Fall 2019 Math 201-003 at Drexel: blackboard notes of 2019-10-23

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1. Determinants

1.1. Definition of a determinant

Fix a nonnegative integer n.

Last time, we defined:

- $[n]$ to be the set $\{1, 2, \ldots, n\}$.
- a **permutation** of $[n]$ to be a bijective map from $[n]$ to $[n]$.
- an **inversion** of a permutation σ of $[n]$ to be a pair (i, j) of two integers such that $1 \leq i < j \leq n$ and $\sigma(i) > \sigma(j)$.
- the **Coxeter length** $\ell(\sigma)$ of a permutation σ of $[n]$ to be the number of its inversions.
- the **sign** $\operatorname{sign}(\sigma)$ of a permutation σ of $[n]$ to be $(-1)^{\ell(\sigma)}$.

Thus, $\operatorname{sign}(\sigma)$ is 1 is σ has an even number of inversions, and is -1 if σ has an odd number of inversions.

[**Warning:** Some books define permutations rather differently! Also, the sign of σ is also known as the **signum** or the **signature** of σ, and is often denoted by $(-1)^\sigma$ or $\operatorname{sgn}(\sigma)$ or $\varepsilon(\sigma)$ or whatever else the author's creativity gives birth to. Really, you should not expect any two texts on this subject to use the same notations – not even if they are written by the same author.]

Now, recall the definition of a determinant:

Definition 1.1.1. Let A be an $n \times n$-matrix. Then, the **determinant** $\det A$ of A is defined to be the sum

$$\sum_{\sigma \text{ is a permutation of } [n]} \operatorname{sign}(\sigma) \cdot A_{1,\sigma(1)} A_{2,\sigma(2)} \cdots A_{n,\sigma(n)}.$$

Example 1.1.2.

$$\det \begin{pmatrix} a & b \\ a' & b' \end{pmatrix} = ab' - ba'.$$

[**Note:** Different authors define determinants in different ways (in particular, Lay defines them differently in [Lay]); but the definitions are all equivalent.]

What can we say about determinants?

First, we will have to understand permutations and their signs better.

1.2. More about permutations and their signs

Let $\beta : X \to Y$ and $\alpha : Y \to Z$ be two maps. Then, $\alpha \circ \beta$ (this is called "α **after** β" or "α **composed with** β" or "the **composition** of α and β") denotes the map from X to Z that first applies β and then applies α to the result; in other words,

$$(\alpha \circ \beta)(x) = \alpha(\beta(x)).$$

Proposition 1.2.1. Let α and β be two permutations of $[n]$. Then, $\alpha \circ \beta$ is a permutation of $[n]$.

Recall:

- If Z is a set, then the **identity map** of Z means the map from Z to Z that sends each $z \in Z$ to z itself. (In other words, it leaves every element unchanged.) It is denoted by id_Z. If Z is clear from the context, we can just call it id (omitting the subscript Z).

- Two maps $\alpha : X \to Y$ and $\beta : Y \to X$ are said to be **(mutually) inverse** if

$$\alpha \circ \beta = \mathrm{id}_Y \qquad \text{and} \qquad \beta \circ \alpha = \mathrm{id}_X.$$

  Roughly speaking, α and β are inverse if each of them undoes the other. If α and β are inverse, then β is said to be the **inverse** of α and is denoted by α^{-1}. (It is easy to see that a map has at most one inverse[1].)

Note that the identity map $\mathrm{id}_{[n]} : [n] \to [n]$ is clearly a permutation of $[n]$, and thus is also known as the **identity permutation**, or simply as the **identity**. This permutation id has no inversions (since $\mathrm{id}(k) = k$ for each k, and thus it is impossible that $i < j$ but $\mathrm{id}(i) > \mathrm{id}(j)$). Thus, it has Coxeter length $\ell(\mathrm{id}) = 0$ and sign $\operatorname{sign}(\mathrm{id}) = (-1)^{\ell(\mathrm{id})} = (-1)^0 = 1$.

