ 0 & -15 & -5 \end{bmatrix} \to \begin{bmatrix} 1 & 4 & 1 \\ 0 & 1 & 0.6 \\ 0 & -10 & -4 \\ 0 & -15 & -5 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1.4 \\ 0 & 1 & 0.6 \\ 0 & 0 & 2 \\ 0 & 0 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

We end up with a pivot in the last column, which indicates that the equation $\lambda_1 a_1 + \lambda_2 a_2 = u$ cannot be solved for λ_1 and λ_2, or equivalently that u is not a linear combination of a_1 and a_2.
(d) Yes, it is easy to see by inspection that $v = (a_1+a_2)/5 = 0.2a_1 + 0.2a_2$, so v is a linear combination of a_1 and a_2, or in other words $v \in \operatorname{span}(a_1, a_2)$.

□

Exercise 67. Put

$$a_1 = \begin{bmatrix}1\\1\\2\\2\end{bmatrix} \qquad a_2 = \begin{bmatrix}2\\2\\1\\1\end{bmatrix} \qquad b_1 = \begin{bmatrix}3\\-3\\4\\-4\end{bmatrix} \qquad b_2 = \begin{bmatrix}4\\-4\\3\\-3\end{bmatrix}.$$

Show that $\text{span}(a_1, a_2) \subseteq \text{ann}(b_1, b_2)$.

Solution: First, we have

$$\begin{aligned}
a_1.b_1 &= 3 - 3 + 8 - 8 = 0\\
a_1.b_2 &= 4 - 4 + 6 - 6 = 0\\
a_2.b_1 &= 6 - 6 + 4 - 4 = 0\\
a_2.b_2 &= 8 - 8 + 3 - 3 = 0.
\end{aligned}$$

Now consider an arbitrary element $v \in \text{span}(a_1, a_2)$. By the definition of $\text{span}(a_1, a_2)$, this means that v can be expressed as $v = \lambda_1 a_1 + \lambda_2 a_2$ for some scalars λ_1 and λ_2. This gives

$$\begin{aligned}
v.b_1 &= (\lambda_1 a_1 + \lambda_2 a_2).b_1 = \lambda_1(a_1.b_1) + \lambda_2(a_2.b_1) = \lambda_1 \times 0 + \lambda_2 \times 0 = 0\\
v.b_2 &= (\lambda_1 a_1 + \lambda_2 a_2).b_2 = \lambda_1(a_1.b_2) + \lambda_2(a_2.b_2) = \lambda_1 \times 0 + \lambda_2 \times 0 = 0.
\end{aligned}$$

As $v.b_1 = v.b_2 = 0$, we have $v \in \text{ann}(b_1, b_2)$. As this holds for every element of $\text{span}(a_1, a_2)$, we have $\text{span}(a_1, a_2) \subseteq \text{ann}(b_1, b_2)$ as claimed. □

Exercise 68. Consider the vectors

$$v_1 = \begin{bmatrix}-1\\2\\-1\\3\end{bmatrix} \qquad v_2 = \begin{bmatrix}1\\-1\\2\\-2\end{bmatrix} \qquad v_3 = \begin{bmatrix}1\\0\\3\\-1\end{bmatrix} \qquad w_1 = \begin{bmatrix}-1\\5\\2\\6\end{bmatrix} \qquad w_2 = \begin{bmatrix}1\\1\\4\\0\end{bmatrix}$$

(a) Show that $\text{span}(v_1, v_2, v_3) = \text{span}(v_1, v_2) = \text{span}(w_1, w_2)$.
(b) Find $\dim(\text{span}(v_1, v_2, v_3, w_1, w_2))$.

Solution: We will first give a solution that involves observing various identities between the given vectors, then a longer but more systematic solution by row-reduction.

First, we observe that $v_3 = v_1 + 2v_2$. This allows us to rewrite any linear combination of v_1, v_2 and v_3 as a linear combination of v_1 and v_2 alone. Thus, we have $\text{span}(v_1, v_2, v_3) = \text{span}(v_1, v_2)$.

Next, we observe that $w_1 = 4v_1 + 3v_2$ and $w_2 = 2v_1 + 3v_2$. This shows that $w_1, w_2 \in \text{span}(v_1, v_2)$ and so $\text{span}(w_1, w_2) \subseteq \text{span}(v_1, v_2)$. In the opposite direction, we have $v_1 = (w_1 - w_2)/2$ and $v_2 = (2w_2 - w_1)/3$, which shows that $v_1, v_2 \in \text{span}(w_1, w_2)$ and so $\text{span}(v_1, v_2) \subseteq \text{span}(w_1, w_2)$.

We now see that all of the given vectors are linear combinations of v_1 and v_2, so the space $V = \text{span}(v_1, v_2, v_3, w_1, w_2)$ is just the same as $\text{span}(v_1, v_2)$. Recall that a list of two nonzero vectors is only linearly dependent if the vectors are scalar multiples of each other. This is clearly not the case for v_1 and v_2, so we see that the list v_1, v_2 is a basis for V, so $\dim(V) = 2$.

