Drexel University, Department of Mathematics

Math 222: Enumerative Combinatorics, Fall 2019: Midterm 1

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1 Exercise 1

1.1 Problem

Let $n \in \mathbb{N}$. Prove that

$$\sum_{k=0}^{n} \binom{2n+1}{k} = 4^n.$$

1.2 First solution

Forget that we fixed n. Recall the following fact ([Math222, Corollary 1.3.27]):

Corollary 1.1. *Let $n \in \mathbb{N}$. Then,* $\sum_{k=0}^{n} \binom{n}{k} = 2^n$.

Also, recall the symmetry property of the binomial coefficients ([Math222, Theorem 1.3.11]):

Theorem 1.2 (Symmetry of the binomial coefficients)**.** *Let $n \in \mathbb{N}$ and $k \in \mathbb{R}$. Then,*

$$\binom{n}{k} = \binom{n}{n-k}.$$

Now, let $n \in \mathbb{N}$. Then, $0 \leq n \leq 2n+1$. Hence, we can split the sum $\sum_{k=0}^{2n+1} \binom{2n+1}{k}$ at $k = n$. We thus obtain

$$\begin{aligned}
\sum_{k=0}^{2n+1} \binom{2n+1}{k} &= \sum_{k=0}^{n} \binom{2n+1}{k} + \sum_{k=n+1}^{2n+1} \underbrace{\binom{2n+1}{k}}_{\substack{= \binom{2n+1}{2n+1-k} \\ \text{(by Theorem 1.2,} \\ \text{applied to } 2n+1 \text{ instead of } n)}} \\
&= \sum_{k=0}^{n} \binom{2n+1}{k} + \sum_{k=n+1}^{2n+1} \binom{2n+1}{2n+1-k} \\
&= \sum_{k=0}^{n} \binom{2n+1}{k} + \sum_{k=0}^{n} \binom{2n+1}{k} \\
&\qquad \text{(here, we have substituted } k \text{ for } 2n+1-k \text{ in the second sum)} \\
&= 2 \cdot \sum_{k=0}^{n} \binom{2n+1}{k}.
\end{aligned}$$

Comparing this with

$$\sum_{k=0}^{2n+1} \binom{2n+1}{k} = 2^{2n+1} \qquad \text{(by Corollary 1.1, applied to } 2n+1 \text{ instead of } n\text{)},$$

we obtain

$$2 \cdot \sum_{k=0}^{n} \binom{2n+1}{k} = 2^{2n+1}.$$

Dividing both sides of this equality by 2, we find

$$\sum_{k=0}^{n} \binom{2n+1}{k} = 2^{2n+1}/2 = 2^{2n} = \Big(\underbrace{2^2}_{=4} \Big)^n = 4^n.$$

This solves the exercise.

1.3 Second solution

Forget that we fixed n. Recall the recurrence of the binomial coefficients ([Math222, Theorem 1.3.8]):

Theorem 1.3 (Recurrence of the binomial coefficients)**.** *Let $n \in \mathbb{R}$ and $k \in \mathbb{R}$. Then,*

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}.$$

Now, let $n \in \mathbb{N}$. Then,

$$\sum_{k=0}^{n} \underbrace{\binom{2n+1}{k}}_{\substack{=\binom{(2n+1)-1}{k-1}+\binom{(2n+1)-1}{k} \\ \text{(by Theorem 1.3, applied to } 2n+1 \text{ instead of } n)}} = \sum_{k=0}^{n} \left(\binom{(2n+1)-1}{k-1} + \binom{(2n+1)-1}{k} \right) = \sum_{k=0}^{n} \left(\binom{2n}{k-1} + \binom{2n}{k} \right)$$
$$\text{(since } (2n+1)-1 = 2n)$$
$$= \sum_{k=0}^{n} \binom{2n}{k-1} + \sum_{k=0}^{n} \binom{2n}{k}. \tag{1}$$

But each $k \in \mathbb{R}$ satisfies

$$\binom{2n}{k-1} = \binom{2n}{2n-(k-1)}$$
$$\text{(by Theorem 1.2, applied to } 2n \text{ and } k-1 \text{ instead of } n \text{ and } k)$$
$$= \binom{2n}{2n+1-k} \tag{2}$$

(since $2n-(k-1) = 2n+1-k$). Now, we can split off the addend for $k=0$ from the sum $\sum_{k=0}^{n} \binom{2n}{k-1}$; we thus find

$$\sum_{k=0}^{n} \binom{2n}{k-1} = \underbrace{\binom{2n}{0-1}}_{\substack{=0 \\ \text{(by the definition of} \\ \text{binomial coefficients,} \\ \text{since } 0-1=-1\notin\mathbb{N})}} + \sum_{k=1}^{n} \binom{2n}{k-1} = \sum_{k=1}^{n} \underbrace{\binom{2n}{k-1}}_{\substack{=\binom{2n}{2n+1-k} \\ \text{(by (2))}}}$$
$$= \sum_{k=1}^{n} \binom{2n}{2n+1-k} = \sum_{k=2n+1-n}^{2n+1-1} \binom{2n}{k}$$
$$\text{(here, we have substituted } k \text{ for } 2n+1-k \text{ in the sum)}$$
$$= \sum_{k=n+1}^{2n} \binom{2n}{k} \tag{3}$$

(since $2n+1-n=n+1$ and $2n+1-1=2n$). Hence, (1) becomes

$$\begin{aligned}\sum_{k=0}^{n}\binom{2n+1}{k} &= \underbrace{\sum_{k=0}^{n}\binom{2n}{k-1}}_{\substack{=\sum_{k=n+1}^{2n}\binom{2n}{k} \\ \text{(by (3))}}}+\sum_{k=0}^{n}\binom{2n}{k} = \sum_{k=n+1}^{2n}\binom{2n}{k}+\sum_{k=0}^{n}\binom{2n}{k} \\ &= \sum_{k=0}^{n}\binom{2n}{k}+\sum_{k=n+1}^{2n}\binom{2n}{k} = \sum_{k=0}^{2n}\binom{2n}{k} \\ &= 2^{2n} \qquad \text{(by Corollary 1.1, applied to } 2n \text{ instead of } n\text{)} \\ &= \Big(\underbrace{2^2}_{=4}\Big)^n = 4^n.\end{aligned}$$

Thus, the exercise is solved again.

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2 Exercise 2

2.1 Problem

Let $n \in \mathbb{N}$. Compute the number of 4-tuples (A, B, C, D) of subsets of $[n]$ satisfying

$$A \cap B = C \cup D.$$

2.2 Solution sketch

The exercise is an analogue of [hw2s, Exercise 2], and the following solution imitates the solution of the latter exercise (except that we are omitting the formal proof, because it should be clear how to construct it if necessary).

We shall say that a 4-tuple (A, B, C, D) of subsets of $[n]$ is *good* if and only if it satisfies $A \cap B = C \cup D$.

We claim the following:

Claim 1: The # of good 4-tuples is 6^n.

Let us first give an informal (but perfectly clear to the experienced reader) proof of this claim:

Informal proof of Claim 1. A 4-tuple (A, B, C, D) of subsets of $[n]$ is good if and only if it satisfies the following property: Each $i \in [n]$ belongs to

- **either** all four sets A, B, C and D,
- **or** the sets A and B and C but not D;
- **or** the sets A and B and D but not C;
- **or** the set A but none of the other three sets,

- **or** the set B but none of the other three sets,
- **or** none of the four sets A, B, C and D.

[1] We shall refer to these 6 possibilities as "Option 1", "Option 2" and so on.