[1]The argument is exactly the same as for the fact that each matrix has at most one inverse!

Proposition 1.2.2. Let α be a permutation of $[n]$. Then, α has an inverse α^{-1}, and this inverse α^{-1} is also a permutation of $[n]$.

Better yet: A map from $[n]$ to $[n]$ has an inverse if and only if it is a permutation.

So the permutations of $[n]$ form a set with some extra structure: You can compose them and invert them, and the results will always be permutations of $[n]$. Once you get to abstract algebra, you will recognize that this means that the permutations of $[n]$ form a group, called the n-th **symmetric group**.

Theorem 1.2.3. If an $n \times n$-matrix A is triangular (i.e., upper-triangular or lower-triangular), then its determinant is the product of its diagonal elements:

$$\det A = A_{1,1} A_{2,2} \cdots A_{n,n}.$$

Example 1.2.4. For $n = 3$, this means

$$\det \begin{pmatrix} a & b & c \\ 0 & b' & c' \\ 0 & 0 & c'' \end{pmatrix} = ab'c'' \qquad \text{and} \qquad \det \begin{pmatrix} a & 0 & 0 \\ a' & b' & 0 \\ a'' & b'' & c'' \end{pmatrix} = ab'c''.$$

In particular, the theorem says that the determinant of a diagonal matrix is the product of its diagonal entries.

Proof of Theorem 1.2.3. Assume that A is upper-triangular. (The lower-triangular case is similar.)

Now, the definition of $\det A$ yields

$$\det A = \sum_{\sigma \text{ is a permutation of } [n]} \operatorname{sign}(\sigma) \cdot A_{1,\sigma(1)} A_{2,\sigma(2)} \cdots A_{n,\sigma(n)}.$$

In this sum, one addend corresponds to $\sigma = \operatorname{id}$, the identity permutation (which sends each number to itself). This addend is

$$\underbrace{\operatorname{sign}(\operatorname{id})}_{\substack{=1 \\ \text{(since id has no inversions)}}} \cdot \underbrace{A_{1,\operatorname{id}(1)} A_{2,\operatorname{id}(2)} \cdots A_{n,\operatorname{id}(n)}}_{\substack{= A_{1,1} A_{2,2} \cdots A_{n,n} \\ \text{(since } \operatorname{id}(k) = k \text{ for each } k)}} = A_{1,1} A_{2,2} \cdots A_{n,n}.$$

This is good! Now we need to prove that all the other addends are 0.

In other words, we need to prove that if σ is a permutation of $[n]$ that is **not** the identity, then $\operatorname{sign}(\sigma) \cdot A_{1,\sigma(1)} A_{2,\sigma(2)} \cdots A_{n,\sigma(n)} = 0$.

So let σ be a permutation of $[n]$ that is **not** the identity. We must prove that $\operatorname{sign}(\sigma) \cdot A_{1,\sigma(1)} A_{2,\sigma(2)} \cdots A_{n,\sigma(n)} = 0$. In other words, we must prove that $A_{1,\sigma(1)} A_{2,\sigma(2)} \cdots A_{n,\sigma(n)} = 0$. In other words, we must prove that at least one of the entries $A_{1,\sigma(1)}, A_{2,\sigma(2)}, \ldots, A_{n,\sigma(n)}$ is 0. It will clearly suffice to show that at least one of these entries lies below the diagonal of A (because A is upper-triangular, so all its entries below the diagonal are 0). In other words, it will suffice to show that at least one $i \in \{1, 2, \ldots, n\}$ satisfies $i > \sigma(i)$ (because then, $A_{i,\sigma(i)}$ will be an entry of A below the diagonal, and thus will be 0).

