Drexel University, Department of Mathematics

Math 222: Enumerative Combinatorics, Fall 2019: Midterm 3

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due date: **Wednesday, 2019-12-04** at the beginning of class,
or before that through Blackboard.

Please solve **3 of the 5 exercises**!
This is a midterm, so **collaboration is not allowed**!

Notations

Here is a list of notations that are used in this homework:

- We shall use the notation $[n]$ for the set $\{1, 2, \ldots, n\}$ (when $n \in \mathbb{Z}$).
- If $n \in \mathbb{N}$, then S_n denotes the set of all permutations of $[n]$.
- If $n \in \mathbb{N}$ and $\sigma \in S_n$, then:
  - the *one-line notation* $\operatorname{OLN} \sigma$ of σ is defined as the n-tuple $(\sigma(1), \sigma(2), \ldots, \sigma(n))$.
  - the *inversions* of σ are defined to be the pairs (i, j) of integers satisfying $1 \leq i < j \leq n$ and $\sigma(i) > \sigma(j)$.
  - the *length* $\ell(\sigma)$ of σ is defined to be the # of inversions of σ.
  - the *sign* $(-1)^{\sigma}$ of σ is defined to be $(-1)^{\ell(\sigma)}$.
  - we say that σ is *even* if $(-1)^{\sigma} = 1$ (that is, if $\ell(\sigma)$ is even).
  - we say that σ is *odd* if $(-1)^{\sigma} = -1$ (that is, if $\ell(\sigma)$ is odd).
  - we let $\operatorname{Fix} \sigma$ denote the set of all fixed points of σ; in other words,

$$\operatorname{Fix} \sigma = \{i \in [n] \mid \sigma(i) = i\}.$$

1 Exercise 1

1.1 Problem

Let n be an integer such that $n \geq 2$. If $w \in S_n$ is a permutation, then the *peaks* of w are defined to be the elements $i \in \{2, 3, \ldots, n-1\}$ satisfying $w(i-1) < w(i) > w(i+1)$. (For example, if $n = 7$ and if OLN $w = (4, 1, 2, 5, 3, 7, 6)$, then the peaks of w are 4 and 6. The name "peak" is explained by a look at the plot of w.)

An *n-peak set* shall mean a subset P of $\{2, 3, \ldots, n-1\}$ such that there exists a $w \in S_n$ satisfying $\{\text{peaks of } w\} = P$. (For example, the example we just gave shows that $\{4, 6\}$ is a 7-peak set.)

Find the # of all n-peak sets (for our given n).

1.2 Solution

[...]

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2 Exercise 2

2.1 Problem

Let n be an integer such that $n \geq 3$. For each $k \in \mathbb{Z}$, set

$$m_k = (\# \text{ of permutations } \sigma \in S_n \text{ such that } \ell(\sigma) \equiv k \mod 3).$$

(*Example:* If $n = 3$, then m_0 counts the two permutations with one-line notations $(1, 2, 3)$ and $(3, 2, 1)$, while m_1 counts the two permutations with one-line notations $(1, 3, 2)$ and $(2, 1, 3)$, and while m_2 counts the two permutations with one-line notations $(2, 3, 1)$ and $(3, 1, 2)$.)

Prove that $m_0 = m_1 = m_2 = n!/3$.

[**Hint:** The Lehmer code (see [Grinbe15, §5.8] or [17f-hw8s, §0.4]) may be of use.]

2.2 Solution

[...]

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3 Exercise 3

3.1 Problem

Let n be an integer such that $n \geq 3$. Find

$$\sum_{w \in S_n \text{ is even}} |\text{Fix}\, w|.$$

[**Hint:** For each $i \in [n]$, compare

$$(\# \text{ of even } w \in S_n \text{ such that } w(i) = i) \text{ with } (\# \text{ of odd } w \in S_n \text{ such that } w(i) = i).$$

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3.2 Solution

[...]

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4 Exercise 4

4.1 Problem

An n-tuple $(i_1, i_2, \ldots, i_n) \in \{0,1\}^n$ (where $n \in \mathbb{N}$) will be called *upsided* if it satisfies $i_1 + i_2 + \cdots + i_p \geq p/2$ for each $p \in [n]$.