Thus, the following simple algorithm constructs every good 4-tuple (A, B, C, D): For each $i \in [n]$, we decide which of the 6 options listed above the element i should satisfy (i.e., whether it satisfies Option 1 or Option 2 etc.). There are 6 choices for it, since these 6 options are mutually exclusive. Thus, in total, there are 6^n good 4-tuples (because we are making this decision once for each of the n elements i of $[n]$). This completes our informal proof of Claim 1. $\square$

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3 Exercise 3

3.1 Problem

Let m and n be two nonnegative integers such that $m \leq n$. Let $f_m, f_{m+1}, \ldots, f_n$ be any $n-m+1$ numbers. Let $g_m, g_{m+1}, \ldots, g_{n+1}$ be any $n-m+2$ numbers. Prove that

$$\sum_{k=m}^{n} f_k \left(g_{k+1} - g_k\right) + \sum_{k=m+1}^{n} g_k \left(f_k - f_{k-1}\right) = f_n g_{n+1} - f_m g_m. \tag{4}$$

3.2 Remark

This is a discrete version of the "integration by parts" formula

$$\int_m^n f g' + \int_m^n g f' = (fg)(n) - (fg)(m)$$

from calculus.

3.3 Solution

We have

$$\begin{aligned}
&\sum_{k=m}^{n} \underbrace{f_k \left(g_{k+1} - g_k\right)}_{= f_k g_{k+1} - f_k g_k} \\
&= \sum_{k=m}^{n} \left(f_k g_{k+1} - f_k g_k\right) = \underbrace{\sum_{k=m}^{n} f_k g_{k+1}}_{\substack{= f_n g_{n+1} + \sum_{k=m}^{n-1} f_k g_{k+1} \\ \text{(here, we have split off the} \\ \text{addend for } k=n \text{ from the sum)}}} - \underbrace{\sum_{k=m}^{n} f_k g_k}_{\substack{= f_m g_m + \sum_{k=m+1}^{n} f_k g_k \\ \text{(here, we have split off the} \\ \text{addend for } k=m \text{ from the sum)}}} \\
&= \left(f_n g_{n+1} + \sum_{k=m}^{n-1} f_k g_{k+1}\right) - \left(f_m g_m + \sum_{k=m+1}^{n} f_k g_k\right)
\end{aligned}$$

[1]This is proved similarly as the analogous claim in the solution to [hw2s, Exercise 2].

and

$$\begin{aligned}
&\sum_{k=m+1}^{n} \underbrace{g_k (f_k - f_{k-1})}_{=g_k f_k - g_k f_{k-1}} \\
&= \sum_{k=m+1}^{n} (g_k f_k - g_k f_{k-1}) = \sum_{k=m+1}^{n} \underbrace{g_k f_k}_{=f_k g_k} - \sum_{k=m+1}^{n} \underbrace{g_k f_{k-1}}_{=f_{k-1} g_k} \\
&= \sum_{k=m+1}^{n} f_k g_k - \sum_{k=m+1}^{n} f_{k-1} g_k = \sum_{k=m+1}^{n} f_k g_k - \sum_{k=m}^{n-1} \underbrace{f_{(k+1)-1}}_{=f_k} g_{k+1} \\
&\qquad \text{(here, we have substituted } k+1 \text{ for } k \text{ in the second sum)} \\
&= \sum_{k=m+1}^{n} f_k g_k - \sum_{k=m}^{n-1} f_k g_{k+1}.
\end{aligned}$$

Adding together these two equalities, we obtain

$$\begin{aligned}
&\sum_{k=m}^{n} f_k (g_{k+1} - g_k) + \sum_{k=m+1}^{n} g_k (f_k - f_{k-1}) \\
&= \left(f_n g_{n+1} + \sum_{k=m}^{n-1} f_k g_{k+1} \right) - \left(f_m g_m + \sum_{k=m+1}^{n} f_k g_k \right) + \sum_{k=m+1}^{n} f_k g_k - \sum_{k=m}^{n-1} f_k g_{k+1} \\
&= f_n g_{n+1} - f_m g_m.
\end{aligned}$$

This solves the exercise.

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4 Exercise 4

4.1 Problem

Let $n \in \mathbb{N}$. Let $T_1, T_2, \ldots, T_n$ be n finite sets of integers. For each $i \in [n]$, we let a_i be the # of even elements of T_i, and we let b_i be the # of odd elements of T_i. Furthermore, for each $i \in [n]$, we set $s_i = a_i + b_i = |T_i|$ and $d_i = a_i - b_i$.

An n-tuple $(i_1, i_2, \ldots, i_n) \in T_1 \times T_2 \times \cdots \times T_n$ is said to be *even* if the sum $i_1 + i_2 + \cdots + i_n$ is even. (For example, the 4-tuple $(1, 0, 4, 1)$ is even, whereas $(1, 0, 3, 1)$ is not.)

Prove that the # of even n-tuples $(i_1, i_2, \ldots, i_n) \in T_1 \times T_2 \times \cdots \times T_n$ equals

$$\frac{s_1 s_2 \cdots s_n + d_1 d_2 \cdots d_n}{2}.$$

4.2 Remark

This generalizes [hw1s, Exercise 6].

4.3 First solution sketch

This first solution is essentially a generalization of [hw1s, solution to Exercise 6]. (Note how the generality makes our life easier, because we have fewer cases to worry about!)

We extend our definition of "even n-tuples" to arbitrary m-tuples of integers (where $m \in \mathbb{N}$ is arbitrary). We do this in the most natural way: We say that an m-tuple $(i_1, i_2, \ldots, i_m)$ of integers (for some $m \in \mathbb{N}$) is *even* if the sum $i_1 + i_2 + \cdots + i_m$ is even. Likewise, we say that an m-tuple $(i_1, i_2, \ldots, i_m)$ of integers (for some $m \in \mathbb{N}$) is *odd* if the sum $i_1 + i_2 + \cdots + i_m$ is odd. Clearly, any m-tuple of integers is either even or odd (but not both at the same time).

We now state the following:

Claim 1: For each $m \in \{0, 1, \ldots, n\}$, we have

$$(\# \text{ of even } m\text{-tuples } (i_1, i_2, \ldots, i_m) \in T_1 \times T_2 \times \cdots \times T_m) = \frac{s_1 s_2 \cdots s_m + d_1 d_2 \cdots d_m}{2}.$$

[*Proof of Claim 1:* We shall prove Claim 1 by induction on m:

Induction base: There is only one 0-tuple $(i_1, i_2, \ldots, i_0) \in T_1 \times T_2 \times \cdots \times T_0$, namely the empty list $()$. This 0-tuple is even (since the sum of its entries is $(\text{empty sum}) = 0$). Thus, there is exactly one even 0-tuple $(i_1, i_2, \ldots, i_0) \in T_1 \times T_2 \times \cdots \times T_0$. In other words,

$$(\# \text{ of even } 0\text{-tuples } (i_1, i_2, \ldots, i_0) \in T_1 \times T_2 \times \cdots \times T_0) = 1.$$

Comparing this with

$$\frac{s_1 s_2 \cdots s_0 + d_1 d_2 \cdots d_0}{2} = \frac{1+1}{2} \qquad (\text{since } s_1 s_2 \cdots s_0 = (\text{empty product}) = 1 \text{ and } d_1 d_2 \cdots d_0 = (\text{empty product}) = 1)$$
$$= 1,$$

we obtain

$$(\# \text{ of even } 0\text{-tuples } (i_1, i_2, \ldots, i_0) \in T_1 \times T_2 \times \cdots \times T_0) = \frac{s_1 s_2 \cdots s_0 + d_1 d_2 \cdots d_0}{2}.$$

In other words, Claim 1 holds for $m = 0$. This completes the induction base.

