Fall 2019 Math 201-003 at Drexel: blackboard notes of 2019-11-13

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1. Eigenvalues and eigenvectors ("eigenstuff")

1.1. Complex numbers

1.1.1. Some definitions done last time

Recall the definition of complex numbers:

Definition 1.1.1. **(a)** A **complex number** is defined as a pair (a, b) of two real numbers.

(b) We let $\mathbb{C}$ be the set of all complex numbers.

(c) For each real number r, we denote the complex number $(r, 0)$ as $r_{\mathbb{C}}$ (and we will later equate it with r).

(d) We let i be the complex number $(0, 1)$.

(e) We define three binary operations $+$, $-$ and $\cdot$ on $\mathbb{C}$ by setting

$$(a,b) + (c,d) = (a+c, b+d), \tag{1}$$

$$(a,b) - (c,d) = (a-c, b-d), \tag{2}$$

$$(a,b) \cdot (c,d) = (ac-bd, ad+bc). \tag{3}$$

(Thus, the operations $+$ and $-$ are just entrywise addition and subtraction, just as for row vectors. The operation $\cdot$ is more complicated, and we will soon see why we have defined it in this particular way.)

(f) As usual, we write $\alpha\beta$ for $\alpha \cdot \beta$ if α and β are complex numbers.

As usual, we write $-\alpha$ for $0_{\mathbb{C}} - \alpha$ if α is a complex number.

We equated each real number r with the complex number $r_{\mathbb{C}} = (r, 0)$.

The complex number $i = (0, 1)$ satisfies $i^2 = -1$.

We represent complex numbers on the **Argand diagram**: To each complex number $a + bi = (a, b)$ corresponds the point (a, b).

Definition 1.1.2. Let z be a complex number. Write z in the form $z = (a, b) = a + bi$ for two real numbers a, b.

Let P_z be the point corresponding to the complex number z on the Argand diagram. (As you remember, this is the point with coordinates (a, b).)

(a) The real numbers a and b are called the **real part** and the **imaginary part** of z. They are the Cartesian coordinates of the point P_z.

(b) The **absolute value** of z is defined to be the real number $\sqrt{a^2 + b^2}$. This is the distance between the origin and P_z. This is the first (radial) **polar coordinate** of P_z. It is denoted by $|z|$.

(c) Assume that $z \neq 0$. Then, consider the angle φ (with $-\pi < \varphi \leq \pi$) at which the ray from 0 to P_z stands to the ray from 0 to 1 (i.e., the positive half-axis). This is the second (angular) **polar coordinate** of P_z. It is denoted by $\arg z$, and is called the **argument** of z.

1.1.2. More about angles

Remark 1.1.3. Let $z = a + bi$ (with $a, b \in \mathbb{R}$ and $z \neq 0$) be a complex number.

How to compute $\arg z$ through a and b ?

Let P_z be the point (a, b) as in the previous definition. Recall that $\arg z$ is the angle at which the ray from 0 to P_z stands to the ray from 0 to 1. The slope of the former ray is b/a (since $P_z = (a, b)$). Thus, the classical relation between the slope of a line and its angle against the x-axis shows that

$$\tan(\arg z) = \frac{b}{a}.$$

Thus, it makes sense to expect $\arg z = \arctan \frac{b}{a}$. But this is not quite the case, since $\frac{b}{a}$ only determines the **line** from 0 to P_z, whereas $\arg z$ depends on the **ray** from 0 to P_z; thus, $\arg z$ depends on "what side of the origin" z lies on. So the correct way to determine $\arg z$ is the following:

- We have $\tan(\arg z) = \frac{b}{a}$. (If $a = 0$, then this must be interpreted as $\tan(\arg z) = \frac{b}{0} = \infty$, which means that $\arg z$ is either $\frac{\pi}{2}$ or $\frac{-\pi}{2}$.)
- We have $\arg z \geq 0$ if and only if $b \geq 0$.