The more systematic approach is just to find the canonical bases for all the spaces involved. We have

$$[v_1|v_2|v_3]^T = \begin{bmatrix}-1 & 2 & -1 & 3\\ 1 & -1 & 2 & -2\\ 1 & 0 & 3 & -1\end{bmatrix} \to \begin{bmatrix}1 & -2 & 1 & -3\\ 0 & 1 & 1 & 1\\ 0 & 2 & 2 & 2\end{bmatrix} \to \begin{bmatrix}1 & 0 & 3 & -1\\ 0 & 1 & 1 & 1\\ 0 & 0 & 0 & 0\end{bmatrix}$$

It follows that the vectors $a_1 = \begin{bmatrix}1 & 0 & 3 & -1\end{bmatrix}^T$ and $a_2 = \begin{bmatrix}0 & 1 & 1 & 1\end{bmatrix}^T$ form the canonical basis for $\text{span}(v_1, v_2, v_3)$. We can perform the same row-reduction leaving out the last row to see that a_1 and a_2 also form the canonical basis for $\text{span}(v_1, v_2)$, so $\text{span}(v_1, v_2, v_3) = \text{span}(v_1, v_2)$. Similarly, we have

$$[w_1|w_2]^T = \begin{bmatrix}-1 & 5 & 2 & 6\\ 1 & 1 & 4 & 0\end{bmatrix} \to \begin{bmatrix}1 & -5 & -2 & -6\\ 0 & 6 & 6 & 6\end{bmatrix} \begin{bmatrix}1 & 0 & 3 & -1\\ 0 & 1 & 1 & 1\end{bmatrix} = [a_1|a_2]^T$$

This shows that a_1 and a_2 also form the canonical basis for $\text{span}(w_1, w_2)$, so $\text{span}(v_1, v_2, v_3) = \text{span}(v_1, v_2) = \text{span}(w_1, w_2)$. From this it follows as before that $\text{span}(v_1, v_2, v_3, w_1, w_2)$ is yet another description of the same space, and the canonical basis has two vectors so the dimension is 2. □

Exercise 69. Put $V = \text{span}(v_1, v_2, v_3)$, where

$$v_1 = \begin{bmatrix} 0 & 2 & 6 & 10 & 1 & 0 \end{bmatrix}^T$$
$$v_2 = \begin{bmatrix} 0 & 1 & 3 & 5 & 1 & -3 \end{bmatrix}^T$$
$$v_3 = \begin{bmatrix} 0 & 3 & 9 & 15 & 1 & 3 \end{bmatrix}^T.$$

(a) What is the dimension of V?
(b) What is the canonical basis for V?
(c) What is the set $J(V)$ of jumps for V?

Solution: We can row-reduce the matrix $A = [v_1|v_2|v_3]^T$ as follows:

$$A = \begin{bmatrix} 0 & 2 & 6 & 10 & 1 & 0 \\ 0 & 1 & 3 & 5 & 1 & -3 \\ 0 & 3 & 9 & 15 & 1 & 3 \end{bmatrix} \to \begin{bmatrix} 0 & 0 & 0 & 0 & -1 & 6 \\ 0 & 1 & 3 & 5 & 1 & -3 \\ 0 & 0 & 0 & 0 & -2 & 12 \end{bmatrix} \to \begin{bmatrix} 0 & 1 & 3 & 5 & 0 & 3 \\ 0 & 0 & 0 & 0 & 1 & -6 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} = B.$$

According to Method 20.14, the canonical basis for V consists of the transposes of the nonzero rows in B, or in other words the vectors

$$u_1 = \begin{bmatrix} 0 & 1 & 3 & 5 & 0 & 3 \end{bmatrix}^T \qquad u_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & -6 \end{bmatrix}.$$

As this basis consists of two vectors, we have $\dim(V) = 2$. According to Lemma 20.13, the jumps for V are the pivot columns for the above matrix B. There are pivots in columns 2 and 5, so $J(V) = \{2, 5\}$. □

Exercise 70. Let V be the set of all vectors of the form

$$v = \begin{bmatrix} p+q & p+2q & p+r & p+3r \end{bmatrix}^T.$$

You may assume that this is a subspace. Find a list of vectors that spans V, and then find the canonical basis for V.

Solution: A general element of V has the form

$$v = \begin{bmatrix} p+q & p+2q & p+r & p+3r \end{bmatrix} = p \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + q \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix} + r \begin{bmatrix} 0 \\ 0 \\ 1 \\ 3 \end{bmatrix}.$$

In other words, the elements of V are precisely the linear combinations of the vectors

$$v_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \qquad v_2 = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix} \qquad v_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 3 \end{bmatrix}.$$

For the canonical basis, we perform the following row-reduction:

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 2 & 2 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & -4 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{bmatrix}.$$

We conclude that the canonical basis consists of the vectors

$$w_1 = \begin{bmatrix} 1 & 0 & 0 & -4 \end{bmatrix}^T \qquad w_2 = \begin{bmatrix} 0 & 1 & 0 & 2 \end{bmatrix}^T \qquad w_3 = \begin{bmatrix} 0 & 0 & 1 & 3 \end{bmatrix}^T.$$

□

Exercise 71. Put $V = \text{span}(e_1 - e_2, e_2 - e_3, \dots, e_{n-1} - e_n) \subseteq \mathbb{R}^n$, where e_i is the i'th standard basis vector for $\mathbb{R}^n$.