Induction step: Let M be a positive integer. Assume that Claim 1 holds for $m = M - 1$. We must prove that Claim 1 holds for $m = M$.

In the following, the word "M-tuple" shall always mean "M-tuple in $T_1 \times T_2 \times \cdots \times T_M$". Likewise, the word "$(M-1)$-tuple" shall always mean "$(M-1)$-tuple in $T_1 \times T_2 \times \cdots \times T_{M-1}$". Thus,

$$(\# \text{ of all } (M-1)\text{-tuples}) = |T_1| \cdot |T_2| \cdot \cdots \cdot |T_{M-1}| = \prod_{i=1}^{M-1} \underbrace{|T_i|}_{\substack{= s_i \\ (\text{since } s_i = |T_i|)}} = \prod_{i=1}^{M-1} s_i$$
$$= s_1 s_2 \cdots s_{M-1}.$$

We have assumed that Claim 1 holds for $m = M - 1$. In other words,

$$(\# \text{ of even } (M-1)\text{-tuples } (i_1, i_2, \ldots, i_{M-1}) \in T_1 \times T_2 \times \cdots \times T_{M-1})$$
$$= \frac{s_1 s_2 \cdots s_{M-1} + d_1 d_2 \cdots d_{M-1}}{2}.$$

Since we have decided to refer to $(M-1)$-tuples $(i_1, i_2, \ldots, i_{M-1}) \in T_1 \times T_2 \times \cdots \times T_{M-1}$ simply as "$(M-1)$-tuples", we can rewrite this as follows:

$$(\# \text{ of even } (M-1)\text{-tuples}) = \frac{s_1 s_2 \cdots s_{M-1} + d_1 d_2 \cdots d_{M-1}}{2}.$$

But each $(M-1)$-tuple is either even or odd, but not both at the same time. Hence,

$$(\# \text{ of all } (M-1)\text{-tuples}) = (\# \text{ of even } (M-1)\text{-tuples}) + (\# \text{ of odd } (M-1)\text{-tuples}).$$

Thus,

$$\begin{aligned}(\# \text{ of odd } (M-1)\text{-tuples}) &= \underbrace{(\# \text{ of all } (M-1)\text{-tuples})}_{=s_1 s_2 \cdots s_{M-1}} - \underbrace{(\# \text{ of even } (M-1)\text{-tuples})}_{=\frac{s_1 s_2 \cdots s_{M-1} + d_1 d_2 \cdots d_{M-1}}{2}} \\ &= s_1 s_2 \cdots s_{M-1} - \frac{s_1 s_2 \cdots s_{M-1} + d_1 d_2 \cdots d_{M-1}}{2} \\ &= \frac{s_1 s_2 \cdots s_{M-1} - d_1 d_2 \cdots d_{M-1}}{2}.\end{aligned}$$

Now, we want to count the even M-tuples $(i_1, i_2, \ldots, i_M) \in T_1 \times T_2 \times \cdots \times T_M$. Each M-tuple $(i_1, i_2, \ldots, i_M) \in T_1 \times T_2 \times \cdots \times T_M$ has a well-defined last entry i_M (since $M > 0$). Now, for each $j \in T_M$, we can count the even M-tuples whose last entry is j:

- Let $j \in T_M$ be even. Then, for any $(M-1)$-tuple $(i_1, i_2, \ldots, i_{M-1})$, we have the following chain of logical equivalences:

$$\begin{aligned}&(\text{the } (M-1)\text{-tuple } (i_1, i_2, \ldots, i_{M-1}) \text{ is even}) \\ \iff &(i_1 + i_2 + \cdots + i_{M-1} \text{ is even}) \qquad (\text{by the definition of "even" for tuples}) \\ \iff &(i_1 + i_2 + \cdots + i_{M-1} + j \text{ is even}) \qquad (\text{since } j \text{ is even}) \\ \iff &(\text{the } M\text{-tuple } (i_1, i_2, \ldots, i_{M-1}, j) \text{ is even}) \\ &\qquad (\text{by the definition of "even" for tuples}).\end{aligned}$$

  Hence, there is a bijection

$$\begin{aligned}\{\text{even } (M-1)\text{-tuples}\} &\to \{\text{even } M\text{-tuples } (i_1, i_2, \ldots, i_M) \text{ with } i_M = j\}, \\ (i_1, i_2, \ldots, i_{M-1}) &\mapsto (i_1, i_2, \ldots, i_{M-1}, j).\end{aligned}$$

  [2] Hence, the bijection principle shows that

$$(\# \text{ of even } (M-1)\text{-tuples}) = (\# \text{ of even } M\text{-tuples } (i_1, i_2, \ldots, i_M) \text{ with } i_M = j).$$

  Hence,

$$\begin{aligned}&(\# \text{ of even } M\text{-tuples } (i_1, i_2, \ldots, i_M) \text{ with } i_M = j) \\ &= (\# \text{ of even } (M-1)\text{-tuples}) = \frac{s_1 s_2 \cdots s_{M-1} + d_1 d_2 \cdots d_{M-1}}{2}.\end{aligned} \qquad (5)$$

  Forget that we fixed j. We thus have proven (5) for each **even** $j \in T_M$.

- Let $j \in T_M$ be odd. Then, for any $(M-1)$-tuple $(i_1, i_2, \ldots, i_{M-1})$, we have the following chain of logical equivalences:

$$\begin{aligned}&(\text{the } (M-1)\text{-tuple } (i_1, i_2, \ldots, i_{M-1}) \text{ is odd}) \\ \iff &(i_1 + i_2 + \cdots + i_{M-1} \text{ is odd}) \qquad (\text{by the definition of "odd" for tuples}) \\ \iff &(i_1 + i_2 + \cdots + i_{M-1} + j \text{ is even}) \qquad (\text{since } j \text{ is odd}) \\ \iff &(\text{the } M\text{-tuple } (i_1, i_2, \ldots, i_{M-1}, j) \text{ is even}) \\ &\qquad (\text{by the definition of "even" for tuples}).\end{aligned}$$

[2] We leave it to the reader to verify that this map is well-defined and is actually a bijection.

Hence, there is a bijection

$$\{\text{odd } (M-1)\text{-tuples}\} \to \{\text{even } M\text{-tuples } (i_1, i_2, \ldots, i_M) \text{ with } i_M = j\},$$
$$(i_1, i_2, \ldots, i_{M-1}) \mapsto (i_1, i_2, \ldots, i_{M-1}, j).$$

[3] Hence, the bijection principle shows that

$$(\# \text{ of odd } (M-1)\text{-tuples}) = (\# \text{ of even } M\text{-tuples } (i_1, i_2, \ldots, i_M) \text{ with } i_M = j).$$

Hence,

$$\begin{aligned}
&(\# \text{ of even } M\text{-tuples } (i_1, i_2, \ldots, i_M) \text{ with } i_M = j) \\
&= (\# \text{ of odd } (M-1)\text{-tuples}) = \frac{s_1 s_2 \cdots s_{M-1} - d_1 d_2 \cdots d_{M-1}}{2}.
\end{aligned} \tag{6}$$

Forget that we fixed j. We thus have proven (6) for each **odd** $j \in T_M$.

The definition of s_M yields $s_M = a_M + b_M$. The definition of d_M yields $d_M = a_M - b_M$. Now, for each even M-tuple $(i_1, i_2, \ldots, i_M)$, there exists a unique $j \in T_M$ satisfying

[3] We leave it to the reader to verify that this map is well-defined and is actually a bijection.