If you know the "two-variable arctangent function" atan2, then you can rewrite this as follows:

$$\arg z = \operatorname{atan2}(b, a).$$

Definition 1.1.4. For any angle φ, we define a complex number $\operatorname{cis}\varphi$ by

$$\operatorname{cis}\varphi = \cos\varphi + i\sin\varphi = (\cos\varphi, \sin\varphi).$$

Geometrically (i.e., on the Argand diagram), the point $P_{\operatorname{cis}\varphi}$ corresponding to this complex number $\operatorname{cis}\varphi$ is the point obtained by rotating $P_1 = (1,0)$ through the angle[1] φ around the origin. (Indeed, the latter point clearly has abscissa $\cos\varphi$ and ordinate $\sin\varphi$, because of how cosine and sine are defined; but so does the point $P_{\operatorname{cis}\varphi}$.) Thus, in particular, the point $P_{\operatorname{cis}\varphi}$ lies on the unit circle (i.e., the circle with center at the origin and radius 1).

Note that every angle φ satisfies $\operatorname{cis}(\varphi + 2\pi) = \operatorname{cis}\varphi$ (since $\cos(\varphi + 2\pi) = \cos\varphi$ and $\sin(\varphi + 2\pi) = \sin\varphi$). More general, two angles α and β satisfy $\operatorname{cis}\alpha = \operatorname{cis}\beta$ if and only if $\alpha - \beta$ is a multiple of 2π. This is often restated as follows: The complex number $\operatorname{cis}\varphi$ uniquely determines the angle φ up to a multiple of 2π.

Proposition 1.1.5. For any two angles α and β, we have

$$\operatorname{cis}(\alpha + \beta) = \operatorname{cis}\alpha \cdot \operatorname{cis}\beta.$$

Proof. This follows by comparing

$$\begin{aligned}\operatorname{cis}(\alpha+\beta) &= \underbrace{\cos(\alpha+\beta)}_{=\cos\alpha\cos\beta-\sin\alpha\sin\beta} + i\underbrace{\sin(\alpha+\beta)}_{=\sin\alpha\cos\beta+\cos\alpha\sin\beta} \\ &= \cos\alpha\cos\beta - \sin\alpha\sin\beta + i(\sin\alpha\cos\beta + \cos\alpha\sin\beta)\end{aligned}$$

with

$$\begin{aligned}&\underbrace{\operatorname{cis}\alpha}_{=\cos\alpha+i\sin\alpha} \cdot \underbrace{\operatorname{cis}\beta}_{=\cos\beta+i\sin\beta} \\ &= (\cos\alpha + i\sin\alpha)\cdot(\cos\beta + i\sin\beta) \\ &= \cos\alpha\cos\beta + \cos\alpha\cdot i\sin\beta + i\sin\alpha\cos\beta + \underbrace{i^2}_{=-1}\sin\alpha\sin\beta \\ &= \cos\alpha\cos\beta - \sin\alpha\sin\beta + i(\sin\alpha\cos\beta + \cos\alpha\sin\beta).\end{aligned}$$

□

Note that Proposition 1.1.5 "packages" both formulas

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta \qquad \text{and}$$
$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta.$$

[1] Angles are always measured counterclockwise in mathematics. For example, the point obtained by rotating P_1 through the angle $\pi/2 = 90^\circ$ around the origin is the point $P_i = (0,1)$.

Proposition 1.1.6. Let z be a nonzero complex number. Then,

$$z = |z| \cdot \operatorname{cis}(\arg z).$$

Proof sketch. Geometrically, this can be seen as follows: The complex numbers $\operatorname{cis}(\arg z)$ and z have the same argument (namely, $\arg z$); thus, the points corresponding to them lie on the same ray from the origin. Hence, $z = r \cdot \operatorname{cis}(\arg z)$ for some nonnegative real r. To find this r, we just compare $|z|$ with $|\operatorname{cis}(\arg z)| = 1$. Thus, we get $r = |z|$. □

Corollary 1.1.7. When we multiply two complex numbers, their absolute values get multiplied, while their arguments get added (modulo 2π).

Proof sketch. We are claiming that any complex numbers z and w satisfy

$$|zw| = |z| \cdot |w| \qquad \text{and} \qquad \arg(zw) \equiv \arg z + \arg w \bmod 2\pi,$$

where "$\alpha \equiv \beta \bmod 2\pi$" means "$\alpha - \beta$ is an integer multiple of 2π".

How do we prove this?

To prove $|zw| = |z| \cdot |w|$, it suffices to recall the definition of absolute value and compute both sides. (This is Exercise 10 on MT2 preparation.)

