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1. Vector spaces

1.1. Linear combinations (continued)

Consider any field $\mathbb{K}$ (for example, $\mathbb{R}$) and any $\mathbb{K}$-vector space V (for example, $\mathbb{K}^n$, or $\mathbb{K}^{n \times m}$, or a vector space of polynomials, or a vector space of functions).

Last time, we defined $\mathbb{K}$-linear combinations and $\mathbb{K}$-linear (in)dependence of a list of vectors in V. These definitions were modeled after the definitions of the corresponding concepts for column vectors, but with the obvious changes (the scalars now come from $\mathbb{K}$ rather than from $\mathbb{R}$, and the concrete zero vector $0_{n\times 1}$ has been replaced by the zero vector $\overrightarrow{0}$ of V).

For the rest of this lecture, we fix an arbitrary field $\mathbb{K}$. From now on, "vector space" always means "$\mathbb{K}$-vector space". You can imagine that $\mathbb{K} = \mathbb{R}$, as this is the most frequently used choice of $\mathbb{K}$.

In Proposition 1.1.1 of the class notes from 2019-10-07, we proved the following proposition for column vectors in $\mathbb{R}^n$; the same proof works for an arbitrary vector space:

> **Proposition 1.1.1.** Let $v_1, v_2, \ldots, v_k$ be k vectors in a vector space V. Then, $v_1, v_2, \ldots, v_k$ are dependent **if and only if** one of them is a combination of the others.

We shall also need the following variant of this proposition:

> **Proposition 1.1.2.** Let $v_1, v_2, \ldots, v_k$ be k independent vectors in a vector space V. Let v be a further vector in V. Then, the $k+1$ vectors $v_1, v_2, \ldots, v_k, v$ are dependent if and only if v is a combination of $v_1, v_2, \ldots, v_k$.

Proof. This is an "if and only if" statement. We shall prove its "$\Longrightarrow$" and "$\Longleftarrow$" directions separately:

$\Longrightarrow$: Assume that the $k+1$ vectors $v_1, v_2, \ldots, v_k, v$ are dependent. We must show that v is a combination of $v_1, v_2, \ldots, v_k$.

By assumption, the $k+1$ vectors $v_1, v_2, \ldots, v_k, v$ are dependent. In other words, there exists a nontrivial relation

$$\lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_k v_k + \lambda v = \overrightarrow{0} \qquad \text{with } \lambda_1, \lambda_2, \ldots, \lambda_k, \lambda \in \mathbb{K}.$$

Since this relation is nontrivial, at least one of the coefficients $\lambda_1, \lambda_2, \ldots, \lambda_k, \lambda$ must be nonzero. However, if $\lambda = 0$, then the relation

$$\lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_k v_k + \lambda v = \overrightarrow{0}$$

rewrites as

$$\lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_k v_k = \overrightarrow{0},$$

which is impossible because $v_1, v_2, \ldots, v_k$ are linearly independent (and at least one of $\lambda_1, \lambda_2, \ldots, \lambda_k$ must be nonzero, since $\lambda = 0$). Thus, λ cannot be 0. So $\lambda \neq 0$. Since $\mathbb{K}$ is a field, this means that λ has a multiplicative inverse $\frac{1}{\lambda}$. Thus, we can solve the equality $\lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_k v_k + \lambda v = \overrightarrow{0}$ for v, obtaining

$$\begin{aligned} v &= -\frac{1}{\lambda}\left(\lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_k v_k\right) \\ &= \frac{-\lambda_1}{\lambda} v_1 + \frac{-\lambda_2}{\lambda} v_2 + \cdots + \frac{-\lambda_k}{\lambda} v_k. \end{aligned}$$

But this shows that v is a combination of $v_1, v_2, \ldots, v_k$. Thus, we have proved the "$\Longrightarrow$" direction of the proposition.

$\Longleftarrow$: Assume that v is a combination of $v_1, v_2, \ldots, v_k$. We must show that the $k+1$ vectors $v_1, v_2, \ldots, v_k, v$ are dependent.