(*Example:* The 3-tuple $(1,0,1)$ is upsided (since $1 \geq 1/2$ and $1+0 \geq 2/2$ and $1+0+1 \geq 3/2$), and so is the 3-tuple $(1,1,0)$ (for similar reasons), but the 3-tuples $(1,0,0)$ and $(0,1,1)$ are not (indeed, $(1,0,0)$ is not upsided because $1+0+0 < 3/2$, whereas $(0,1,1)$ is not upsided because $0 < 1/2$). The 0-tuple $()$ is upsided (for vacuous reasons).)

For given $n \in \mathbb{N}$ and $k \in \mathbb{Z}$, let $U(n,k)$ denote the # of upsided n-tuples $(i_1, i_2, \ldots, i_n) \in \{0,1\}^n$ satisfying $i_1 + i_2 + \cdots + i_n = k$.

(a) Prove that if $n \in \mathbb{N}$ and $k \in \mathbb{Z}$ satisfy $k < n/2$, then $U(n,k) = 0$.

(b) Prove that if $n \in \mathbb{N}$ and $k \in \mathbb{Z}$ satisfy $k \geq (n-1)/2$, then

$$U(n,k) = \binom{n}{k} - \binom{n}{k+1}.$$

(c) Prove that $\binom{n}{0} < \binom{n}{1} < \cdots < \binom{n}{\lfloor n/2 \rfloor}$ for each $n \in \mathbb{N}$.

[**Hint:** Induction can be helpful. There are many ways to solve part **(c)**, but the one using part **(b)** is perhaps the nicest.]

4.2 Solution

[...]

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5 Exercise 5

5.1 Problem

Let $n \in \mathbb{N}$. Recall that a *composition of* n means a tuple $(a_1, a_2, \ldots, a_k)$ of positive integers satisfying $a_1 + a_2 + \cdots + a_k = n$. Such a composition $(a_1, a_2, \ldots, a_k)$ is called *odd* if all of $a_1, a_2, \ldots, a_k$ are odd.

Let us also say that a composition $(a_1, a_2, \ldots, a_k)$ is *odd-but-one* if a_i is even for exactly one $i \in [k]$. (For example, the composition $(3, 5, 5)$ of 13 is odd; the composition $(3, 4, 1, 5)$ of 13 is odd-but-one; the composition $(6, 6, 1)$ of 13 is neither.)

Prove that

$$\begin{aligned} & \sum_{\substack{(a_1,a_2,\ldots,a_k) \text{ is an} \\ \text{odd composition of } n}} k \\ & = (\# \text{ of odd-but-one compositions of } n+1) \\ & = \frac{(n+4) f_n + 2n f_{n-1}}{5}, \end{aligned}$$

where $(f_0, f_1, f_2, \ldots)$ is the Fibonacci sequence (defined in [Math222, Definition 1.1.10]).

5.2 Solution

[...]

References

[17f-hw8s] Darij Grinberg, *UMN Fall 2017 Math 4990 homework set #8 with solutions*, `http://www.cip.ifi.lmu.de/~grinberg/t/17f/hw8os.pdf`
Also available on the mirror server `http://darijgrinberg.gitlab.io/t/17f/hw8os.pdf`

[Grinbe15] Darij Grinberg, *Notes on the combinatorial fundamentals of algebra*, 10 January 2019.
`http://www.cip.ifi.lmu.de/~grinberg/primes2015/sols.pdf`
Also available on the mirror server `http://darijgrinberg.gitlab.io/primes2015/sols.pdf`
The numbering of theorems and formulas in this link might shift when the project gets updated; for a "frozen" version whose numbering is guaranteed to match that in the citations above, see `https://github.com/darijgr/detnotes/releases/tag/2019-01-10` .

[Math222] Darij Grinberg, *Enumerative Combinatorics: class notes*, 16 December 2019.
`http://www.cip.ifi.lmu.de/~grinberg/t/19fco/n/n.pdf` Also available on the mirror server `http://darijgrinberg.gitlab.io/t/19fco/n/n.pdf`
Caution: The numbering of theorems and formulas in this link might shift when the project gets updated; for a "frozen" version whose numbering is guaranteed to match that in the citations above, see `https://gitlab.com/darijgrinberg/darijgrinberg.gitlab.io/blob/2dab2743a181d5ba8fc145a661fd274bc37d03be/public/t/19fco/n/n.pdf`