$i_M = j$. Hence, the sum rule yields

$$\begin{aligned}
&(\# \text{ of even } M\text{-tuples } (i_1, i_2, \ldots, i_M) \in T_1 \times T_2 \times \cdots \times T_M) \\
&= \sum_{j \in T_M} (\# \text{ of even } M\text{-tuples } (i_1, i_2, \ldots, i_M) \text{ with } i_M = j) \\
&= \sum_{\substack{j \in T_M; \\ j \text{ is odd}}} \underbrace{(\# \text{ of even } M\text{-tuples } (i_1, i_2, \ldots, i_M) \text{ with } i_M = j)}_{\substack{= \frac{s_1 s_2 \cdots s_{M-1} - d_1 d_2 \cdots d_{M-1}}{2} \\ \text{(by (6))}}} \\
&\qquad + \sum_{\substack{j \in T_M; \\ j \text{ is even}}} \underbrace{(\# \text{ of even } M\text{-tuples } (i_1, i_2, \ldots, i_M) \text{ with } i_M = j)}_{\substack{= \frac{s_1 s_2 \cdots s_{M-1} + d_1 d_2 \cdots d_{M-1}}{2} \\ \text{(by (5))}}} \\
&\qquad\qquad (\text{since each } j \in T_M \text{ is either odd or even (but not both)}) \\
&= \underbrace{\sum_{\substack{j \in T_M; \\ j \text{ is odd}}} \frac{s_1 s_2 \cdots s_{M-1} - d_1 d_2 \cdots d_{M-1}}{2}}_{= (\# \text{ of all odd } j \in T_M) \cdot \frac{s_1 s_2 \cdots s_{M-1} - d_1 d_2 \cdots d_{M-1}}{2}} \\
&\qquad + \underbrace{\sum_{\substack{j \in T_M; \\ j \text{ is even}}} \frac{s_1 s_2 \cdots s_{M-1} + d_1 d_2 \cdots d_{M-1}}{2}}_{= (\# \text{ of all even } j \in T_M) \cdot \frac{s_1 s_2 \cdots s_{M-1} + d_1 d_2 \cdots d_{M-1}}{2}} \\
&= \underbrace{(\# \text{ of all odd } j \in T_M)}_{\substack{= (\# \text{ of all odd elements of } T_M) \\ = b_M \\ \text{(since } b_M \text{ was defined as the } \# \text{ of all odd elements of } T_M)}} \cdot \frac{s_1 s_2 \cdots s_{M-1} - d_1 d_2 \cdots d_{M-1}}{2} \\
&\qquad + \underbrace{(\# \text{ of all even } j \in T_M)}_{\substack{= (\# \text{ of all even elements of } T_M) \\ = a_M \\ \text{(since } a_M \text{ was defined as the } \# \text{ of all even elements of } T_M)}} \cdot \frac{s_1 s_2 \cdots s_{M-1} + d_1 d_2 \cdots d_{M-1}}{2} \\
&= b_M \cdot \frac{s_1 s_2 \cdots s_{M-1} - d_1 d_2 \cdots d_{M-1}}{2} + a_M \cdot \frac{s_1 s_2 \cdots s_{M-1} + d_1 d_2 \cdots d_{M-1}}{2} \\
&= \frac{1}{2} (s_1 s_2 \cdots s_{M-1}) \underbrace{(a_M + b_M)}_{\substack{= s_M \\ \text{(since } s_M = a_M + b_M)}} + \frac{1}{2} (d_1 d_2 \cdots d_{M-1}) \underbrace{(a_M - b_M)}_{\substack{= d_M \\ \text{(since } d_M = a_M - b_M)}} \\
&= \frac{1}{2} \underbrace{(s_1 s_2 \cdots s_{M-1}) s_M}_{= s_1 s_2 \cdots s_M} + \frac{1}{2} \underbrace{(d_1 d_2 \cdots d_{M-1}) d_M}_{= d_1 d_2 \cdots d_M} \\
&= \frac{1}{2} s_1 s_2 \cdots s_M + \frac{1}{2} d_1 d_2 \cdots d_M = \frac{s_1 s_2 \cdots s_M + d_1 d_2 \cdots d_M}{2}.
\end{aligned}$$

In other words, Claim 1 holds for $m = M$. This completes the induction step. Hence, Claim 1 is proven.]

Applying Claim 1 to $m = n$, we conclude that

$$(\# \text{ of even } n\text{-tuples } (i_1, i_2, \ldots, i_n) \in T_1 \times T_2 \times \cdots \times T_n) = \frac{s_1 s_2 \cdots s_n + d_1 d_2 \cdots d_n}{2}.$$

This solves the exercise.

4.4 Second solution

Here is a solution by "destructive interference". We shall see more of this kind of reasoning later in the class; until then, you are well forgiven for considering it magic (unless you have seen the discrete Fourier transform, in which case you should be at least vaguely familiar with this kind of magic).

We will use the following fact about finite sums and products:

Lemma 4.1 (The product rule)**.** *Let $n \in \mathbb{N}$. For every $i \in \{1, 2, \ldots, n\}$, let Z_i be a finite set. For every $i \in \{1, 2, \ldots, n\}$ and every $k \in Z_i$, let $p_{i,k}$ be a number. Then,*

$$\prod_{i=1}^{n} \sum_{k \in Z_i} p_{i,k} = \sum_{(k_1, k_2, \ldots, k_n) \in Z_1 \times Z_2 \times \cdots \times Z_n} \prod_{i=1}^{n} p_{i,k_i}.$$

This lemma is a straightforward generalization of identities like

$$(a+b)(c+d) = ac + ad + bc + bd \qquad \text{and}$$
$$(a+b+c)(d+e+f) = ad + ae + af + bd + be + bf + cd + ce + cf \qquad \text{and}$$
$$(a+b)(c+d)(e+f) = ace + acf + ade + adf + bce + bcf + bde + bdf \qquad \text{and}$$
$$(a+b)(c+d+e) = ac + ad + ae + bc + bd + be$$

to a product of arbitrarily many sums with arbitrarily many addends. It is proven rigorously in [Grinbe16, Lemma 7.160]; here we restrict ourselves to making it plausible on an example:

Example 4.2. Let $n = 3$, $Z_1 = \{1, 2\}$, $Z_2 = \{1, 2, 3\}$ and $Z_3 = \{1, 2\}$. Then, Lemma 4.1 says that

$$(p_{1,1} + p_{1,2})(p_{2,1} + p_{2,2} + p_{2,3})(p_{3,1} + p_{3,2}) = \sum_{(k_1, k_2, k_3) \in \{1,2\} \times \{1,2,3\} \times \{1,2\}} p_{1,k_1} p_{2,k_2} p_{3,k_3}.$$

This equality is exactly what you get if you expand the left hand side into a huge sum (using the distributivity law for finite sums). The huge sum consists of all possible products consisting of one addend from the $(p_{1,1} + p_{1,2})$ parenthesis, one addend from the $(p_{2,1} + p_{2,2} + p_{2,3})$ parenthesis, and one addend from the $(p_{3,1} + p_{3,2})$ parenthesis. The right hand side is simply a compact way of expressing this huge sum. (The summation index (k_1, k_2, k_3) encodes which addends we are picking from which parenthesis: namely, we pick the k_1-th addend from the $(p_{1,1} + p_{1,2})$ parenthesis, the k_2-th addend from the $(p_{2,1} + p_{2,2} + p_{2,3})$ parenthesis, and the k_3-th addend from the $(p_{3,1} + p_{3,2})$ parenthesis.)