Since v is a combination of $v_1, v_2, \ldots, v_k$, we can write v as

$$v = \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_k v_k \qquad \text{with } \alpha_1, \alpha_2, \ldots, \alpha_k \in \mathbb{K}.$$

We can rewrite this equality as

$$\begin{aligned} \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_k v_k - v &= \overrightarrow{0}, \qquad \text{i.e., as} \\ \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_k v_k + (-1)\, v &= \overrightarrow{0}. \end{aligned}$$

This is a nontrivial relation between $v_1, v_2, \ldots, v_k, v$ (nontrivial because the last coefficient is $-1 \neq 0$). Hence, the $k+1$ vectors $v_1, v_2, \ldots, v_k, v$ are dependent. Thus, we have proved the "$\Longleftarrow$" direction of the proposition. □

1.2. Subspaces

When U is a subset of a vector space V, we can try to make U itself into a vector space by "inheriting" the addition, the scaling and the zero vector from V: That is, we define the sum $v + w$ of two vectors $v, w \in U$ to be the result of adding v and w as elements of V; similarly we define scaling and the zero vector. However, this all works only if the vectors appearing in these definitions actually lie in U. So let us gather the necessary conditions in the definition of a **subspace**:

Definition 1.2.1. Let U be a subset of a vector space V. We say that U is a **subspace** (or **vector subspace**) of V if the following three conditions hold:

- **(a)** We have $\overrightarrow{0} \in U$ (where $\overrightarrow{0}$ is the zero vector of V). (In other words, U contains the zero vector.)
- **(b)** We have $v + w \in U$ for all $v, w \in U$. (In other words, U is closed under addition.)
- **(c)** We have $\lambda v \in U$ for all $\lambda \in \mathbb{K}$ and $v \in U$. (In other words, U is closed under scaling.)

Proposition 1.2.2. Let U be a subspace of a vector space V. Then, U becomes a vector space, if we let it "inherit" the addition $+$, the scaling $\cdot$ and the zero vector $\overrightarrow{0}$ from V. Here, "inheriting" means that:

- we define the sum $v + w$ of two vectors $v, w \in U$ to be the result of adding v and w as elements of V.
- we define λv for $\lambda \in \mathbb{K}$ and $v \in U$ to be the result of scaling v by λ as element of V.
- we define the zero vector $\overrightarrow{0}$ of U as the zero vector $\overrightarrow{0}$ of V.

Proof. We just need to check that the axioms in the definition of a vector space are satisfied for U. But this is clear, because they are satisfied for V and because the operations of U are simply restrictions of the corresponding operations of V. □

What are some examples of subspaces? Let us first go for the lowest-hanging fruits:

Proposition 1.2.3. Let V be a vector space.
(a) The subset V of V is a subspace of V.
(b) The subset $\left\{ \overrightarrow{0} \right\}$ of V is a subspace of V.

Proof. **(a)** This is clear, since V contains the zero vector and is closed under addition and is closed under scaling.

(b) The subset $\left\{ \overrightarrow{0} \right\}$ contains the zero vector (indeed, it contains the zero vector and nothing else). It is closed under addition (since $\overrightarrow{0} + \overrightarrow{0} = \overrightarrow{0} \in \left\{ \overrightarrow{0} \right\}$) and closed under scaling (since $\lambda \cdot \overrightarrow{0} = \overrightarrow{0} \in \left\{ \overrightarrow{0} \right\}$ for each $\lambda \in \mathbb{K}$). Thus, $\left\{ \overrightarrow{0} \right\}$ is a subspace of V. □

The two subspaces $\left\{ \overrightarrow{0} \right\}$ and V are the "two extremes" for how large a subspace of V can be. Any subspace W of V is somewhere between $\left\{ \overrightarrow{0} \right\}$ and V (in the sense that it satisfies $\left\{ \overrightarrow{0} \right\} \subseteq W \subseteq V$).

Let us next explore some more specific examples of subspaces (see [lina, Example 4.25] for details).

Recall that $\mathbb{R}^n = \{\text{column vectors of size } n \text{ with real entries}\}$ for each $n \in \mathbb{N}$. As we know, this is a vector space. Its zero vector $\overrightarrow{0}$ is $0_{n\times 1} = (0, 0, \ldots, 0)^T$. Its addition is just usual addition of vectors, and its scaling is just usual scaling of vectors.

Now, let us construct a few subsets of $\mathbb{R}^3$ and check whether they are subspaces of $\mathbb{R}^3$.

Example 1.2.4. The subset $A := \left\{ (x_1, x_2, x_3)^T \in \mathbb{R}^3 \mid x_1 - x_2 + 2x_3 = 0 \right\}$ is a subspace of $\mathbb{R}^3$.