We will prove this by contradiction: Assume the contrary. Thus, no $i \in \{1, 2, \ldots, n\}$ satisfies $i > \sigma(i)$. In other words, each $i \in \{1, 2, \ldots, n\}$ satisfies $i \leq \sigma(i)$. Hence:

- we have $n \leq \sigma(n)$, and thus $\sigma(n) = n$ (since $\sigma(n)$ must be in $[n]$);
- we have $n - 1 \leq \sigma(n-1)$, and thus $\sigma(n-1) = n - 1$ (since $\sigma(n-1)$ must be in $[n]$, but n is already taken as a value of σ);
- we have $n - 2 \leq \sigma(n-2)$, and thus $\sigma(n-2) = n - 2$ (since $\sigma(n-2)$ must be in $[n]$, but n and $n-1$ are already taken as values of σ);
- and so on.

Thus, $\sigma(i) = i$ for all i. In other words, σ is the identity permutation. This contradicts the fact that σ is not the identity permutation.

This contradiction completes our proof. $\square$

Recall that we have used the elementary matrices as "building blocks" for matrices. In a similar way, the transpositions (which we will now define) serve as "building blocks" for permutations:

Definition 1.2.5. Let p and q be two distinct elements of $[n]$. Then, the **transposition** $t_{p,q}$ denotes the permutation of $[n]$ that interchanges p with q and leaves all other elements of $[n]$ unchanged.

Thus, if $n = 7$, then

$$t_{2,4} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 1 & 4 & 3 & 2 & 5 & 6 & 7 \end{pmatrix} \qquad \text{(in two-line notation)}.$$

Example 1.2.6. Let $n = 7$. Consider a permutation σ of $[7]$, written in two-line notation as follows:

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \sigma(1) & \sigma(2) & \sigma(3) & \sigma(4) & \sigma(5) & \sigma(6) & \sigma(7) \end{pmatrix}.$$

(a) The two-line notation of $\sigma \circ t_{2,4}$ is

$$\sigma \circ t_{2,4} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \sigma(1) & \sigma(4) & \sigma(3) & \sigma(2) & \sigma(5) & \sigma(6) & \sigma(7) \end{pmatrix}$$

or, equivalently,

$$\sigma \circ t_{2,4} = \begin{pmatrix} 1 & 4 & 3 & 2 & 5 & 6 & 7 \\ \sigma(1) & \sigma(2) & \sigma(3) & \sigma(4) & \sigma(5) & \sigma(6) & \sigma(7) \end{pmatrix}.$$

In other words, the two-line notation of $\sigma \circ t_{2,4}$ is obtained from σ by swapping 2 with 4 in the top line.

(b) The two-line notation of $t_{2,4} \circ \sigma$ is obtained from σ by swapping 2 with 4 in the bottom line.

More generally:

Proposition 1.2.7. Let σ be any permutation of $[n]$. Let p and q be two distinct elements of $[n]$. Then:

(a) The two-line notation of $\sigma \circ t_{p,q}$ is obtained from the two-line notation of σ by swapping p with q in the top line.

(b) The two-line notation of $t_{p,q} \circ \sigma$ is obtained from the two-line notation of σ by swapping p with q in the bottom line.

You can easily convince yourself that $\sigma \circ t_{p,q}$ and $t_{p,q} \circ \sigma$ are two different permutations in general. (For example, $t_{1,2} \circ t_{2,3} \neq t_{2,3} \circ t_{1,2}$.)

Recall: The composition of two maps α and β is $\alpha \circ \beta$. We can also define the composition $\alpha_1 \circ \alpha_2 \circ \cdots \circ \alpha_k$ of k maps $\alpha_1, \alpha_2, \ldots, \alpha_k$. (This will be the map that applies α_k first, then α_{k-1}, then α_{k-2}, and so on.[2]) A composition of 0 maps is understood to mean the identity map.

We won't directly use the following theorem, but it explains the role of transpositions as "building blocks" for arbitrary permutations:

Theorem 1.2.8. Any permutation of $[n]$ is a composition of some transpositions.