(a) What is the dimension of V?
(b) What is the canonical basis for V?
(c) What is the set $J(V)$ of jumps for V?

(You can start by doing the case $n = 5$ by row-reduction if you like, but ideally you should give an answer for the general case, together with a more abstract proof that your answer is correct.)

Solution: Put $v_i = e_i - e_{i+1}$, so $V = \operatorname{span}(v_1, \dots, v_{n-1})$. For the case $n = 5$ we have can row-reduce the matrix $A = [v_1|v_2|v_3|v_4]^T$ as follows:

$$\begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix}$$

The final matrix B can be described as $[w_1|w_2|w_3|w_4]^T$, where $w_i = e_i - e_4$. It follows that these vectors w_i form the canonical basis for V, so $\dim(V) = 4$. Moreover, the set of jumps for V is the set of pivot columns for B, namely $\{1, 2, 3, 4\}$.

The same pattern works for general n. In more detail, we can define vectors $w_1, \dots, w_{n-1}$ by $w_i = e_i - e_n$, and we set $W = \operatorname{span}(w_1, \dots, w_{n-1})$. For $i < n - 1$ we have

$$v_i = e_i - e_{i+1} = (e_i - e_n) - (e_{i+1} - e_n) = w_i - w_{i+1},$$

whereas v_{n-1} is just equal to w_{n-1}. This shows that $v_i \in W$ for all i, and it follows that $V \subseteq W$. In the opposite direction, we have

$$v_i + v_{i+1} + \cdots + v_{n-1} = (e_i - e_{i+1}) + (e_{i+1} - e_{i+2}) + \cdots + (e_{n-1} - e_n) = e_i - e_n = w_i,$$

which shows that $w_i \in V$ for all i, and thus that $W \subseteq V$. It follows that $W = V$, so the list $\mathcal{W} = w_1, \dots, w_{n-1}$ spans V. The corresponding matrix $B = [w_1|\cdots|w_{n-1}]^T$ is clearly in RREF (and has no zero rows), so $\mathcal{W}$ is in fact the canonical basis for V. It follows that $\dim(V) = n - 1$ and $J(V) = \{1, 2, \dots, n - 1\}$. □

Exercise 72. Put $V = \operatorname{ann}(a_1, a_2, a_3) \subseteq \mathbb{R}^6$, where

$$a_1 = \begin{bmatrix} 1 & 1 & 2 & 3 & 3 & 2 \end{bmatrix}^T$$
$$a_2 = \begin{bmatrix} 3 & 3 & 2 & 1 & 1 & 2 \end{bmatrix}^T$$
$$a_3 = \begin{bmatrix} 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}^T.$$

Find the canonical basis for V.

Solution: The equations $a_3.x = a_2.x = a_1.x = 0$ can be written as

$$x_6 + x_5 + x_4 + x_3 = 0$$
$$2x_6 + x_5 + x_4 + 2x_3 + 3x_2 + x_1 = 0$$
$$2x_6 + 3x_5 + 3x_4 + 2x_3 + x_2 + x_1 = 0.$$

The matrix A on the left below is $[a_1|a_2|a_3]^T$; the matrix A^* on the right is obtained by turning A through 180° and is the matrix of coefficients in the above system of equations.

$$A = \begin{bmatrix} 1 & 1 & 2 & 3 & 3 & 2 \\ 3 & 3 & 2 & 1 & 1 & 2 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix} \qquad A^* = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 2 & 1 & 1 & 2 & 3 & 3 \\ 2 & 3 & 3 & 2 & 1 & 1 \end{bmatrix}.$$

We can row-reduce A^* as follows:

$$A^* \to \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & -1 & -1 & 0 & 3 & 3 \\ 0 & 1 & 1 & 0 & 1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 1 & 3 & 3 \\ 0 & 1 & 1 & 0 & -3 & -3 \\ 0 & 0 & 0 & 0 & 4 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix} = B^*$$

The matrix B^* corresponds to the system of equations

$$x_6 + x_3 = 0$$
$$x_5 + x_4 = 0$$
$$x_2 + x_1 = 0,$$

which can be rewritten as $x_6 = -x_3$ and $x_5 = -x_4$ and $x_2 = -x_1$. This gives

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} x_1 \\ -x_1 \\ x_3 \\ x_4 \\ -x_4 \\ -x_3 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ -1 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}.$$

It follows that the vectors

$$v_1 = \begin{bmatrix}1 & -1 & 0 & 0 & 0 & 0\end{bmatrix}^T$$
$$v_2 = \begin{bmatrix}0 & 0 & 1 & 0 & 0 & -1\end{bmatrix}^T$$
$$v_3 = \begin{bmatrix}0 & 0 & 0 & 1 & -1 & 0\end{bmatrix}^T$$

form the canonical basis for V.