Proof. Let's check the three requirements for a subspace:

- $\overrightarrow{0} \in A$, because $\overrightarrow{0} = 0_{3\times 1} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ satisfies $0 - 0 + 2 \cdot 0 = 0$.
- A is closed under addition: If $v = (x_1, x_2, x_3)^T$ and $w = (y_1, y_2, y_3)^T$ both belong to A, then $x_1 - x_2 + 2x_3 = 0$ and $y_1 - y_2 + 2y_3 = 0$. Now, $v + w = (x_1 + y_1, x_2 + y_2, x_3 + y_3)^T$ satisfies

$$(x_1 + y_1) - (x_2 + y_2) + 2(x_3 + y_3) = \underbrace{(x_1 - x_2 + 2x_3)}_{=0} + \underbrace{(y_1 - y_2 + 2y_3)}_{=0} = 0$$

  and thus belongs to A.
- A is closed under scaling: If $\lambda \in \mathbb{K}$ and if $v = (x_1, x_2, x_3)^T$ belongs to A, then $x_1 - x_2 + 2x_3 = 0$. Now, $\lambda v = (\lambda x_1, \lambda x_2, \lambda x_3)^T$ satisfies

$$\lambda x_1 - \lambda x_2 + 2\lambda x_3 = \lambda \underbrace{(x_1 - x_2 + 2x_3)}_{=0} = \lambda 0 = 0$$

  and thus belongs to A.

Thus, A is a subspace of $\mathbb{R}^3$. □

Example 1.2.5. The subset $B := \left\{ (x_1, x_2, x_3)^T \in \mathbb{R}^3 \mid x_1 = 2x_2 = 3x_3 \right\}$ is a subspace of $\mathbb{R}^3$.

Proof. Similar to the proof above. □

Example 1.2.6. The subset $C := \left\{ (x_1, x_2, x_3)^T \in \mathbb{R}^3 \mid x_1 - x_2 + 2x_3 = 1 \right\}$ is **not** a subspace of $\mathbb{R}^3$.

Proof. We don't have $\overrightarrow{0} \in C$, because $\overrightarrow{0} = 0_{3\times 1} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ does not satisfy $0 - 0 + 2 \cdot 0 = 1$. □

Example 1.2.7. The subset $D := \left\{ (a, 0, 2a + b)^T \mid a, b \in \mathbb{R} \right\}$ is a subspace of $\mathbb{R}^3$.

Proof. Let us again check all three requirements in the definition of a subspace:

- We have $\overrightarrow{0} \in D$, since $\overrightarrow{0}$ has the form $(a, 0, 2a + b)^T$ for some $a, b \in \mathbb{R}$ (namely, $a = 0$ and $b = 0$).
- D is closed under addition (since any two vectors $v = (a_1, 0, 2a_1 + b_1)^T$ and $w = (a_2, 0, 2a_2 + b_2)^T$ in D satisfy

$$\begin{aligned} v + w &= (a_1, 0, 2a_1 + b_1)^T + (a_2, 0, 2a_2 + b_2)^T \\ &= (a_1 + a_2, 0 + 0, (2a_1 + b_1) + (2a_2 + b_2))^T \\ &= (a_1 + a_2, 0, 2(a_1 + a_2) + (b_1 + b_2))^T \\ &= (a, 0, 2a + b)^T \qquad \text{for } a = a_1 + a_2 \text{ and } b = b_1 + b_2; \end{aligned}$$

  thus, they satisfy $v + w \in D$).
- D is closed under scaling (this is proved similarly).

□

Example 1.2.8. The subset $E := \left\{ (a, 0, a + 1)^T \mid a \in \mathbb{R} \right\}$ is **not** a subspace of $\mathbb{R}^3$.

Proof. We have $\overrightarrow{0} \notin E$, because we cannot write $\overrightarrow{0}$ in the form $(a, 0, a + 1)^T$ for any $a \in \mathbb{R}$. □

Example 1.2.9. The subset $F := \left\{ (x_1, x_2, x_3)^T \in \mathbb{R}^3 \mid x_1 x_2 x_3 = 0 \right\}$ is **not** a subspace of $\mathbb{R}^3$.