In other words, if σ is a permutation of $[n]$, then we can write σ in the form

$$\sigma = t_{p_1,q_1} \circ t_{p_2,q_2} \circ \cdots \circ t_{p_k,q_k} \qquad \text{for some } p_1, q_1, p_2, q_2, \ldots, p_k, q_k \in [n] .$$

(It is possible that $k = 0$ here, in which case the right hand side of this equation should be understood to be a composition of 0 maps, i.e., the identity permutation of $[n]$.)

Proof idea. We can rewrite

$$\sigma = t_{p_1,q_1} \circ t_{p_2,q_2} \circ \cdots \circ t_{p_k,q_k}$$

as

$$\sigma = t_{p_1,q_1} \circ t_{p_2,q_2} \circ \cdots \circ t_{p_k,q_k} \circ \mathrm{id} .$$

But this can be restated as "σ is obtained from id by a sequence of swaps of entries in the bottom row (of the two-line notation)" (because of Proposition 1.2.7 **(b)**).

For example, let $n = 5$ and $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 5 & 4 & 1 & 3 \end{pmatrix}$. Then, let us perform some swaps of entries in the bottom row to turn σ into id:

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 5 & 4 & 1 & 3 \end{pmatrix}$$

[2]Of course, $\alpha_1 \circ \alpha_2 \circ \cdots \circ \alpha_k = \alpha_1 \circ (\alpha_2 \circ \alpha_3 \circ \cdots \circ \alpha_k) = (\alpha_1 \circ \alpha_2 \circ \cdots \circ \alpha_{k-1}) \circ \alpha_k$, so you can view a composition of k maps as a result of repeatedly composing them in pairs.

$$\overset{\text{swap 1 with 2}}{\longrightarrow} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 5 & 4 & 2 & 3 \end{pmatrix} \qquad \begin{pmatrix} \text{we have done this swap} \\ \text{in order to put 1 in its place} \end{pmatrix}$$
$$\overset{\text{swap 2 with 5}}{\longrightarrow} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 4 & 5 & 3 \end{pmatrix} \qquad \begin{pmatrix} \text{we have done this swap} \\ \text{in order to put 2 in its place} \end{pmatrix}$$
$$\overset{\text{swap 3 with 4}}{\longrightarrow} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 5 & 4 \end{pmatrix} \qquad \begin{pmatrix} \text{we have done this swap} \\ \text{in order to put 3 in its place} \end{pmatrix}$$
$$\overset{\text{swap 4 with 5}}{\longrightarrow} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 & 5 \end{pmatrix} \qquad \begin{pmatrix} \text{we have done this swap} \\ \text{in order to put 4 in its place} \end{pmatrix}$$
$$= \mathrm{id}.$$

This means that

$$\sigma = t_{1,2} \circ t_{2,5} \circ t_{3,4} \circ t_{4,5} \circ \mathrm{id} = t_{1,2} \circ t_{2,5} \circ t_{3,4} \circ t_{4,5}.$$

The same logic works in the general case: We just need to find a way to transform the bottom row of σ into $(1, 2, 3, \ldots, n)$ by a sequence of swaps. One way to do this is by repeatedly swapping elements that are in wrong places to bring at least one element into its right place in each swap.[3] Another way to do this is by repeatedly picking an inversion (i.e., a bigger element that is placed left of a smaller element) and swapping them.[4] □

Now we shall compute the sign of a transposition ([Strickland, Example B.6]):

Proposition 1.2.9. Let p and q be two distinct elements of $[n]$. Then, $\operatorname{sign}\left(t_{p,q}\right) = -1$.

Proof. Notice that $t_{p,q} = t_{q,p}$. Therefore, we can WLOG assume that $p < q$ (since otherwise, we can do the same argument with p and q interchanged). Assume this, and observe that the inversions of $t_{p,q}$ are:

- (p, q);
- (p, i) is an inversion for each $i \in \{p+1, p+2, \ldots, q-1\}$;
- (i, q) is an inversion for each $i \in \{p+1, p+2, \ldots, q-1\}$.