It is not hard to prove Lemma 4.1 by induction. (First prove the $n = 2$ case by induction on $|Z_2|$; then prove the general case by induction on n.)

Let us now solve the problem. Lemma 4.1 (applied to $Z_i = T_i$ and $p_{i,k} = (-1)^k$) yields

$$\begin{aligned}
&\prod_{i=1}^{n} \sum_{k \in T_i} (-1)^k \\
&= \sum_{(k_1,k_2,\ldots,k_n) \in T_1 \times T_2 \times \cdots \times T_n} \underbrace{\prod_{i=1}^{n} (-1)^{k_i}}_{\substack{=(-1)^{k_1}(-1)^{k_2}\cdots(-1)^{k_n} \\ =(-1)^{k_1+k_2+\cdots+k_n}}} = \sum_{(k_1,k_2,\ldots,k_n) \in T_1 \times T_2 \times \cdots \times T_n} (-1)^{k_1+k_2+\cdots+k_n} \\
&= \sum_{\substack{(k_1,k_2,\ldots,k_n) \in T_1 \times T_2 \times \cdots \times T_n; \\ k_1+k_2+\cdots+k_n \text{ is even}}} \underbrace{(-1)^{k_1+k_2+\cdots+k_n}}_{\substack{=1 \\ (\text{since } k_1+k_2+\cdots+k_n \text{ is even})}} + \sum_{\substack{(k_1,k_2,\ldots,k_n) \in T_1 \times T_2 \times \cdots \times T_n; \\ k_1+k_2+\cdots+k_n \text{ is odd}}} \underbrace{(-1)^{k_1+k_2+\cdots+k_n}}_{\substack{=-1 \\ (\text{since } k_1+k_2+\cdots+k_n \text{ is odd})}} \\
&\qquad \left(\begin{array}{c} \text{because for each } (k_1,k_2,\ldots,k_n) \in T_1 \times T_2 \times \cdots \times T_n, \\ \text{the integer } k_1+k_2+\cdots+k_n \text{ is either even or odd (but not both)} \end{array}\right) \\
&= \sum_{\substack{(k_1,k_2,\ldots,k_n) \in T_1 \times T_2 \times \cdots \times T_n; \\ k_1+k_2+\cdots+k_n \text{ is even}}} 1 + \sum_{\substack{(k_1,k_2,\ldots,k_n) \in T_1 \times T_2 \times \cdots \times T_n; \\ k_1+k_2+\cdots+k_n \text{ is odd}}} (-1) \\
&= \sum_{\substack{(k_1,k_2,\ldots,k_n) \in T_1 \times T_2 \times \cdots \times T_n; \\ k_1+k_2+\cdots+k_n \text{ is even}}} 1 - \sum_{\substack{(k_1,k_2,\ldots,k_n) \in T_1 \times T_2 \times \cdots \times T_n; \\ k_1+k_2+\cdots+k_n \text{ is odd}}} 1. \qquad (7)
\end{aligned}$$

But each $i \in [n]$ satisfies

$$\begin{aligned}
\sum_{k \in T_i} (-1)^k &= \sum_{\substack{k \in T_i; \\ k \text{ is even}}} \underbrace{(-1)^k}_{\substack{=1 \\ (\text{since } k \text{ is even})}} + \sum_{\substack{k \in T_i; \\ k \text{ is odd}}} \underbrace{(-1)^k}_{\substack{=-1 \\ (\text{since } k \text{ is odd})}} \\
&= \underbrace{\sum_{\substack{k \in T_i; \\ k \text{ is even}}} 1}_{\substack{=(\#\text{ of all even } k \in T_i) \cdot 1 \\ =(\#\text{ of all even } k \in T_i)}} + \underbrace{\sum_{\substack{k \in T_i; \\ k \text{ is odd}}} (-1)}_{\substack{=(\#\text{ of all odd } k \in T_i) \cdot (-1) \\ =-(\#\text{ of all odd } k \in T_i)}} \\
&= \underbrace{(\#\text{ of all even } k \in T_i)}_{\substack{=(\#\text{ of even elements of } T_i) \\ =a_i \\ (\text{by the definition of } a_i)}} - \underbrace{(\#\text{ of all odd } k \in T_i)}_{\substack{=(\#\text{ of odd elements of } T_i) \\ =b_i \\ (\text{by the definition of } b_i)}} \\
&= a_i - b_i = d_i \qquad (\text{since } d_i \text{ was defined as } a_i - b_i).
\end{aligned}$$

Hence, (7) rewrites as

$$\prod_{i=1}^{n} d_i = \sum_{\substack{(k_1,k_2,\ldots,k_n) \in T_1 \times T_2 \times \cdots \times T_n; \\ k_1+k_2+\cdots+k_n \text{ is even}}} 1 - \sum_{\substack{(k_1,k_2,\ldots,k_n) \in T_1 \times T_2 \times \cdots \times T_n; \\ k_1+k_2+\cdots+k_n \text{ is odd}}} 1. \qquad (8)$$

On the other hand, Lemma 4.1 (applied to $Z_i = T_i$ and $p_{i,k} = 1$) yields

$$\begin{aligned}
\prod_{i=1}^{n} \sum_{k \in T_i} 1 &= \sum_{(k_1,k_2,\ldots,k_n) \in T_1 \times T_2 \times \cdots \times T_n} \underbrace{\prod_{i=1}^{n} 1}_{=1} = \sum_{(k_1,k_2,\ldots,k_n) \in T_1 \times T_2 \times \cdots \times T_n} 1 \\
&= \sum_{\substack{(k_1,k_2,\ldots,k_n) \in T_1 \times T_2 \times \cdots \times T_n; \\ k_1+k_2+\cdots+k_n \text{ is even}}} 1 + \sum_{\substack{(k_1,k_2,\ldots,k_n) \in T_1 \times T_2 \times \cdots \times T_n; \\ k_1+k_2+\cdots+k_n \text{ is odd}}} 1 \\
&\qquad \left(\begin{array}{c} \text{because for each } (k_1,k_2,\ldots,k_n) \in T_1 \times T_2 \times \cdots \times T_n, \\ \text{the integer } k_1+k_2+\cdots+k_n \text{ is either even or odd (but not both)} \end{array}\right).
\end{aligned}$$

Comparing this with

$$\prod_{i=1}^{n} \underbrace{\sum_{k\in T_i} 1}_{\substack{=|T_i|=s_i \\ (\text{since } s_i=|T_i|)}} = \prod_{i=1}^{n} s_i,$$

we obtain

$$\prod_{i=1}^{n} s_i = \sum_{\substack{(k_1,k_2,\ldots,k_n)\in T_1\times T_2\times\cdots\times T_n; \\ k_1+k_2+\cdots+k_n \text{ is even}}} 1 + \sum_{\substack{(k_1,k_2,\ldots,k_n)\in T_1\times T_2\times\cdots\times T_n; \\ k_1+k_2+\cdots+k_n \text{ is odd}}} 1.$$