Proof. The set F is not closed under addition, because the two vectors $v = (1,1,0)^T$ and $w = (0,0,1)^T$ belong to F, but their sum $v + w = (1,1,1)^T$ does not.

(That said, F contains $\overrightarrow{0}$ and is closed under scaling.) □

> **Example 1.2.10.** The subset $G := \left\{ (x_1, x_2, x_3)^T \in \mathbb{R}^3 \mid x_1, x_2, x_3 \in \mathbb{Q} \right\}$ is **not** a subspace of $\mathbb{R}^3$.

Proof. The set G is not closed under scaling, because $\lambda = \sqrt{2}$ and $v = (1,1,1)^T \in G$ lead to $\lambda v = \left(\sqrt{2}, \sqrt{2}, \sqrt{2}\right)^T \notin G$.

(That said, G contains $\overrightarrow{0}$ and is closed under addition.) □

> **Example 1.2.11.** The subset $H := \left\{ (a, 0, 2a + b + 1)^T \mid a, b \in \mathbb{R} \right\}$ is a subspace of $\mathbb{R}^3$.

Proof. This is a trick question. The "+1" makes it look like it's not a subspace. But we can simply substitute c for $b + 1$, and then get

$$\begin{aligned} H &= \left\{ (a, 0, 2a + c)^T \mid a, c \in \mathbb{R} \right\} \\ &= \left\{ (a, 0, 2a + b)^T \mid a, b \in \mathbb{R} \right\} = D. \end{aligned}$$

We know that D is a subspace of $\mathbb{R}^3$, so we conclude that H is a subspace of $\mathbb{R}^3$. □

From the geometric point of view,

- the subspaces of $\mathbb{R}^2$ are $\left\{ \overrightarrow{0} \right\}$, $\mathbb{R}^2$ and all lines through the origin.
- the subspaces of $\mathbb{R}^3$ are $\left\{ \overrightarrow{0} \right\}$, $\mathbb{R}^3$, all lines through the origin, and all planes through the origin.

We can also take a look at subspaces of vector spaces other than $\mathbb{R}^n$:

> **Example 1.2.12.** Let P be the $\mathbb{R}$-vector space of all polynomials in one variable x with real coefficients.
>
> **(a)** Let $P_{1\to 0} = \{ f \in P \mid f(1) = 0 \}$. Then, $P_{1\to 0}$ is a subspace of P.
>
> (This is easy to check. For example, $P_{1\to 0}$ is closed under addition, because if two polynomials f, g satisfy $f(1) = 0$ and $g(1) = 0$, then their sum $f + g$ also satisfies $(f + g)(1) = 0$.)
>
> **(b)** Let $P_{0\to 1} = \{ f \in P \mid f(0) = 1 \}$. Then, $P_{0\to 1}$ is **not** a subspace of P.
>
> (For example, it does not contain the zero vector, which is the zero polynomial 0.)
>
> **(c)** Let $P_5 = \{ f \in P \mid \deg f \leq 5 \}$ (where $\deg f$ denotes the degree of the polynomial f, and we understand the zero polynomial to have negative degree). In

other words, $P_5 = \{a_0 + a_1 x + a_2 x^2 + \cdots + a_5 x^5 \mid a_0, a_1, \ldots, a_5 \in \mathbb{R}\}$. Then, P_5 is a subspace of P.

(Again, this is easy to check. For example, P_5 is closed under addition, because the sum of two polynomials of degree ≤ 5 is again of degree ≤ 5.)

(d) Let $P_5^\circ = \{f \in P \mid \deg f = 5\}$. Then, P_5° is **not** a subspace of P.

(Again, this is because it does not contain the zero vector. But even if it did, it would also fail the "closed under addition" axiom.)

1.3. Spans

Recall the definition of a span of k column vectors. We can use the same definition to define the span of k vectors in a vector space V:

Definition 1.3.1. Let $v_1, v_2, \ldots, v_k$ be some vectors in a vector space V.

(a) The **span** of $v_1, v_2, \ldots, v_k$ is the set of all combinations of $v_1, v_2, \ldots, v_k$. It is called $\operatorname{span}(v_1, v_2, \ldots, v_k)$ (or sometimes $\langle v_1, v_2, \ldots, v_k \rangle$).