Thus, the total number of inversions of $t_{p,q}$ is $1 + (q-p-1) + (q-p-1) = 2(q-p) - 1$, which is clearly odd. In other words, $\ell\left(t_{p,q}\right)$ is odd. Hence, $\operatorname{sign}\left(t_{p,q}\right) = (-1)^{\ell\left(t_{p,q}\right)} = (-1)^{\text{something odd}} = -1$. □

The following fact ([Strickland, Proposition B.13]) is the most important property of signs of permutations:

[3] This method is called "insertion sort".

[4] This method (or one specific implementation of it) is known as "bubble sort".

Theorem 1.2.10. Let σ and τ be two permutations of $[n]$. Then,

$$\operatorname{sign}(\sigma \circ \tau) = \operatorname{sign}(\sigma) \cdot \operatorname{sign}(\tau).$$

Proof idea. The definition of signs yields $\operatorname{sign}(\sigma \circ \tau) = (-1)^{\ell(\sigma\circ\tau)}$ and $\operatorname{sign}(\sigma) \cdot \operatorname{sign}(\tau) = (-1)^{\ell(\sigma)} \cdot (-1)^{\ell(\tau)} = (-1)^{\ell(\sigma)+\ell(\tau)}$. Hence, it suffices to prove that $\ell(\sigma \circ \tau)$ and $\ell(\sigma) + \ell(\tau)$ differ by a multiple of 2. This is not obvious![5]

I've shown a "proof" using braids on the whiteboard. For a formal proof, see [Strickland, Proposition B.13]. □

Corollary 1.2.11. Let σ be a permutation of $[n]$. Let p and q be two distinct elements of $[n]$. Then,

$$\operatorname{sign}(\sigma \circ t_{p,q}) = -\operatorname{sign}(\sigma) \qquad \text{and} \qquad \operatorname{sign}(t_{p,q} \circ \sigma) = -\operatorname{sign}(\sigma).$$

Proof. Applying Theorem 1.2.10 to $\tau = t_{p,q}$, we find

$$\operatorname{sign}(\sigma \circ t_{p,q}) = \operatorname{sign}(\sigma) \cdot \underbrace{\operatorname{sign}(t_{p,q})}_{\substack{=-1 \\ \text{(by Proposition 1.2.9)}}} = -\operatorname{sign}(\sigma).$$

This proves the first equality. Similarly we can prove the second equality. □

[Note the connection to the "15-game": see `http://migo.sixbit.org/puzzles/fifteen/` and various other sources.]

Proposition 1.2.12. Let σ be a permutation of $[n]$. Then, $\operatorname{sign}(\sigma^{-1}) = \operatorname{sign}(\sigma)$.

Proof. Apply Theorem 1.2.10 to $\tau = \sigma^{-1}$. This results in

$$\operatorname{sign}\left(\sigma \circ \sigma^{-1}\right) = \operatorname{sign}(\sigma) \cdot \operatorname{sign}\left(\sigma^{-1}\right).$$

But $\operatorname{sign}\left(\underbrace{\sigma \circ \sigma^{-1}}_{=\mathrm{id}}\right) = \operatorname{sign}(\mathrm{id}) = 1$. So we get $1 = \operatorname{sign}(\sigma) \cdot \operatorname{sign}(\sigma^{-1})$. Thus, $\operatorname{sign}(\sigma^{-1}) = 1/\operatorname{sign}(\sigma) = \operatorname{sign}(\sigma)$ (since $\operatorname{sign}(\sigma)$ is either 1 or -1). □

This all is not really linear algebra; we have rather been doing some basic combinatorics of permutations that will be used in linear algebra. If you like this kind of combinatorics, check out [Mulhol16], [Bump02] and [Joyner08].

1.3. Some determinant identities

Let us see what all these properties of signs imply for matrices.

[5] It is not true that $\ell(\sigma \circ \tau) = \ell(\sigma) + \ell(\tau)$.

Theorem 1.3.1. If A is any $n \times n$-matrix, then $\det\left(A^T\right) = \det A$.