The calculation can be written more compactly in terms of Method 20.23. The matrix B^* has pivot columns 1, 2 and 5, and non-pivot columns 3, 4 and 6. Deleting the pivot columns leaves the matrix

$$C^* = \begin{bmatrix} c_1^T \\ \hline c_2^T \\ \hline c_3^T \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

We then construct the matrix

$$D^* = \begin{bmatrix} -c_1 & -c_2 & e_1 & e_2 & -c_3 & e_3 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 \end{bmatrix}$$

and rotate it to get

$$D = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & -1 & 0 \end{bmatrix}.$$

The canonical basis vectors v_i appear as the rows of D. □

Exercise 73. Put

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 1 \\ 1 & 3 & 3 & 1 \\ 1 & 4 & 4 & 1 \end{bmatrix}.$$

Find the canonical basis for $\operatorname{img}(A)$, and the canonical basis for $\ker(A)$.

Solution: First, let $a_1, \dots, a_4$ be the columns of A. Proposition 19.19 tellus us that $\operatorname{img}(A) = \operatorname{span}(a_1, \dots, a_4)$. To find the canonical basis for this space, Method 20.14 tells us that we should form the matrix whose rows are $a_1^T, \dots, a_4^T$, but that matrix is just A^T. We can row-reduce A^T as follows:

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

By looking at the transposed rows of the final matrix, we see that the canonical basis for $\operatorname{img}(A)$ consists of the vectors

$$u_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ -2 \end{bmatrix} \qquad \text{and} \qquad u_2 = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix}.$$

Next, we recall that $\ker(A)$ is the set of vectors x that satisfy $Ax = 0$. After noting that

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 1 \\ 1 & 3 & 3 & 1 \\ 1 & 4 & 4 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 + x_3 + x_4 \\ x_1 + 2x_2 + 2x_3 + x_4 \\ x_1 + 3x_2 + 3x_3 + x_4 \\ x_1 + 4x_2 + 4x_3 + x_4 \end{bmatrix},$$

we see that $\ker(A)$ is the set of solutions to the equations

$$x_1 + x_2 + x_3 + x_4 = 0$$
$$x_1 + 2x_2 + 2x_3 + x_4 = 0$$
$$x_1 + 3x_2 + 3x_3 + x_4 = 0$$
$$x_1 + 4x_2 + 4x_3 + x_4 = 0.$$

These are easily solved to give $x_4 = -x_1$ and $x_3 = -x_2$ with x_1 and x_2 arbitrary. (In order to get the canonical basis rather than any other basis, we need to write things this way around, with the higher-numbered variables on the left written in terms of the lower-numbered variables on the right.) This gives

$$x = \begin{bmatrix} x_1 \\ x_2 \\ -x_2 \\ -x_1 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}.$$

From this we see that the canonical basis for $\ker(A)$ consists of the vectors

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix} \qquad \text{and} \qquad v_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}.$$

□

17. Lecture 17

Exercise 74. Put

$$v_1 = \begin{bmatrix} 1 \\ 3 \\ 5 \\ 3 \end{bmatrix} \; v_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ -3 \end{bmatrix} \; w_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \; w_2 = \begin{bmatrix} 3 \\ 2 \\ 1 \\ 0 \end{bmatrix}$$

and $V = \text{span}(v_1, v_2)$ and $W = \text{span}(w_1, w_2)$.

(a) Find the canonical basis for $V + W$.
(b) Find vectors a_1 and a_2 such that $V = \text{ann}(a_1, a_2)$.
(c) Find vectors b_1 and b_2 such that $W = \text{ann}(b_1, b_2)$.
(d) Find the canonical basis for $V \cap W$.

Solution:

(a) We can row-reduce the matrix $[v_1|v_2|w_1|w_2]^T$ as follows:

$$\begin{bmatrix} 1 & 3 & 5 & 3 \\ 1 & 1 & 1 & -3 \\ 1 & 2 & 3 & 4 \\ 3 & 2 & 1 & 0 \end{bmatrix} \to \begin{bmatrix} 0 & 2 & 4 & 6 \\ 1 & 1 & 1 & -3 \\ 0 & 1 & 2 & 7 \\ 0 & -1 & -2 & 9 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 & -3 \\ 0 & 1 & 2 & 7 \\ 0 & 0 & 0 & -8 \\ 0 & 0 & 0 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

We deduce that the vectors

$$p_1 = \begin{bmatrix} 1 & 0 & -1 & 0 \end{bmatrix}^T \qquad p_2 = \begin{bmatrix} 0 & 1 & 2 & 0 \end{bmatrix}^T \qquad p_3 = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}^T$$

form the canonical basis for $V + W$.