(b) We say that the vectors $v_1, v_2, \ldots, v_k$ **span** V if and only if $\operatorname{span}(v_1, v_2, \ldots, v_k) = V$. In other words, they span V if and only if each vector in V is a combination of $v_1, v_2, \ldots, v_k$.

Just as we did for column vectors (in Proposition 1.2.1 of the class notes from 2019-10-07), we can show the following proposition:

Proposition 1.3.2. Let $v_1, v_2, \ldots, v_k$ be some vectors in a vector space V. Then, any combination of combinations of $v_1, v_2, \ldots, v_k$ is a combination of $v_1, v_2, \ldots, v_k$.

From this, we can obtain the following:

Theorem 1.3.3. Let $v_1, v_2, \ldots, v_k$ be some vectors in a vector space V. Then, $\operatorname{span}(v_1, v_2, \ldots, v_k)$ is a subspace of V.

Proof. We need to check that $\operatorname{span}(v_1, v_2, \ldots, v_k)$ contains $\overrightarrow{0}$ and is closed under addition and closed under scaling.

Indeed:

- $\overrightarrow{0} \in \operatorname{span}(v_1, v_2, \ldots, v_k)$ (since $\overrightarrow{0} = 0v_1 + 0v_2 + \cdots + 0v_k$).
- $\operatorname{span}(v_1, v_2, \ldots, v_k)$ is closed under addition; i.e., if $a, b \in \operatorname{span}(v_1, v_2, \ldots, v_k)$, then $a + b \in \operatorname{span}(v_1, v_2, \ldots, v_k)$. [*Proof:* Let $a, b \in \operatorname{span}(v_1, v_2, \ldots, v_k)$. Thus, a and b are combinations of $v_1, v_2, \ldots, v_k$. Hence, $a + b$ is a combination of combinations of $v_1, v_2, \ldots, v_k$ (since $a + b$ is clearly a combination of a and b); but thus, by Proposition 1.3.2, we conclude that $a + b$ is a combination of $v_1, v_2, \ldots, v_k$. In other words, $a + b \in \operatorname{span}(v_1, v_2, \ldots, v_k)$. Qed.]
- $\operatorname{span}(v_1, v_2, \ldots, v_k)$ is closed under scaling (for similar reasons).

□

By the way, the following is fundamental:

Proposition 1.3.4. Let U be a subspace of a vector space V. Let $u_1, u_2, \ldots, u_k$ be any vectors in U. Then, any combination of $u_1, u_2, \ldots, u_k$ must belong to U.

(In other words, a subspace of V is closed under linear combination.)

Proof of Proposition 1.3.4. We must prove that $\lambda_1 u_1 + \lambda_2 u_2 + \cdots + \lambda_k u_k \in U$ for any $\lambda_1, \lambda_2, \ldots, \lambda_k \in \mathbb{K}$.

Fix $\lambda_1, \lambda_2, \ldots, \lambda_k \in \mathbb{K}$.

Recall that U is a subspace of V. Hence, U contains $\overrightarrow{0}$, is closed under addition and is closed under scaling. In particular, a sum of **two** elements of U is always an element of U (because U is closed under addition). Thus, it is easy to see that a sum of **any number of** elements of U is always an element of U (indeed, you can prove this by induction; the induction base follows from $\overrightarrow{0} \in U$).

For each $i \in \{1, 2, \ldots, k\}$, we have $\lambda_i \in \mathbb{K}$ and $u_i \in U$ and therefore $\lambda_i u_i \in U$ (since U is a subspace of V). Thus, $\lambda_1 u_1, \lambda_2 u_2, \ldots, \lambda_k u_k$ are k elements of U. Hence, $\lambda_1 u_1 + \lambda_2 u_2 + \cdots + \lambda_k u_k$ is a sum of k elements of U, and therefore is an element of U (because a sum of any number of elements of U is always an element of U). This completes our proof of Proposition 1.3.4. □

1.4. Finite-dimensional vector spaces and dimension

Definition 1.4.1. Let V be a vector space. We say that V is **finite-dimensional** (for short, **fin-dim**) if there exists a finite list of vectors $(v_1, v_2, \ldots, v_k)$ that spans V.

Example 1.4.2. (a) Let $n \in \mathbb{N}$. Then, the space $\mathbb{K}^n$ of column vectors of size n is fin-dim. Indeed, the finite list $(e_1, e_2, \ldots, e_n)$ (where $e_i = (0, 0, \ldots, 0, 1, 0, 0, \ldots, 0)^T$ with the 1 being in position i) spans $\mathbb{K}^n$.