It remains to prove that

$$\arg(zw) \equiv \arg z + \arg w \bmod 2\pi.$$

In other words, it remains to prove that

$$\operatorname{cis}(\arg(zw)) = \operatorname{cis}(\arg z + \arg(w)),$$

because $\operatorname{cis}\varphi$ uniquely determines φ up to a multiple of 2π.

Proposition 1.1.6 yields $z = |z| \cdot \operatorname{cis}(\arg z)$, thus

$$\operatorname{cis}(\arg z) = \frac{z}{|z|}.$$

Similarly,

$$\operatorname{cis}(\arg w) = \frac{w}{|w|} \qquad \text{and} \qquad \operatorname{cis}(\arg(zw)) = \frac{zw}{|zw|}.$$

Hence,

$$\begin{aligned} \operatorname{cis}(\arg(zw)) &= \frac{zw}{|zw|} = \frac{zw}{|z| \cdot |w|} \qquad (\text{since } |zw| = |z| \cdot |w|) \\ &= \underbrace{\frac{z}{|z|}}_{=\operatorname{cis}(\arg z)} \cdot \underbrace{\frac{w}{|w|}}_{=\operatorname{cis}(\arg w)} = \operatorname{cis}(\arg z) \cdot \operatorname{cis}(\arg w) \\ &= \operatorname{cis}(\arg z + \arg w) \end{aligned}$$

(since Proposition 1.1.5 yields $\operatorname{cis}(\arg z + \arg w) = \operatorname{cis}(\arg z) \cdot \operatorname{cis}(\arg w)$). □

Thus, for a fixed complex number z, the map

$$\mathbb{C} \to \mathbb{C},$$
$$w \mapsto zw$$

is a rotation (around the origin, with angle $\arg z$) composed with a homothety (around the origin, with scaling factor $|z|$).

Remark 1.1.8. Here is a digression for those who know a bit of complex analysis (or at least the complex exponential function).

The complex exponential function $\exp : \mathbb{C} \to \mathbb{C}$ is defined by

$$\exp t = 1 + \frac{t}{1!} + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + \cdots . \qquad (4)$$

The infinite sum on the right hand side of this equation converges (once you have appropriately defined convergence of sums of complex numbers), so this function exp is well-defined. If t is a real number, then $\exp t = e^t$ (where $e \approx 2.7183$ is the famous "number e"); thus, it is common to write e^t for $\exp t$ even when t is a complex number (although there is no general well-defined concept of a^b for complex numbers a and b).

Now, Euler's formula says that $\operatorname{cis} \varphi = \exp(i\varphi)$ for any angle φ. Restated in explicit language, this is saying that if you plug $i\varphi$ for t in the power series (4), then you get a power series that converges to $\operatorname{cis} \varphi$. In other words,

$$1 + \frac{i\varphi}{1!} + \frac{(i\varphi)^2}{2!} + \frac{(i\varphi)^3}{3!} + \frac{(i\varphi)^4}{4!} + \cdots = \operatorname{cis} \varphi = \cos \varphi + i \sin \varphi.$$

Since the powers of i are

$$\begin{array}{llll} i^0 = 1, & i^1 = i, & i^2 = -1, & i^3 = -i, \\ i^4 = 1, & i^5 = i, & i^6 = -1, & i^7 = -i, \\ i^8 = 1, & \ldots & (\text{so } i^{n+4} = i^n \text{ for each } n), & \end{array}$$

you can rewrite this as

$$1 + \frac{i\varphi}{1!} - \frac{\varphi^2}{2!} - \frac{i\varphi^3}{3!} + \frac{\varphi^4}{4!} + \frac{i\varphi^5}{5!} - \frac{\varphi^6}{6!} - \frac{i\varphi^7}{7!} + \cdots = \cos \varphi + i \sin \varphi.$$

Splitting this equality into its real and imaginary parts, we obtain

$$1 - \frac{\varphi^2}{2!} + \frac{\varphi^4}{4!} - \frac{\varphi^6}{6!} + \cdots = \cos \varphi \qquad \text{and}$$
$$\frac{\varphi}{1!} - \frac{\varphi^3}{3!} + \frac{\varphi^5}{5!} - \frac{\varphi^7}{7!} + \cdots = \sin \varphi.$$

These are the classical Maclaurin power series for cos and sin.

Also, $e^{i\varphi} = \exp(i\varphi) = \operatorname{cis} \varphi$ yields $e^{i\pi} = \operatorname{cis} \pi = -1$, the famous Euler identity.