(b) The equations $x.v_2 = x.v_1 = 0$ can be written as

$$-3x_4 + x_3 + x_2 + x_1 = 0$$
$$3x_4 + 5x_3 + 3x_2 + x_1 = 0.$$

These can be solved in the usual way to give $x_4 = x_2/9 + 2x_1/9$ and $x_3 = -2x_2/3 - x_1/3$. This in turn gives

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ -2x_2/3 - x_1/3 \\ x_2/9 + 2x_1/9 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ -1/3 \\ 2/9 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ -2/3 \\ 1/9 \end{bmatrix}.$$

It follows that $V = \text{ann}(a_1, a_2)$, where

$$a_1 = \begin{bmatrix} 1 & 0 & -1/3 & 2/9 \end{bmatrix}^T \qquad a_2 = \begin{bmatrix} 0 & 1 & -2/3 & 1/9 \end{bmatrix}^T.$$

(c) The method is the same as for part (b). The equations $x.w_2 = x.w_1 = 0$ can be written as

$$x_3 + 2x_2 + 3x_1 = 0$$
$$4x_4 + 3x_3 + 2x_2 + x_1 = 0$$

and these can be solved to give $x_4 = x_2 + 2x_1$ and $x_3 = -2x_2 - 3x_1$. This in turn gives

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ -2x_2 - 3x_1 \\ x_2 + 2x_1 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ -3 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ -2 \\ 1 \end{bmatrix}.$$

It follows that $W = \operatorname{ann}(b_1, b_2)$, where

$$b_1 = \begin{bmatrix} 1 & 0 & -3 & 2 \end{bmatrix}^T \qquad b_2 = \begin{bmatrix} 0 & 1 & -2 & 1 \end{bmatrix}^T.$$

(d) Now $V \cap W = \operatorname{ann}(a_1, a_2) \cap \operatorname{ann}(b_1, b_2) = \operatorname{ann}(a_1, a_2, b_1, b_2)$. To save writing we will use the pure matrix method to calculate this. The relevant matrix A^* has rows consisting of the vectors b_2, b_1, a_2 and a_1 written backwards:

$$A^* = \begin{bmatrix} 1 & -2 & 1 & 0 \\ 2 & -3 & 0 & 1 \\ 1/9 & -2/3 & 1 & 0 \\ 2/9 & -1/3 & 0 & 1 \end{bmatrix}$$

This can be row-reduced as follows:

$$A^* \to \begin{bmatrix} 1 & -2 & 1 & 0 \\ 2 & -3 & 0 & 1 \\ 1 & -6 & 9 & 0 \\ 2 & -3 & 0 & 9 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & -4 & 8 & 0 \\ 0 & 1 & -2 & 9 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -3 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = B^*$$

The matrix B^* corresponds to the system of equations $x_4 = 3x_2$ and $x_3 = 2x_2$ and $x_1 = 0$, so $x = x_2 \begin{bmatrix} 0 & 1 & 2 & 3 \end{bmatrix}$. It follows that $V \cap W$ is the set of multiples of the vector $q = \begin{bmatrix} 0 & 1 & 2 & 3 \end{bmatrix}^T$, so q on its own is the canonical basis for $V \cap W$.

□

Exercise 75. Put

$$U = \{x \in \mathbb{R}^3 \mid x_1 + 2x_2 + 2x_3 = 0\}$$
$$V = \{x \in \mathbb{R}^3 \mid 4x_1 - x_2 - x_3 = 0\}.$$

Find the canonical bases for U, V, $U + V$ and $U \cap V$.

Solution: First, we put $a = \begin{bmatrix} 1 & 2 & 2 \end{bmatrix}$ and $b = \begin{bmatrix} 4 & -1 & -1 \end{bmatrix}$. We have $a.x = x_1 + 2x_2 + 2x_3$, so U can be described as $U = \{x \mid x.a = 0\}$ or equivalently $U = \operatorname{ann}(a)$. Similarly, we have $V = \operatorname{ann}(b)$.

For $x \in U$ we have $x_3 = -x_1/2 - x_2$, so

$$x = \begin{bmatrix} x_1 \\ x_2 \\ -x_1/2 - x_2 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ -1/2 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}.$$

It follows that the vectors $u_1 = \begin{bmatrix} 1 & 0 & -1/2 \end{bmatrix}^T$ and $u_2 = \begin{bmatrix} 0 & 1 & -1 \end{bmatrix}^T$ form the canonical basis for U.

Similarly, for $x \in V$ we have $x_3 = 4x_1 - x_2$ so

$$x = \begin{bmatrix} x_1 \\ x_2 \\ 4x_1 - x_2 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix},$$

so the vectors $v_1 = \begin{bmatrix} 1 & 0 & 4 \end{bmatrix}^T$ and $v_2 = \begin{bmatrix} 0 & 1 & -1 \end{bmatrix}^T$ form the canonical basis for V.

