Drexel University, Department of Mathematics

Math 222: Enumerative Combinatorics, Fall 2019: Homework 4

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due date: **Monday, 2019-11-25** at the beginning of class, or before that through Blackboard.

Please solve **3 of the 4 exercises**!

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1 Exercise 1

1.1 Problem

Let n be a positive integer. Let $A_1, A_2, \ldots, A_n$ be n finite sets. Prove that

$$|A_1 \cap A_2 \cap \cdots \cap A_n| = \sum_{m=1}^{n} (-1)^{m-1} \sum_{\substack{(i_1, i_2, \ldots, i_m) \in [n]^m; \\ i_1 < i_2 < \cdots < i_m}} |A_{i_1} \cup A_{i_2} \cup \cdots \cup A_{i_m}|.$$

1.2 Remark

This is similar to [Math222, Theorem 2.9.1], but with $\cup$ and $\cap$ instead of $\cap$ and $\cup$.

1.3 Solution

[...]

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2 Exercise 2

2.1 Problem

Let $n \in \mathbb{N}$ and $k \in \mathbb{N}$. Prove that

$$\left(\# \text{ of } (x_1, x_2, \ldots, x_k) \in \{0,1,2\}^k \text{ such that } x_1 + x_2 + \cdots + x_k = n\right)$$
$$= \sum_{j=0}^{k} (-1)^j \binom{k}{j} \binom{n-3j+k-1}{n-3j}.$$

[**Hint:** $\{0,1,2\}$ is the set of all $i \in \mathbb{N}$ that don't satisfy $i \geq 3$.]

2.2 Solution

[...]

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3 Exercise 3

3.1 Problem

Let k and m be positive integers. Prove that

$$\sum_{\substack{(a_1,a_2,\ldots,a_k)\in\mathbb{N}^k;\\ a_1+a_2+\cdots+a_k=m}} a_1 a_2 \cdots a_k = \binom{k+m-1}{2k-1}.$$

3.2 Solution

[...]

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4 Exercise 4

4.1 Problem

Let $n \in \mathbb{N}$ and $k \in \mathbb{N}$. Prove that

$$\sum_{i=0}^{k} \binom{n+i-1}{i} \binom{n}{k-2i} = \binom{n+k-1}{k}.$$

[**Hint:** Any nonnegative integer u can be written as $u = 2q + r$ for a unique $q \in \mathbb{N}$ and a unique $r \in \{0,1\}$. What does this mean for weak compositions of k into n parts?]

4.2 Solution

[...]

References

[Math222] Darij Grinberg, *Enumerative Combinatorics: class notes*, 16 December 2019. `http://www.cip.ifi.lmu.de/~grinberg/t/19fco/n/n.pdf` Also available on the mirror server `http://darijgrinberg.gitlab.io/t/19fco/n/n.pdf`
Caution: The numbering of theorems and formulas in this link might shift when the project gets updated; for a "frozen" version whose numbering is guaranteed to match that in the citations above, see `https://gitlab.com/darijgrinberg/darijgrinberg.gitlab.io/blob/2dab2743a181d5ba8fc145a661fd274bc37d03be/public/t/19fco/n/n.pdf`