Example 1.3.2. For $n = 3$, this theorem says that

$$\det \begin{pmatrix} a & b & c \\ a' & b' & c' \\ a'' & b'' & c'' \end{pmatrix} = \det \begin{pmatrix} a & a' & a'' \\ b & b' & b'' \\ c & c' & c'' \end{pmatrix}.$$

Proof of Theorem 1.3.1. The definition of the determinant yields

$$\det A = \sum_{\sigma \text{ is a permutation of } [n]} \operatorname{sign}(\sigma) \cdot A_{1,\sigma(1)} A_{2,\sigma(2)} \cdots A_{n,\sigma(n)}$$

and

$$\begin{aligned} &\det\left(A^T\right) \\ &= \sum_{\sigma \text{ is a permutation of } [n]} \underbrace{\operatorname{sign}(\sigma)}_{\substack{=\operatorname{sign}\left(\sigma^{-1}\right) \\ \text{(by Proposition 1.2.12)}}} \cdot \underbrace{\left(A^T\right)_{1,\sigma(1)} \left(A^T\right)_{2,\sigma(2)} \cdots \left(A^T\right)_{n,\sigma(n)}}_{\substack{=A_{\sigma(1),1} A_{\sigma(2),2} \cdots A_{\sigma(n),n} \\ =A_{1,\sigma^{-1}(1)} A_{2,\sigma^{-1}(2)} \cdots A_{n,\sigma^{-1}(n)} \\ \text{(indeed, the numbers } A_{1,\sigma^{-1}(1)}, A_{2,\sigma^{-1}(2)}, \ldots, A_{n,\sigma^{-1}(n)} \\ \text{are the same as the numbers } A_{\sigma(1),1}, A_{\sigma(2),2}, \ldots, A_{\sigma(n),n}, \\ \text{except they are listed in a different order)}}} \\ &= \sum_{\sigma \text{ is a permutation of } [n]} \operatorname{sign}\left(\sigma^{-1}\right) \cdot A_{1,\sigma^{-1}(1)} A_{2,\sigma^{-1}(2)} \cdots A_{n,\sigma^{-1}(n)}. \end{aligned}$$

The right hand sides of these two equalities are actually the same sum, except that the addends are in a different order. Indeed, the inverses of all permutations of $[n]$ are precisely the permutations of $[n]$. Thus, the right hand sides are equal. Hence, the left hand sides are equal, i.e., we have $\det A = \det\left(A^T\right)$. $\square$

Theorem 1.3.3. If an $n \times n$-matrix A has two equal rows, then $\det A = 0$.

Example 1.3.4. Let us check this for a 3×3-matrix whose first two rows are equal:

$$\det \begin{pmatrix} a & b & c \\ a & b & c \\ a' & b' & c' \end{pmatrix} = abc' + bca' + cab' - acb' - bac' - cba' = 0,$$

since the addends cancel each other in pairs.

Proof of Theorem 1.3.3 (sketched). Let us assume that the p-th and the q-th row of A are equal, where p and q are two elements of $[n]$ with $p < q$. Now, the definition of the determinant says

$$\begin{aligned}\det A &= \sum_{\sigma \text{ is a permutation of } [n]} \operatorname{sign}(\sigma) \cdot A_{1,\sigma(1)} A_{2,\sigma(2)} \cdots A_{n,\sigma(n)} \\ &= \sum_{\substack{\sigma \text{ is a permutation of } [n]; \\ \operatorname{sign}(\sigma)=1}} A_{1,\sigma(1)} A_{2,\sigma(2)} \cdots A_{n,\sigma(n)} \\ &\qquad - \sum_{\substack{\sigma \text{ is a permutation of } [n]; \\ \operatorname{sign}(\sigma)=-1}} A_{1,\sigma(1)} A_{2,\sigma(2)} \cdots A_{n,\sigma(n)}\end{aligned}$$

(because each permutation σ of $[n]$ must satisfy $\operatorname{sign}(\sigma) = 1$ or $\operatorname{sign}(\sigma) = -1$, and thus we can split the sum up into the part containing all addends with $\operatorname{sign}(\sigma) = 1$ and the part containing all addends with $\operatorname{sign}(\sigma) = -1$).