It now follows that $U + V = \operatorname{span}(u_1, u_2, v_1, v_2)$. However, we can omit v_2 because it is the same as u_2, so $U + V = \operatorname{span}(u_1, u_2, v_1)$. To find the canonical basis for this space we row-reduce the matrix $[u_1|u_2|v_1]^T$:

$$\begin{bmatrix} 1 & 0 & -1/2 \\ 0 & 1 & -1 \\ 1 & 0 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1/2 \\ 0 & 1 & -1 \\ 0 & 0 & 9/2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1/2 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} e_1^T \\ \hline e_2^T \\ \hline e_3^T \end{bmatrix}$$

It follows that e_1, e_2, e_3 is the canonical basis for $U + V$ and so $U + V = \mathbb{R}^3$.

The dimension formula now gives

$$\dim(U \cap V) = \dim(U) + \dim(V) - \dim(U + V) = 2 + 2 - 3 = 1.$$

It follows that any nonzero vector in $U \cap V$ (considered as a list of length 1) forms a basis for $U \cap V$. We have seen that the vector $w = \begin{bmatrix} 0 & 1 & -1 \end{bmatrix}^T = u_2 = v_2$ lies in both U and V, so it forms a basis for $U \cap V$. The first nonzero entry in w is 1, so this is the canonical basis.

For a more direct approach, we can use the fact that

$$U \cap V = \operatorname{ann}(a) \cap \operatorname{ann}(b) = \operatorname{ann}(a, b).$$

The equations $x.b = x.a = 0$ can be written with the variables in decreasing order as

$$\begin{aligned} 2x_3 + 2x_2 + x_1 &= 0 \\ -x_3 - x_2 + 4x_1 &= 0. \end{aligned}$$

These equations can be solved to give $x_3 = -x_2$ and $x_1 = 0$, so

$$x = \begin{bmatrix} 0 \\ x_2 \\ -x_2 \end{bmatrix} = x_2 \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} = x_2 w.$$

Form this we again see that w is the canonical basis for $U \cap V$. □

Exercise 76. Let V be the set of all vectors of the form

$$v = \begin{bmatrix} p+q & 2p-2q & 3p+3q & 4p-4q \end{bmatrix}^T.$$

(a) Find vectors v_1 and v_2 such that $V = \operatorname{span}(v_1, v_2)$.
(b) Find vectors w_1 and w_2 such that $V = \operatorname{ann}(w_1, w_2)$.

Solution:

(a) A general element $v \in V$ can be written as

$$v = \begin{bmatrix} p+q \\ 2p-2q \\ 3p+3q \\ 4p-4q \end{bmatrix} = p \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} + q \begin{bmatrix} 1 \\ -2 \\ 3 \\ -4 \end{bmatrix}.$$

It follows that if we put $v_1 = \begin{bmatrix} 1 & 2 & 3 & 4 \end{bmatrix}^T$ and $v_2 = \begin{bmatrix} 1 & -2 & 3 & -4 \end{bmatrix}^T$ then the elements of V are precisely the linear combinations of v_1 and v_2, or in other words $V = \operatorname{span}(v_1, v_2)$.

If we want we can tidy this up by row-reduction:

$$[v_1|v_2]^T = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & -2 & 3 & -4 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -4 & 0 & -8 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 0 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & 0 & 2 \end{bmatrix}.$$

It follows that V can also be described as $\operatorname{span}(v_1', v_2')$, where $v_1' = \begin{bmatrix} 1 & 0 & 3 & 0 \end{bmatrix}^T$ and $v_2' = \begin{bmatrix} 0 & 1 & 0 & 2 \end{bmatrix}^T$. (In fact, v_1' and v_2' form the canonical basis for V.)

(b) The equations $x.v_2 = 0$ and $x.v_1 = 0$ can be written as

$$\begin{aligned} -4x_4 + 3x_3 - 2x_2 + x_1 &= 0 \\ 4x_4 + 3x_3 + 2x_2 + x_1 &= 0. \end{aligned}$$

By adding the above equations we get $6x_2 + 2x_1 = 0$ or $x_3 = -x_1/3$. By subtracting the above equations we get $8x_4 + 4x_2 = 0$ or $x_4 = -x_2/2$. This gives

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ -x_1/3 \\ -x_2/2 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ -1/3 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1/2 \end{bmatrix}.$$

It follows that $V = \operatorname{ann}(w_1, w_2)$, where $w_1 = \begin{bmatrix} 1 & 0 & -1/3 & 0 \end{bmatrix}^T$ and $w_2 = \begin{bmatrix} 0 & 1 & 0 & -1/2 \end{bmatrix}$.

Note that we could also have started with the equations $x.v_2' = x.v_1' = 0$ instead of $x.v_2 = x.v_1 = 0$ and we would still have obtained the same vectors w_i.

□

Exercise 77. For each of the following configurations, either find an example, or show that no example is possible.