1.2. Application of diagonalization to ODEs

Let us talk a bit about an application of diagonalization of matrices.

This section follows [Strickland, §15].

Recall one of the simplest forms of an ODE (ordinary differential equation): We are looking for a function x in a single variable t that satisfies $x' = ax$, where a is a real constant, and where x' denotes the derivative of x in t. (People often write $\dot{x}$ for x'.)

The solutions of this ODE are all functions of the form ce^{at} for constants $c \in \mathbb{R}$.

Now, imagine we want to solve a system of two ODEs for two functions x and y. For example, let us solve

$$\begin{cases} x' = x + y \\ y' = x + y \end{cases}.$$

Rewrite this system as

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix}, \qquad \text{where } A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

Note that the column vectors now have functions as entries (rather than numbers), but this doesn't change anything.

Now, let us diagonalize A:

$$A = UDU^{-1}, \qquad U = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \qquad D = \operatorname{diag}(2,0) = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}.$$

Note that

$$U^{-1} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix}.$$

Thus, our system rewrites as

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = UDU^{-1} \begin{pmatrix} x \\ y \end{pmatrix}.$$

Rewrite this further by multiplying both sides by U^{-1}:

$$U^{-1} \begin{pmatrix} x' \\ y' \end{pmatrix} = DU^{-1} \begin{pmatrix} x \\ y \end{pmatrix}.$$

Note that

$$U^{-1} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{x+y}{2} \\ \frac{x-y}{2} \end{pmatrix}$$

and

$$U^{-1}\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \frac{x'+y'}{2} \\ \frac{x'-y'}{2} \end{pmatrix} = \begin{pmatrix} \left(\frac{x+y}{2}\right)' \\ \left(\frac{x-y}{2}\right)' \end{pmatrix}.$$

Setting

$$z = \frac{x+y}{2} \qquad \text{and} \qquad w = \frac{x-y}{2},$$

we can rewrite these as

$$U^{-1}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} z \\ w \end{pmatrix} \qquad \text{and} \qquad U^{-1}\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} z' \\ w' \end{pmatrix}.$$

So our system becomes

$$\begin{pmatrix} z' \\ w' \end{pmatrix} = D \begin{pmatrix} z \\ w \end{pmatrix}.$$

But

$$D = \operatorname{diag}(2,0), \qquad \text{so} \qquad D\begin{pmatrix} z \\ w \end{pmatrix} = \begin{pmatrix} 2z \\ 0w \end{pmatrix}.$$

So our system becomes

$$\begin{pmatrix} z' \\ w' \end{pmatrix} = \begin{pmatrix} 2z \\ 0w \end{pmatrix}.$$

This is tantamount to $z' = 2z$ and $w' = 0w$. The first of these equations involves z only, while the second involves w only. Solving them, we obtain $z = ce^{2t}$ and $w = d \underbrace{e^{0t}}_{=1} = d$ (for two real constants c and d).

So we know that

$$\frac{x+y}{2} = z = ce^{2t} \qquad \text{and} \qquad \frac{x-y}{2} = w = d.$$

We can recover x and y from these by treating these equations as a system of linear equations and solving them by Gaussian elimination. We get

$$x = ce^{2t} + d; \qquad y = ce^{2t} - d.$$

See [Strickland, §15] for further examples.

See also [Strickland, §16] for another application of diagonalization: the study of Markov chains. See [Strickland, §17] for an application to ranking websites for web search (Google's PageRank algorithm).

2. More about determinants

Let us state (without proof) a few more facts about determinants.

2.1. Determinants of block matrices

Theorem 2.1.1. Let $\left[\begin{array}{cc} A & B \\ 0_{q\times p} & D \end{array}\right]$ be a block matrix, where A is a $p \times p$-matrix and D is a $q \times q$-matrix. Then,

$$\det \left[\begin{array}{cc} A & B \\ 0_{q\times p} & D \end{array}\right] = \det A \cdot \det D.$$

Example 2.1.2. Let $p = 2$ and $q = 2$. Then, this theorem is saying

$$\det \left(\begin{array}{cccc} a & b & c & d \\ a' & b' & c' & d' \\ 0 & 0 & c'' & d'' \\ 0 & 0 & c''' & d''' \end{array}\right) = \det \left(\begin{array}{cc} a & b \\ a' & b' \end{array}\right) \cdot \det \left(\begin{array}{cc} c'' & d'' \\ c''' & d''' \end{array}\right).$$

