Drexel University, Department of Mathematics

Math 222: Enumerative Combinatorics, Fall 2019: Midterm 1

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due date: **Friday, 2019-10-25** at the beginning of class,
or before that through Blackboard.

Please solve **5 of the 6 exercises**!

This is a midterm, so **collaboration is not allowed**!

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1 Exercise 1

1.1 Problem

Let $n \in \mathbb{N}$. Prove that

$$\sum_{k=0}^{n} \binom{2n+1}{k} = 4^n.$$

1.2 Solution

[...]

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2 Exercise 2

2.1 Problem

Let $n \in \mathbb{N}$. Compute the number of 4-tuples (A, B, C, D) of subsets of $[n]$ satisfying

$$A \cap B = C \cup D.$$

2.2 Solution

[...]

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3 Exercise 3

3.1 Problem

Let m and n be two nonnegative integers such that $m \leq n$. Let $f_m, f_{m+1}, \ldots, f_n$ be any $n - m + 1$ numbers. Let $g_m, g_{m+1}, \ldots, g_{n+1}$ be any $n - m + 2$ numbers. Prove that

$$\sum_{k=m}^{n} f_k \left(g_{k+1} - g_k\right) + \sum_{k=m+1}^{n} g_k \left(f_k - f_{k-1}\right) = f_n g_{n+1} - f_m g_m. \tag{1}$$

3.2 Remark

This is a discrete version of the "integration by parts" formula

$$\int_m^n f g' + \int_m^n g f' = (fg)(n) - (fg)(m)$$

from calculus.

3.3 Solution

[...]

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4 Exercise 4

4.1 Problem

Let $n \in \mathbb{N}$. Let $T_1, T_2, \ldots, T_n$ be n finite sets of integers. For each $i \in [n]$, we let a_i be the $\#$ of even elements of T_i, and we let b_i be the $\#$ of odd elements of T_i. Furthermore, for each $i \in [n]$, we set $s_i = a_i + b_i = |T_i|$ and $d_i = a_i - b_i$.

An n-tuple $(i_1, i_2, \ldots, i_n) \in T_1 \times T_2 \times \cdots \times T_n$ is said to be *even* if the sum $i_1 + i_2 + \cdots + i_n$ is even. (For example, the 4-tuple $(1, 0, 4, 1)$ is even, whereas $(1, 0, 3, 1)$ is not.)

Prove that the $\#$ of even n-tuples $(i_1, i_2, \ldots, i_n) \in T_1 \times T_2 \times \cdots \times T_n$ equals

$$\frac{s_1 s_2 \cdots s_n + d_1 d_2 \cdots d_n}{2}.$$

4.2 Remark

This generalizes [hw1s, Exercise 6].

4.3 Solution

[...]

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5 Exercise 5

5.1 Problem

Let A and B be two finite sets. Let $n = |A|$ and $m = |B|$. Prove the following:

(a) For each $b \in B$, we have

$$(\# \text{ of maps } f : A \to B \text{ such that } b \in f(A)) = m^n - (m-1)^n .$$

(b) We have

$$\sum_{f:A\to B} |f(A)| = m(m^n - (m-1)^n).$$

5.2 Remark

If $f : A \to B$ is a map, then $f(A)$ denotes the image of f (that is, the subset $\{f(a) \mid a \in A\}$ of B). More generally, if $f : A \to B$ is a map and S is any subset of A, then $f(S)$ denotes the subset $\{f(a) \mid a \in S\}$ of B.

The sum in part **(b)** of this exercise is a sum over all maps f from A to B. It can also be written as $\sum\limits_{f \in B^A} |f(A)|$. See [Math222, Example 1.2.4 **(b)**] for an example.

5.3 Solution

[...]

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6 Exercise 6

6.1 Problem

Recall the Fibonacci sequence $(f_0, f_1, f_2, \ldots)$. Prove that

$$\sum_{i=0}^{n} \binom{n}{i} f_{i+j} = f_{2n+j} \tag{2}$$

for each $n \in \mathbb{N}$ and $j \in \mathbb{N}$.

6.2 Solution

[...]

References

[Math222] Darij Grinberg, *Enumerative Combinatorics: class notes*, 16 December 2019. `http://www.cip.ifi.lmu.de/~grinberg/t/19fco/n/n.pdf` Also available on the mirror server `http://darijgrinberg.gitlab.io/t/19fco/n/n.pdf`
Caution: The numbering of theorems and formulas in this link might shift when the project gets updated; for a "frozen" version whose numbering is guaranteed to match that in the citations above, see `https://gitlab.com/darijgrinberg/darijgrinberg.gitlab.io/blob/2dab2743a181d5ba8fc145a661fd274bc37d03be/public/t/19fco/n/n.pdf`

[hw1s] Darij Grinberg, *Drexel Fall 2019 Math 222 homework set #1 with solutions*, `http://www.cip.ifi.lmu.de/~grinberg/t/19fco/hw1s.pdf`