(a) Subspaces $U, V \leq \mathbb{R}^4$ with $\dim(U) = \dim(V) = 3$ and $\dim(U \cap V) = 1$.
(b) Subspaces $U, V \leq \mathbb{R}^4$ with $\dim(U) = \dim(V) = 3$ and $\dim(U \cap V) = 2$.
(c) Subspaces $U, V \leq \mathbb{R}^5$ with $\dim(U) = \dim(V) = 2$ and $\dim(U + V) = 5$.

(d) Subspaces $U, V \leq \mathbb{R}^3$ with $\dim(U) = \dim(V) = \dim(U+V) = \dim(U \cap V)$.

Solution: We will repeatedly use the dimension formula

$$\dim(U) + \dim(V) = \dim(U+V) + \dim(U \cap V).$$

(a) This is not possible. Indeed, the dimension formula can be rearranged to give $\dim(U+V) = \dim(U) + \dim(V) - \dim(U \cap V) = 3 + 3 - 1 = 5$, but $U + V$ is a subspace of $\mathbb{R}^4$, so it cannot have dimension greater than 4.

(b) The simplest example is

$$\begin{aligned} U &= \text{span}(e_1, e_2, e_3) = \{\begin{bmatrix} w & x & y & 0 \end{bmatrix}^T \mid w, x, y \in \mathbb{R}\} \\ V &= \text{span}(e_1, e_2, e_4) = \{\begin{bmatrix} w & x & 0 & z \end{bmatrix}^T \mid w, x, z \in \mathbb{R}\} \\ U \cap V &= \text{span}(e_1, e_2) = \{\begin{bmatrix} w & x & 0 & 0 \end{bmatrix}^T \mid w, x \in \mathbb{R}\}. \end{aligned}$$

(c) This is not possible. Indeed, the dimension formula can be rearranged to give $\dim(U \cap V) = \dim(U) + \dim(V) - \dim(U+V) = 2 + 2 - 5 = -1$, but no subspace can have negative dimension.

(d) The minimal example here is to take $U = V = \{0\}$, so $U + V = U \cap V = \{0\}$ and $\dim(U) = \dim(V) = \dim(U+V) = \dim(U \cap V) = 0$. More generally, we can choose U to be any subspace of $\mathbb{R}^3$ (of dimension d, say) and take $V = U$. We then have $U + V = U + U = U$ and $U \cap V = U \cap U = U$ so $\dim(U) = \dim(V) = \dim(U+V) = \dim(U \cap V) = d$.

□

18. Lecture 18

Exercise 78. Find the ranks of the following matrices:

$$A = \begin{bmatrix} 0 & 1 & 2 \\ -1 & 0 & 3 \\ -2 & -3 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 4 & 5 & 6 & 7 \\ 3 & 4 & 5 & 6 & 7 & 8 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 10 & 100 \\ 10 & 100 & 1000 \\ 100 & 1000 & 10000 \end{bmatrix} \qquad D = \begin{bmatrix} 1 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \end{bmatrix}$$

Solution: The rank of a matrix M is the number of nonzero rows in the row-reduced form of M. We have row-reductions as follows:

$$A = \begin{bmatrix} 0 & 1 & 2 \\ -1 & 0 & 3 \\ -2 & -3 & 0 \end{bmatrix} \to \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & -3 \\ -2 & -3 & 0 \end{bmatrix} \to \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & -3 \\ 0 & -3 & -6 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 2 \\ 0 & -3 & -6 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

and

$$B = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 4 & 5 & 6 & 7 \\ 3 & 4 & 5 & 6 & 7 & 8 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 0 & -1 & -2 & -3 & -4 & -5 \\ 0 & -2 & -4 & -6 & -8 & -10 \end{bmatrix} \to$$

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 0 & 1 & 2 & 3 & 4 & 5 \\ 0 & -2 & -4 & -6 & -8 & -10 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 & -2 & -3 & -4 \\ 0 & 1 & 2 & 3 & 4 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

and

$$C = \begin{bmatrix} 1 & 10 & 100 \\ 10 & 100 & 1000 \\ 100 & 1000 & 10000 \end{bmatrix} \to \begin{bmatrix} 1 & 10 & 100 \\ 0 & 0 & 0 \\ 100 & 1000 & 10000 \end{bmatrix} \to \begin{bmatrix} 1 & 10 & 100 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and

$$D = \begin{bmatrix} 1 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & -1 & -1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & -1 & -1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \to$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

From this we see that $\mathrm{rank}(A) = \mathrm{rank}(B) = 2$ and $\mathrm{rank}(C) = 1$ and $\mathrm{rank}(D) = 3$. □

Exercise 79. Give examples as follows, or explain why no such examples are possible.

(a) A 3×5 matrix of rank 4.
(b) A 3×3 matrix of rank 1, in which none of the entries are zero.
(c) A 2×4 matrix A such that A has rank 1 and A^T has rank 2.
(d) A 3×3 matrix A such that $A + A^T = 0$ and A has rank 2.
(e) An invertible 3×3 matrix of rank 2.
(f) A matrix in RREF with rank 1 and 4 nonzero columns.