Adding (8) to this equality, we obtain

$$\begin{aligned}
\prod_{i=1}^{n} s_i + \prod_{i=1}^{n} d_i &= \left(\sum_{\substack{(k_1,k_2,\ldots,k_n)\in T_1\times T_2\times\cdots\times T_n; \\ k_1+k_2+\cdots+k_n \text{ is even}}} 1 + \sum_{\substack{(k_1,k_2,\ldots,k_n)\in T_1\times T_2\times\cdots\times T_n; \\ k_1+k_2+\cdots+k_n \text{ is odd}}} 1 \right) \\
&\quad + \left(\sum_{\substack{(k_1,k_2,\ldots,k_n)\in T_1\times T_2\times\cdots\times T_n; \\ k_1+k_2+\cdots+k_n \text{ is even}}} 1 - \sum_{\substack{(k_1,k_2,\ldots,k_n)\in T_1\times T_2\times\cdots\times T_n; \\ k_1+k_2+\cdots+k_n \text{ is odd}}} 1 \right) \\
&= 2 \cdot \underbrace{\sum_{\substack{(k_1,k_2,\ldots,k_n)\in T_1\times T_2\times\cdots\times T_n; \\ k_1+k_2+\cdots+k_n \text{ is even}}} 1}_{\substack{=(\#\text{ of all } (k_1,k_2,\ldots,k_n)\in T_1\times T_2\times\cdots\times T_n \text{ such that } k_1+k_2+\cdots+k_n \text{ is even}) \\ =(\#\text{ of all } (i_1,i_2,\ldots,i_n)\in T_1\times T_2\times\cdots\times T_n \text{ such that } i_1+i_2+\cdots+i_n \text{ is even}) \\ =(\#\text{ of all even } n\text{-tuples } (i_1,i_2,\ldots,i_n)\in T_1\times T_2\times\cdots\times T_n) \\ (\text{by the definition of "even" for } n\text{-tuples})}} \\
&= 2 \cdot (\#\text{ of all even } n\text{-tuples } (i_1,i_2,\ldots,i_n)\in T_1\times T_2\times\cdots\times T_n).
\end{aligned}$$

Solving this equation for $(\#\text{ of all even } n\text{-tuples } (i_1,i_2,\ldots,i_n)\in T_1\times T_2\times\cdots\times T_n)$, we find

$$\begin{aligned}
&(\#\text{ of all even } n\text{-tuples } (i_1,i_2,\ldots,i_n)\in T_1\times T_2\times\cdots\times T_n) \\
&= \frac{1}{2} \cdot \left(\underbrace{\prod_{i=1}^{n} s_i}_{=s_1s_2\cdots s_n} + \underbrace{\prod_{i=1}^{n} d_i}_{=d_1d_2\cdots d_n} \right) = \frac{1}{2}(s_1s_2\cdots s_n + d_1d_2\cdots d_n) = \frac{s_1s_2\cdots s_n + d_1d_2\cdots d_n}{2}.
\end{aligned}$$

This solves the exercise.

---

5 Exercise 5

5.1 Problem

Let A and B be two finite sets. Let $n = |A|$ and $m = |B|$. Prove the following:

(a) For each $b \in B$, we have

$$(\#\text{ of maps } f : A \to B \text{ such that } b \in f(A)) = m^n - (m-1)^n.$$

(b) We have

$$\sum_{f:A\to B} |f(A)| = m(m^n - (m-1)^n).$$

5.2 Remark

If $f : A \to B$ is a map, then $f(A)$ denotes the image of f (that is, the subset $\{f(a) \mid a \in A\}$ of B). More generally, if $f : A \to B$ is a map and S is any subset of A, then $f(S)$ denotes the subset $\{f(a) \mid a \in S\}$ of B.

The sum in part **(b)** of this exercise is a sum over all maps f from A to B. It can also be written as $\sum\limits_{f\in B^A} |f(A)|$. See [Math222, Example 1.2.4 **(b)**] for an example.

5.3 Solution sketch

(a) Let $b \in B$. Thus, $|B \setminus \{b\}| = \underbrace{|B|}_{=m} - 1 = m - 1$.

Each map $f : A \to B$ satisfies either $b \in f(A)$ or $b \notin f(A)$ (but not both at the same time). Thus, the sum rule yields

$$\begin{aligned}(\#\text{ of maps } f : A \to B) &= (\#\text{ of maps } f : A \to B \text{ such that } b \in f(A)) \\ &\qquad + (\#\text{ of maps } f : A \to B \text{ such that } b \notin f(A)).\end{aligned}$$

Hence,

$$\begin{aligned}&(\#\text{ of maps } f : A \to B \text{ such that } b \in f(A)) \\ &= (\#\text{ of maps } f : A \to B) - (\#\text{ of maps } f : A \to B \text{ such that } b \notin f(A)). \qquad (9)\end{aligned}$$

Now, let us compute $(\#\text{ of maps } f : A \to B \text{ such that } b \notin f(A))$. Indeed, if $f : A \to B$ is a map such that $b \notin f(A)$, then all values of f belong to the set $B \setminus \{b\}$ (since $b \notin f(A)$), and thus f can be regarded as a map from A to $B \setminus \{b\}$. More pedantically: If $f : A \to B$ is a map such that $b \notin f(A)$, then we can define a map $\widetilde{f} : A \to B \setminus \{b\}$ by setting

$$\widetilde{f}(a) = f(a) \qquad \text{for all } a \in A.$$

(This map $\widetilde{f}$ has the exact same values as f, but it has a different target set, so rigor demands us to distinguish it from f. But you should think of it as just being f wearing a tighter cloak.)

Thus, whenever $f : A \to B$ is a map such that $b \notin f(A)$, we have defined a new map $\widetilde{f}$ from A to $B \setminus \{b\}$. Thus, we obtain a map

$$\begin{aligned}T : \{\text{maps } f : A \to B \text{ such that } b \notin f(A)\} &\to \{\text{maps from } A \text{ to } B \setminus \{b\}\}, \\ f &\mapsto \widetilde{f}\end{aligned}$$

(yes, this is a map between sets of maps). This map T is bijective[4]. Hence, the bijection principle yields

$$\begin{aligned}&(\#\text{ of maps } f : A \to B \text{ such that } b \notin f(A)) \\ &= (\#\text{ of maps from } A \text{ to } B \setminus \{b\}) = |\{\text{maps } A \to B \setminus \{b\}\}| = \left|(B \setminus \{b\})^A\right| \\ &= |B \setminus \{b\}|^{|A|} = (m-1)^n \qquad (\text{since } |B \setminus \{b\}| = m-1 \text{ and } |A| = n).\end{aligned}$$

[4]Indeed, T is injective, because f is clearly uniquely determined by its image $T(f) = \widetilde{f}$ (just recall that $f(a) = \widetilde{f}(a)$ for all $a \in A$). And furthermore, T is surjective, because if $g : A \to B \setminus \{b\}$ is any map, then there is a map $f : A \to B$ such that $b \notin f(A)$ and $\widetilde{f} = g$ (indeed, this map f is defined simply by setting $f(a) = g(a)$ for all $a \in A$).

Hence, (9) becomes

$$\begin{aligned}&(\#\text{ of maps } f : A \to B \text{ such that } b \in f(A)) \\ &= \underbrace{(\#\text{ of maps } f : A \to B)}_{\substack{=(\#\text{ of maps from } A \text{ to } B) \\ =|\{\text{maps from } A \text{ to } B\}| \\ =\left|B^A\right|=|B|^{|A|}=m^n \\ (\text{since } |B|=m \text{ and } |A|=n)}} - \underbrace{(\#\text{ of maps } f : A \to B \text{ such that } b \notin f(A))}_{=(m-1)^n} = m^n - (m-1)^n .\end{aligned}$$

This solves part **(a)** of the exercise.