We want to show that each addend of the first sum on the right hand side equals some addend of the second sum, and actually there is a 1-to-1 correspondence that makes all addends cancel each other.

Recall Corollary 1.2.11, which stated that $\operatorname{sign}(\sigma \circ t_{p,q}) = -\operatorname{sign}(\sigma)$ for each permutation σ of $[n]$. Thus, if σ is any permutation of $[n]$, then the two permutations σ and $\sigma \circ t_{p,q}$ have opposite signs. Hence, if the first sum has an addend corresponding to some σ, then the second sum has an addend corresponding to $\sigma \circ t_{p,q}$ (and vice versa). We claim that these two addends will cancel each other. Why?

Consider a permutation σ of $[n]$, and the corresponding product

$$A_{1,\sigma(1)} A_{2,\sigma(2)} \cdots A_{n,\sigma(n)}.$$

For the permutation $\sigma \circ t_{p,q}$, the corresponding product will be the same, except that the factors $A_{p,\sigma(p)}$ and $A_{q,\sigma(q)}$ get replaced by $A_{p,\sigma(q)}$ and $A_{q,\sigma(p)}$. But this replacement does not change the product, because

$$\underbrace{A_{p,\sigma(p)}}_{\substack{=A_{q,\sigma(p)} \\ \text{(since the } p\text{-th} \\ \text{and } q\text{-th rows} \\ \text{of } A \text{ are equal)}}} \cdot \underbrace{A_{q,\sigma(q)}}_{\substack{=A_{p,\sigma(q)} \\ \text{(since the } q\text{-th} \\ \text{and } p\text{-th rows} \\ \text{of } A \text{ are equal)}}} = A_{q,\sigma(p)} \cdot A_{p,\sigma(q)} = A_{p,\sigma(q)} \cdot A_{q,\sigma(p)}.$$

Thus, the products corresponding to σ and to $\sigma \circ t_{p,q}$ are the same. But these products belong to different sums, and thus cancel each other. This cancellation kills all addends of both sums. Thus, the result will be 0. In other words, $\det A = 0$. $\square$

Theorem 1.3.5. If we scale a row of an $n \times n$-matrix A by a number λ, then $\det A$ gets multiplied by λ.

Example 1.3.6. Let us scale the 2-nd row of a 3×3-matrix by λ, and see what happens to its determinant:

$$\begin{aligned}
&\det \begin{pmatrix} a & b & c \\ \lambda a' & \lambda b' & \lambda c' \\ a'' & b'' & c'' \end{pmatrix} \\
&= a \left(\lambda b'\right) c'' + b \left(\lambda c'\right) a'' + c \left(\lambda a'\right) b'' - a \left(\lambda c'\right) b'' - b \left(\lambda a'\right) c'' - c \left(\lambda b'\right) a'' \\
&= \lambda \underbrace{\left(ab'c'' + bc'a'' + ca'b'' - ac'b'' - ba'c'' - cb'a''\right)}_{=\det \begin{pmatrix} a & b & c \\ a' & b' & c' \\ a'' & b'' & c'' \end{pmatrix}} \\
&= \lambda \det \begin{pmatrix} a & b & c \\ a' & b' & c' \\ a'' & b'' & c'' \end{pmatrix}.
\end{aligned}$$

Proof of Theorem 1.3.5 (sketched). The same idea as in the example works in the general case: Scale a row of A by λ, and write the determinant of the resulting matrix as a sum over all permutations of $[n]$ (as in the definition of a determinant). Each addend of this sum will contain exactly one copy of λ, because it will contain exactly one factor from the row that has been scaled by λ. Thus, a λ factors out from the determinant, and what remains is the determinant of the original matrix. □

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