Solution:

(a) This is not possible, because the rank of any $m \times n$ matrix is at most the minimum of n and m, so a 3×5 matrix cannot have rank larger than 3.
(b) The simplest example is $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$.
(c) This is not possible, because A and A^T always have the same rank.
(d) The simplest example is $A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}$.
(e) This is not possible. If A is an *invertible* $n \times n$ matrix, then the columns form a basis for $\mathbb{R}^n$, which means that the rank must be n.
(f) One example is the matrix $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

□

Exercise 80. Consider the following matrices, which depend on a parameter t.

$$A = \begin{bmatrix} 1 & 0 \\ 0 & (t-3)(t-4) \end{bmatrix} \qquad B = \begin{bmatrix} 1 & t \\ t & 2t-1 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & t \\ 1 & 4 & t^2 \end{bmatrix} \qquad D = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 \\ 1 & 2 & t & 3 & t \\ 1 & 4 & t^2 & 7 & 3 \end{bmatrix}$$

It should be clear that A usually has rank 2, except that when $t = 3$ or $t = 4$ the second row becomes zero and so the rank is only 1. In the same way, for each of the other matrices, there is a usual value for the rank, but the rank drops for some exceptional values of t.

(1) Simplify B by row and column operations. Do not divide any row or column by anything that depends on t, but make B as simple as you can without such divisions.
(2) What is the usual rank of B?
(3) What is the exceptional value of t for which the rank of B is lower? What is the rank in that case?
(4) What is the usual rank of C, and what are the exceptional cases? (Use the same method as for B.)
(5) What is the usual rank of D, and what are the exceptional cases? (**Hint:** how is D related to C?)

Solution:

(1) Subtract t times the first row from the second row, then subtract t times the first column from the second column:
$$B = \begin{bmatrix} 1 & t \\ t & 2t-1 \end{bmatrix} \to \begin{bmatrix} 1 & t \\ 0 & -t^2+2t-1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 \\ 0 & -t^2+2t-1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -(t-1)^2 \end{bmatrix} = B'.$$
We might now be tempted to divide the second row by $-(t-1)^2$ to get the identity matrix. However, that would not be valid when $t = 1$, because then we would be dividing by zero. It is for this reason that the question tells you not to divide by anyhing that depends on t.
(2) As row and column operations do not affect the rank, we have $\mathrm{rank}(B) = \mathrm{rank}(B')$. If $t \neq 1$ then it is clear that the two rows in B' are linearly independent and so $\mathrm{rank}(B) = \mathrm{rank}(B') = 2$; this is the usual case.
(3) In the exceptional case where $t = 1$ we have $B' = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and it is clear that $\mathrm{rank}(B) = \mathrm{rank}(B') = 1$.

(4) We can simplify C by row and column operations as follows.

$$C = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & t \\ 1 & 4 & t^2 \end{bmatrix} \xrightarrow{1} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & t-1 \\ 0 & 3 & t^2-1 \end{bmatrix} \xrightarrow{2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & t-1 \\ 0 & 3 & t^2-1 \end{bmatrix} \xrightarrow{3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 3 & t^2-3t+2 \end{bmatrix} \xrightarrow{4} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & t^2-3t+2 \end{bmatrix} = C'$$

(Step 1: subtract row 1 from the other two rows; Step 2: subtract column 1 from the other two columns; Step 3: add $1-t$ times column 2 to column 3; Step 4: subtract 3 times row 2 from row 3.) Note also that $t^2-3t+2 = (t-1)(t-2)$. For most values of t this will be nonzero, so $\text{rank}(C) = \text{rank}(C') = 3$. The exceptional cases are where $t=1$ or $t=2$, in which case $C' = \left[\begin{smallmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right]$ and $\text{rank}(C) = \text{rank}(C') = 2$.

(5) C consists of the first three columns of D. If $t \neq 1, 2$ then $\text{rank}(C) = 3$ so the columns of C span $\mathbb{R}^3$, so the columns of D certainly span $\mathbb{R}^3$, so $\text{rank}(D) = 3$. In the case $t = 1$ we can write down D and simplify by column operations as follows:

$$D = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 \\ 1 & 2 & 1 & 3 & 1 \\ 1 & 4 & 1 & 7 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 2 & 1 \\ 1 & 3 & 0 & 6 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ -2 & 3 & 0 & 0 & 0 \end{bmatrix} = D'.$$

It is clear that in this case we have $\text{rank}(D) = \text{rank}(D') = 2$. In the other exceptional case where $t = 2$ we can write down D and simplify by column operations as follows:

$$D = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 \\ 1 & 2 & 2 & 3 & 2 \\ 1 & 4 & 4 & 7 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 2 & 2 \\ 1 & 3 & 0 & 6 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 3 & 0 & 0 & -3 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix} = D''.$$

It is clear from this that the case $t = 2$ is not in fact exceptional for D, because we have $\text{rank}(D) = \text{rank}(D'') = 3$ in that case (which is the same answer as for every other value of t except $t = 1$).

□