Remark 2.1.3. It is **not** true that if A is a $p \times p$-matrix and D is a $q \times q$-matrix, then

$$\det \left[\begin{array}{cc} A & B \\ C & D \end{array}\right] = \det A \cdot \det D - \det B \cdot \det C. \tag{5}$$

Indeed, if $p \neq q$, then $\det B$ and $\det C$ make no sense to begin with (since B and C are not square matrices). But even when $p = q$, you can find counterexamples to (5).

However, here is something that is true:

Theorem 2.1.4. (The Schur complement theorem for determinants.)

Let $\left[\begin{array}{cc} A & B \\ C & D \end{array}\right]$ be a block matrix, where A is an invertible $p \times p$-matrix and D is a $q \times q$-matrix. Then,

$$\det \left[\begin{array}{cc} A & B \\ C & D \end{array}\right] = \det A \cdot \det \left(D - CA^{-1}B\right).$$

There is also a way to compute the inverse of a block matrix $\left[\begin{array}{cc} A & B \\ C & D \end{array}\right]^{-1}$ using its blocks. See Wikipedia: Schur complement.

2.2. An application of determinants

The following exercise is an example of how identifying a polynomial as a determinant can sometimes help us understand this polynomial better.

Exercise 2.2.1. Factor the polynomial

$$ab^2 + bc^2 + ca^2 - ac^2 - ba^2 - cb^2$$

(in three indeterminates a, b, c).

Solution. The polynomial $ab^2 + bc^2 + ca^2 - ac^2 - ba^2 - cb^2$ looks very much like the determinant of a 3×3-matrix (it is a sum of 6 terms, 3 of which have negative signs). A bit of experimentation confirms that it is indeed the determinant of a very simple 3×3-matrix:

$$ab^2 + bc^2 + ca^2 - ac^2 - ba^2 - cb^2 = \det \begin{pmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{pmatrix}.$$

We shall now compute this determinant in a different way: Rather than expanding it, we will use row operations to gradually simplify it:[2]

$$\begin{aligned}
&\det \begin{pmatrix} \boxed{1} & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{pmatrix} \\
&= \det \begin{pmatrix} \boxed{1} & 1 & 1 \\ 0 & \boxed{b-a} & c-a \\ a^2 & b^2 & c^2 \end{pmatrix} \\
&\qquad \left(\text{here, we have subtracted } a \cdot \text{row}\,1 \text{ from } \text{row}\,2\right) \\
&= \det \begin{pmatrix} \boxed{1} & 1 & 1 \\ 0 & \boxed{b-a} & c-a \\ 0 & b^2-a^2 & c^2-a^2 \end{pmatrix} \\
&\qquad \left(\text{here, we have subtracted } a^2 \cdot \text{row}\,1 \text{ from } \text{row}\,3\right) \\
&= \det \begin{pmatrix} \boxed{1} & 1 & 1 \\ 0 & \boxed{b-a} & c-a \\ 0 & 0 & c^2-a^2-(b+a)(c-a) \end{pmatrix} \\
&\qquad \left(\text{here, we have subtracted } (b+a) \cdot \text{row}\,2 \text{ from } \text{row}\,3\right) \\
&= 1\,(b-a) \underbrace{\left(c^2-a^2-(b+a)(c-a)\right)}_{\substack{=c^2-a^2-cb-ca+ab+a^2 \\ =c^2-cb-ca+ab=(c-a)(c-b)}} \\
&\qquad \left(\begin{array}{c}\text{since the determinant of an upper-triangular matrix} \\ \text{is the product of its diagonal entries}\end{array}\right)
\end{aligned}$$

[2]As we have already done many times, we draw boxes around pivot entries.

$$= (b-a)(c-a)(c-b).$$

Thus,

$$ab^2 + bc^2 + ca^2 - ac^2 - ba^2 - cb^2 = \det \begin{pmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{pmatrix} = (b-a)(c-a)(c-b).$$

□

References

[Strickland] Neil Strickland, *MAS201 Linear Mathematics for Applications*, lecture notes, 28 September 2013.
`http://neil-strickland.staff.shef.ac.uk/courses/MAS201/`