(b) Let $n, m \in \mathbb{N}$. Then, the space $\mathbb{K}^{n \times m}$ of $n \times m$-matrices is fin-dim. Indeed, if $n = 2$ and $m = 2$, then the finite list

$$\left(\underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}}_{\substack{\text{the four matrices, each of which has a 1 in} \\ \text{some position, and zeroes everywhere else}}} \right)$$

spans $\mathbb{K}^{n \times m}$ (because every 2×2-matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ can be written as

$$a \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + b \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + c \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + d \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

). This shows that $\mathbb{K}^{n\times m}$ is fin-dim when $n = 2$ and $m = 2$. Similarly you can prove this for any n and m (but there will be nm rather than 4 matrices in the spanning list).

(c) Let P be the $\mathbb{R}$-vector space of all polynomials in one variable x with real coefficients. Then, P is **not** fin-dim.

(d) Fix $k \in \mathbb{N}$. Let P_k be the $\mathbb{R}$-vector space of all polynomials of degree $\leq k$ in one variable x with real coefficients. Then, P_k is a subspace of P, and is fin-dim. Indeed, the list $\left(x^0, x^1, \ldots, x^k\right)$ spans P_k.

(e) Fix $a \in \mathbb{R}$. Let $P_{a\to 0}$ be the $\mathbb{R}$-vector space of all polynomials f in one variable x with real coefficients satisfying $P(a) = 0$ (in other words, having a as a root). Then, $P_{a\to 0}$ is a subspace of P, but is **not** fin-dim.

We shall soon extend some properties of column vectors to properties of vectors in arbitrary fin-dim vector spaces. The following theorem (due to Steinitz) will be our main tool:

Theorem 1.4.3 (Steinitz's theorem)**.** Let V be a vector space. Let $n \in \mathbb{N}$. Let $v_1, v_2, \ldots, v_n$ be n vectors in V. Then, any $n+1$ combinations of $v_1, v_2, \ldots, v_n$ are dependent.

Proof. We use induction on n.

The *base case* ($n = 0$) is trivial: In this case, we are just saying that any 1 combination of 0 vectors is dependent. But the only combination of 0 vectors is $\overrightarrow{0}$ (since the only combination we can form without having any vectors is the empty sum), which is clearly dependent[1].

Induction step: Let $n > 0$, and assume (as the induction hypothesis) that any n combinations of any $n-1$ given vectors are dependent.

We must now prove that any $n+1$ combinations of any n given vectors are dependent. So let $v_1, v_2, \ldots, v_n$ be n vectors in V. We must prove that any $n+1$ combinations of $v_1, v_2, \ldots, v_n$ are dependent.

Let

$$\begin{aligned} y_1 &= k_{1,1}v_1 + k_{1,2}v_2 + \cdots + k_{1,n}v_n, \\ y_2 &= k_{2,1}v_1 + k_{2,2}v_2 + \cdots + k_{2,n}v_n, \\ &\vdots \\ y_{n+1} &= k_{n+1,1}v_1 + k_{n+1,2}v_2 + \cdots + k_{n+1,n}v_n \end{aligned}$$

be $n+1$ combinations of $v_1, v_2, \ldots, v_n$ (where all $k_{i,j}$ are scalars). We must prove that $y_1, y_2, \ldots, y_{n+1}$ are dependent.

If all the coefficients $k_{i,j}$ are 0, then $y_1, y_2, \ldots, y_{n+1}$ all equal the zero vector $\overrightarrow{0}$, and thus are clearly dependent. Hence, from now on, we assume that not all the

[1] To be more precise: the list $\left(\overrightarrow{0}\right)$ consisting of the zero vector is dependent (since $1 \cdot \overrightarrow{0} = \overrightarrow{0}$ is a nontrivial relation for it).

coefficients $k_{i,j}$ are 0. In other words, at least one coefficient $k_{i,j}$ is $\neq 0$. Without loss of generality, we thus assume that $k_{1,1} \neq 0$ (since otherwise, we can move the nonzero $k_{i,j}$ to the position of $k_{1,1}$ by swapping the vectors v_1 and v_j and swapping the combinations y_1 and y_i). Consider the n vectors

$$\begin{aligned} z_2 &= k_{1,1}y_2 - k_{2,1}y_1, \\ z_3 &= k_{1,1}y_3 - k_{3,1}y_1, \\ &\vdots \\ z_{n+1} &= k_{1,1}y_{n+1} - k_{n+1,1}y_1 \end{aligned}$$