(b) Recall the following fact ([Math222, Proposition 1.6.3]):

Proposition 5.1 ("Counting by roll call")**.** *Let S be a finite set.*

(a) *If T is a subset of S, then*

$$|T| = \sum_{s \in S} [s \in T] .$$

(b) *For each $s \in S$, let $\mathcal{A}(s)$ be a logical statement (which can be either true or false depending on s; for example, $\mathcal{A}(s)$ could be "s is even" if S is a set of integers, or "s is empty" if S is a set of sets). Then,*

$$(\#\text{ of } s \in S \text{ that satisfy } \mathcal{A}(s)) = \sum_{s \in S} [\mathcal{A}(s)] .$$

Now, if $f : A \to B$ is any map, then $f(A)$ is a subset of B, and thus Proposition 5.1 **(a)** (applied to $S = B$ and $T = f(A)$) yields

$$|f(A)| = \sum_{s \in B} [s \in f(A)] = \sum_{b \in B} [b \in f(A)]$$

(here, we have renamed the summation index s as b). Thus,

$$\sum_{f:A\to B} \underbrace{|f(A)|}_{=\sum_{b\in B}[b\in f(A)]} = \sum_{f:A\to B} \sum_{b\in B} [b \in f(A)] = \sum_{b\in B} \sum_{f:A\to B} [b \in f(A)] \qquad (10)$$

(here, we have interchanged the summation signs, due to the Fubini principle).

Now, let $b \in B$. Then, Proposition 5.1 **(b)** (applied to $S = B^A$ and $\mathcal{A}(s) = $ ("$b \in s(A)$")) yields

$$\left(\#\text{ of } s \in B^A \text{ that satisfy } b \in s(A)\right) = \sum_{s \in B^A} [b \in s(A)] = \sum_{s:A\to B} [b \in s(A)]$$

(since the $s \in B^A$ are precisely the maps $s : A \to B$). Renaming the index s as f everywhere in this equality, we obtain

$$\left(\#\text{ of } f \in B^A \text{ that satisfy } b \in f(A)\right) = \sum_{f:A\to B} [b \in f(A)] .$$

Hence,

$$\begin{aligned}\sum_{f:A\to B} [b \in f(A)] &= \left(\#\text{ of } f \in B^A \text{ that satisfy } b \in f(A)\right) \\ &= (\#\text{ of maps } f : A \to B \text{ such that } b \in f(A)) \\ &\qquad \left(\text{since the } f \in B^A \text{ are precisely the maps } f : A \to B\right) \\ &= m^n - (m-1)^n \qquad (\text{by part } \textbf{(a)} \text{ of this exercise}) . \qquad (11)\end{aligned}$$

Now, forget that we fixed b. We thus have proved (11) for each $b \in B$. Now, (10) becomes

$$\sum_{f:A\to B} |f(A)| = \sum_{b\in B} \underbrace{\sum_{f:A\to B} [b \in f(A)]}_{\substack{=m^n-(m-1)^n \\ \text{(by (11))}}} = \sum_{b\in B} (m^n - (m-1)^n) = \underbrace{|B|}_{=m} \cdot (m^n - (m-1)^n)$$
$$= m(m^n - (m-1)^n).$$

This solves part **(b)** of the exercise.

---

6 Exercise 6

6.1 Problem

Recall the Fibonacci sequence $(f_0, f_1, f_2, \ldots)$. Prove that

$$\sum_{i=0}^{n} \binom{n}{i} f_{i+j} = f_{2n+j} \qquad (12)$$

for each $n \in \mathbb{N}$ and $j \in \mathbb{N}$.

6.2 First solution

Recall the following fact ([Math222, Theorem 1.1.12]):

Theorem 6.1 (Binet's formula)**.** *For each $n \in \mathbb{N}$, we have*

$$f_n = \frac{1}{\sqrt{5}} (\varphi^n - \psi^n), \qquad (13)$$

where

$$\varphi = \frac{1+\sqrt{5}}{2} \approx 1.618\ldots \qquad \text{and} \qquad \psi = \frac{1-\sqrt{5}}{2} \approx -0.618\ldots.$$

Now, let $n \in \mathbb{N}$ and $j \in \mathbb{N}$. Let $\varphi = \frac{1+\sqrt{5}}{2}$ and $\psi = \frac{1-\sqrt{5}}{2}$. Straightforward computations reveal that $\varphi^2 = \varphi + 1$ and $\psi^2 = \psi + 1$. (Actually, φ and ψ are the two solutions of the quadratic equation $x^2 = x + 1$.)

On the other hand, for each $x \in \mathbb{R}$, we have

$$\sum_{i=0}^{n} \binom{n}{i} x^i = (x+1)^n \qquad (14)$$

(because the binomial formula yields $(x+1)^n = \sum_{i=0}^{n} \binom{n}{i} x^i \underbrace{1^{n-i}}_{=1} = \sum_{i=0}^{n} \binom{n}{i} x^i$).

Now,

$$\sum_{i=0}^{n}\dbinom{n}{i}\underbrace{f_{i+j}}_{\substack{=\frac{1}{\sqrt{5}}\left(\varphi^{i+j}-\psi^{i+j}\right)\\ \text{(by (13), applied to } i+j\\ \text{instead of } n)}}=\sum_{i=0}^{n}\underbrace{\dbinom{n}{i}\cdot\frac{1}{\sqrt{5}}\left(\varphi^{i+j}-\psi^{i+j}\right)}_{=\frac{1}{\sqrt{5}}\binom{n}{i}\varphi^{i+j}-\frac{1}{\sqrt{5}}\binom{n}{i}\psi^{i+j}}$$

$$=\sum_{i=0}^{n}\left(\frac{1}{\sqrt{5}}\dbinom{n}{i}\varphi^{i+j}-\frac{1}{\sqrt{5}}\dbinom{n}{i}\psi^{i+j}\right)=\sum_{i=0}^{n}\frac{1}{\sqrt{5}}\dbinom{n}{i}\underbrace{\varphi^{i+j}}_{=\varphi^i\varphi^j}-\sum_{i=0}^{n}\frac{1}{\sqrt{5}}\dbinom{n}{i}\underbrace{\psi^{i+j}}_{=\psi^i\psi^j}$$

$$=\sum_{i=0}^{n}\frac{1}{\sqrt{5}}\dbinom{n}{i}\varphi^i\varphi^j-\sum_{i=0}^{n}\frac{1}{\sqrt{5}}\dbinom{n}{i}\psi^i\psi^j$$

$$=\frac{1}{\sqrt{5}}\varphi^j\underbrace{\sum_{i=0}^{n}\dbinom{n}{i}\varphi^i}_{\substack{=(\varphi+1)^n\\ \text{(by (14), applied to } x=\varphi)}}-\frac{1}{\sqrt{5}}\psi^j\underbrace{\sum_{i=0}^{n}\dbinom{n}{i}\psi^i}_{\substack{=(\psi+1)^n\\ \text{(by (14), applied to } x=\psi)}}$$

$$=\frac{1}{\sqrt{5}}\varphi^j\left(\underbrace{\varphi+1}_{=\varphi^2}\right)^n-\frac{1}{\sqrt{5}}\psi^j\left(\underbrace{\psi+1}_{=\psi^2}\right)^n=\frac{1}{\sqrt{5}}\underbrace{\varphi^j\left(\varphi^2\right)^n}_{=\varphi^j\varphi^{2n}=\varphi^{2n}\varphi^j=\varphi^{2n+j}}-\frac{1}{\sqrt{5}}\underbrace{\psi^j\left(\psi^2\right)^n}_{=\psi^j\psi^{2n}=\psi^{2n}\psi^j=\psi^{2n+j}}$$

$$=\frac{1}{\sqrt{5}}\varphi^{2n+j}-\frac{1}{\sqrt{5}}\psi^{2n+j}=\frac{1}{\sqrt{5}}\left(\varphi^{2n+j}-\psi^{2n+j}\right).$$

Comparing this with

$$f_{2n+j}=\frac{1}{\sqrt{5}}\left(\varphi^{2n+j}-\psi^{2n+j}\right)\qquad\text{(by (13), applied to } 2n+j \text{ instead of } n),$$

we obtain $\sum\limits_{i=0}^{n}\dbinom{n}{i}f_{i+j}=f_{2n+j}$. This solves the exercise.

