Drexel University, Department of Mathematics

Math 222: Enumerative Combinatorics, Fall 2019: Midterm 2

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due date: **Wednesday, 2019-11-13** at the beginning of class, or before that through Blackboard.

Please solve **4 of the 6 exercises**!
This is a midterm, so **collaboration is not allowed**!

1 Exercise 1

1.1 Problem

Let $u \in \mathbb{R}$ and $n \in \mathbb{N}$. Prove that

$$\sum_{i=0}^{n} \binom{2i-u}{i} \binom{2(n-i)+u}{n-i} = 4^n. \tag{1}$$

1.2 Remark

For $u = 0$, this simplifies to

$$\sum_{i=0}^{n} \binom{2i}{i} \binom{2(n-i)}{n-i} = 4^n, \tag{2}$$

a famous identity which is probably easiest to prove by applying the Chu–Vandermonde identity to $x = -1/2$ and $y = -1/2$ (see [18f-hw3s, solution to Exercise 3 **(b)**] or [Grinbe15, solution to Exercise 3.23 **(a)**] for details). But to my knowledge, the more general equality (1) resists this clever trick. Instead, proceed as follows: Rewrite $\dbinom{2i-u}{i}$ using upper negation, and rewrite $\dbinom{2(n-i)+u}{n-i}$ as $\sum\limits_{k=0}^{n-i} \dbinom{2n+1}{k} \dbinom{u-1-2i}{n-i-k}$ using Chu–Vandermonde. Then use trinomial revision to turn $\dbinom{u-i-1}{i} \dbinom{u-1-2i}{n-i-k}$ into $\dbinom{u-i-1}{n-k} \dbinom{n-k}{i}$. Does the result remind you of anything?

1.3 Solution

[...]

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2 Exercise 2

2.1 Problem

Let S be a finite set. Let X and Y be two **distinct** subsets of S. Prove that

$$\sum_{I \subseteq S} (-1)^{|X \cap I| + |Y \cap I|} = 0.$$

2.2 Solution

[...]

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3 Exercise 3

3.1 Problem

A map $f : A \to B$ between two sets A and B will be called a 2-*surjection* if each $b \in B$ satisfies ($\#$ of $a \in A$ satisfying $f(a) = b) \geq 2$. (That is, if each element of B is taken as a value by f at least twice.)

Let $m, n \in \mathbb{N}$. Find a formula (similar to [Math222, Theorem 2.4.17]) for the $\#$ of 2-surjections from $[m]$ to $[n]$.

3.2 Solution

[...]

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4 Exercise 4

4.1 Problem

Let $n \in \mathbb{N}$ and $k \in \mathbb{N}$ be such that $n \geq k$.

Let S_n denote the set of all permutations of $[n]$. For each permutation $w \in S_n$, let $\operatorname{Fix} w$ denote the set of all fixed points of w (that is, the set $\{i \in [n] \mid w(i) = i\}$).

Prove that

$$\sum_{w \in S_n} \binom{|\operatorname{Fix} w|}{k} = (n-k)! \binom{n}{k} = \frac{n!}{k!}.$$

4.2 Remark

The $k = 1$ case of this is saying that $\sum_{w \in S_n} |\operatorname{Fix} w| = n!$ (or, equivalently: a permutation of $[n]$ has exactly 1 fixed point on average). This was proved in [17f-hw7s, §0.2, Exercise 2]. That argument may be helpful.

4.3 Solution

[...]

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5 Exercise 5

5.1 Problem

Let $n \in \mathbb{N}$. A permutation w of $[n]$ will be called *domino-free* if there exists no $i \in [n-1]$ satisfying

$$w(i) = i + 1 \qquad \text{and} \qquad w(i+1) = i.$$

Find a formula for the # of domino-free permutations of $[n]$.

5.2 Solution

[...]

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6 Exercise 6

6.1 Problem

Recall that a *composition* means a finite list of positive integers. (For example, $(2, 3, 2)$ is a composition, but $(1, 0, 4)$ is not.)

If $n \in \mathbb{N}$, then a *composition of* n means a composition $(i_1, i_2, \ldots, i_k)$ satisfying $i_1 + i_2 + \cdots + i_k = n$.

(a) A composition $(i_1, i_2, \ldots, i_k)$ is said to be *even* if all its entries $i_1, i_2, \ldots, i_k$ are even. Find a formula for the # of even compositions of a given $n \in \mathbb{N}$.

(b) A composition $(i_1, i_2, \ldots, i_k)$ is said to be *odd* if all its entries $i_1, i_2, \ldots, i_k$ are odd. Find a formula for the # of odd compositions of a given $n \in \mathbb{N}$.

6.2 Solution

[...]

References

[17f-hw7s] Darij Grinberg, *UMN Fall 2017 Math 4990 homework set #7 with solutions*, `http://www.cip.ifi.lmu.de/~grinberg/t/17f/hw7os.pdf`

[18f-hw3s] Darij Grinberg, *UMN Fall 2018 Math 5705 homework set #3 with solutions*, `http://www.cip.ifi.lmu.de/~grinberg/t/18f/hw3s.pdf`

[Grinbe15] Darij Grinberg, *Notes on the combinatorial fundamentals of algebra*, 10 January 2019.
`http://www.cip.ifi.lmu.de/~grinberg/primes2015/sols.pdf`
The numbering of theorems and formulas in this link might shift when the project gets updated; for a "frozen" version whose numbering is guaranteed to match that in the citations above, see `https://github.com/darijgr/detnotes/releases/tag/2019-01-10` .

[Math222] Darij Grinberg, *Enumerative Combinatorics: class notes*, 16 December 2019.
`http://www.cip.ifi.lmu.de/~grinberg/t/19fco/n/n.pdf` Also available on the mirror server `http://darijgrinberg.gitlab.io/t/19fco/n/n.pdf`
Caution: The numbering of theorems and formulas in this link might shift when the project gets updated; for a "frozen" version whose numbering is guaranteed to match that in the citations above, see `https://gitlab.com/darijgrinberg/darijgrinberg.gitlab.io/blob/2dab2743a181d5ba8fc145a661fd274bc37d03be/public/t/19fco/n/n.pdf`