(so $z_i = k_{1,1}y_i - k_{i,1}y_1$ for each $i \in \{2, 3, \ldots, n+1\}$). These n vectors $z_2, z_3, \ldots, z_{n+1}$ are linear combinations of the $n-1$ vectors $v_2, v_3, \ldots, v_n$, because for each $i \in \{2, 3, \ldots, n+1\}$, we have

$$\begin{aligned} z_i &= k_{1,1} \underbrace{y_i}_{=k_{i,1}v_1 + k_{i,2}v_2 + \cdots + k_{i,n}v_n} - k_{i,1} \underbrace{y_1}_{=k_{1,1}v_1 + k_{1,2}v_2 + \cdots + k_{1,n}v_n} \\ &= k_{1,1} \left(k_{i,1}v_1 + k_{i,2}v_2 + \cdots + k_{i,n}v_n\right) - k_{i,1} \left(k_{1,1}v_1 + k_{1,2}v_2 + \cdots + k_{1,n}v_n\right) \\ &= \left(k_{1,1}k_{i,1}v_1 + k_{1,1}k_{i,2}v_2 + \cdots + k_{1,1}k_{i,n}v_n\right) - \left(k_{i,1}k_{1,1}v_1 + k_{i,1}k_{1,2}v_2 + \cdots + k_{i,1}k_{1,n}v_n\right) \\ &= \left(k_{1,1}k_{i,2}v_2 + \cdots + k_{1,1}k_{i,n}v_n\right) - \left(k_{i,1}k_{1,2}v_2 + \cdots + k_{i,1}k_{1,n}v_n\right) \\ &\qquad \left(\begin{array}{c}\text{here, we have cancelled the first addends in both parentheses,} \\ \text{as they were equal}\end{array}\right) \\ &= \left(k_{1,1}k_{i,2} - k_{i,1}k_{1,2}\right) v_2 + \cdots + \left(k_{1,1}k_{i,n} - k_{i,1}k_{1,n}\right) v_n. \end{aligned}$$

Thus, by the induction hypothesis, these n vectors $z_2, z_3, \ldots, z_{n+1}$ are dependent. In other words, there exists a nontrivial relation

$$\ell_2 z_2 + \ell_3 z_3 + \cdots + \ell_{n+1} z_{n+1} = \overrightarrow{0}$$

between them. Substituting the definition of z_i into this relation, we obtain

$$\ell_2 \left(k_{1,1}y_2 - k_{2,1}y_1\right) + \ell_3 \left(k_{1,1}y_3 - k_{3,1}y_1\right) + \cdots + \ell_{n+1} \left(k_{1,1}y_{n+1} - k_{n+1,1}y_1\right) = \overrightarrow{0}.$$

Expanding the left hand side and re-grouping the addends according to the y_j vector appearing in them, we transform this equality into

$$\left(-\ell_2 k_{2,1} - \ell_3 k_{3,1} - \cdots - \ell_{n+1} k_{n+1,1}\right) y_1 + \ell_2 k_{1,1} y_2 + \ell_3 k_{1,1} y_3 + \cdots + \ell_{n+1} k_{1,1} y_{n+1} = \overrightarrow{0}.$$

This is a relation between $y_1, y_2, \ldots, y_{n+1}$. Since the relation

$$\ell_2 z_2 + \ell_3 z_3 + \cdots + \ell_{n+1} z_{n+1} = \overrightarrow{0}$$

is nontrivial, there exists at least one $i \in \{2, 3, \ldots, n+1\}$ such that $\ell_i \neq 0$. Hence, this i also satisfies $\ell_i k_{1,1} \neq 0$ (since $k_{1,1} \neq 0$). Thus, our relation between $y_1, y_2, \ldots, y_{n+1}$ is also nontrivial. This shows that $y_1, y_2, \ldots, y_{n+1}$ are dependent. This completes the induction step.

Hence, Steinitz's theorem is proved by induction. □

I learned the above proof from [Charli19, Démonstration de Théorème 2.8].

References

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