6.3 Second solution

Recall the definition of the Fibonacci sequence. This definition entails that

$$f_n=f_{n-1}+f_{n-2}\qquad\text{for all integers } n\geq 2. \tag{15}$$

Now, let us solve the exercise by induction on n:

Induction base: It is easy to see that $\sum\limits_{i=0}^{0}\dbinom{0}{i}f_{i+j}=f_{2\cdot 0+j}$ for each $j\in\mathbb{N}$ [5]. In other words, the claim of the exercise holds for $n=0$. This completes the induction base.

Induction step: Let $m\in\mathbb{N}$. Assume that the claim of the exercise holds for $n=m$. We must prove that the claim of the exercise holds for $n=m+1$.

We have assumed that the claim of the exercise holds for $n=m$. In other words, we have

$$\sum_{i=0}^{m}\dbinom{m}{i}f_{i+j}=f_{2m+j}\qquad\text{for each } j\in\mathbb{N}. \tag{16}$$

[5] *Proof.* Let $j\in\mathbb{N}$. Then, $\sum\limits_{i=0}^{0}\dbinom{0}{i}f_{i+j}=\underbrace{\dbinom{0}{0}}_{=1}f_{0+j}=f_{0+j}=f_j$. Comparing this with $f_{2\cdot 0+j}=f_{0+j}=f_j$, we obtain $\sum\limits_{i=0}^{0}\dbinom{0}{i}f_{i+j}=f_{2\cdot 0+j}$. Qed.

Theorem 1.2 (applied to $n = m$ and $k = m+1$) yields $\binom{m}{m+1} = \binom{m}{m-(m+1)} = 0$ (by the definition of binomial coefficients, because $m - (m+1) = -1 \notin \mathbb{N}$).

Now, let $j \in \mathbb{N}$. Then,

$$\begin{aligned}
&\sum_{i=0}^{m+1} \underbrace{\binom{m+1}{i}}_{\substack{=\binom{(m+1)-1}{i-1}+\binom{(m+1)-1}{i} \\ \text{(by Theorem 1.3, applied to } n=m+1 \text{ and } k=i\text{)}}} f_{i+j} \\
&= \sum_{i=0}^{m+1} \left(\binom{(m+1)-1}{i-1} + \binom{(m+1)-1}{i} \right) f_{i+j} \\
&= \sum_{i=0}^{m+1} \left(\binom{m}{i-1} + \binom{m}{i} \right) f_{i+j} \qquad (\text{since } (m+1)-1 = m) \\
&= \underbrace{\sum_{i=0}^{m+1} \binom{m}{i-1} f_{i+j}}_{\substack{=\binom{m}{0-1} f_{0+j} + \sum_{i=1}^{m+1} \binom{m}{i-1} f_{i+j} \\ \text{(here, we have split off the addend for } i=0 \text{ from the sum)}}} + \underbrace{\sum_{i=0}^{m+1} \binom{m}{i} f_{i+j}}_{\substack{=\binom{m}{m+1} f_{(m+1)+j} + \sum_{i=0}^{m} \binom{m}{i} f_{i+j} \\ \text{(here, we have split off the addend for } i=m+1 \text{ from the sum)}}} \\
&= \underbrace{\binom{m}{0-1}}_{\substack{=0 \\ \text{(by the definition of} \\ \text{binomial coefficients,} \\ \text{since } 0-1=-1\notin\mathbb{N})}} f_{0+j} + \underbrace{\sum_{i=1}^{m+1} \binom{m}{i-1} f_{i+j}}_{\substack{=\sum_{i=0}^{m} \binom{m}{(i+1)-1} f_{(i+1)+j} \\ \text{(here, we have substituted } i+1 \\ \text{for } i \text{ in the sum)}}} + \underbrace{\binom{m}{m+1}}_{=0} f_{(m+1)+j} + \sum_{i=0}^{m} \binom{m}{i} f_{i+j} \\
&= \sum_{i=0}^{m} \underbrace{\binom{m}{(i+1)-1}}_{=\binom{m}{i}} \underbrace{f_{(i+1)+j}}_{\substack{=f_{i+(j+1)} \\ \text{(since } (i+1)+j=i+(j+1))}} + \sum_{i=0}^{m} \binom{m}{i} f_{i+j} \\
&= \underbrace{\sum_{i=0}^{m} \binom{m}{i} f_{i+(j+1)}}_{\substack{=f_{2m+(j+1)} \\ \text{(by (16),} \\ \text{applied to } j+1 \text{ instead of } j)}} + \underbrace{\sum_{i=0}^{m} \binom{m}{i} f_{i+j}}_{\substack{=f_{2m+j} \\ \text{(by (16))}}} = f_{2m+(j+1)} + f_{2m+j}.
\end{aligned}$$

Comparing this with

$$\begin{aligned}
f_{2(m+1)+j} &= f_{2m+j+2} \qquad (\text{since } 2(m+1)+j = 2m+j+2) \\
&= f_{2m+j+2-1} + f_{2m+j+2-2} \qquad (\text{by (15), applied to } n = 2m+j+2) \\
&= f_{2m+(j+1)} + f_{2m+j} \\
&\qquad (\text{since } 2m+j+2-1 = 2m+(j+1) \text{ and } 2m+j+2-2 = 2m+j),
\end{aligned}$$

we obtain

$$\sum_{i=0}^{m+1} \binom{m+1}{i} f_{i+j} = f_{2(m+1)+j}.$$

Now, forget that we fixed j. We thus have proved that

$$\sum_{i=0}^{m+1} \binom{m+1}{i} f_{i+j} = f_{2(m+1)+j} \qquad \text{for each } j \in \mathbb{N}.$$

In other words, the claim of the exercise holds for $n = m + 1$. This completes the induction step. Thus, the exercise is solved by induction.

6.4 Remark

1. The above two solutions are not as different as they look. The first uses the binomial formula; the second imitates the proof of the binomial formula.

2. The second solution shows that the exercise holds for any sequence $(f_0, f_1, f_2, \ldots)$ of numbers that satisfies the recurrence relation (15) (not just for the Fibonacci sequence). (The first solution can also be adapted to prove this, once you show a more general version of Binet's formula.)

References

[Grinbe16] Darij Grinberg, *Notes on the combinatorial fundamentals of algebra*, 10 January 2019.
`http://www.cip.ifi.lmu.de/~grinberg/primes2015/sols.pdf`
The numbering of theorems and formulas in this link might shift when the project gets updated; for a "frozen" version whose numbering is guaranteed to match that in the citations above, see `https://github.com/darijgr/detnotes/releases/tag/2019-01-10` .

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[hw2s] Darij Grinberg, *Drexel Fall 2019 Math 222 homework set #2 with solutions*, `http://www.cip.ifi.lmu.de/~grinberg/t/19fco/hw2s.pdf`

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Caution: The numbering of theorems and formulas in this link might shift when the project gets updated; for a "frozen" version whose numbering is guaranteed to match that in the citations above, see `https://gitlab.com/darijgrinberg/darijgrinberg.gitlab.io/blob/2dab2743a181d5ba8fc145a661fd274bc37d03be/public/t/19fco/n/n.pdf`