Drexel University, Department of Mathematics

Math 222: Enumerative Combinatorics, Fall 2019: Homework 3

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1 Exercise 1

1.1 Problem

Let $n \in \mathbb{N}$. Prove that

$$\sum_{k=0}^{n} \binom{2n+1}{k}^2 = \binom{4n+1}{2n}.$$

1.2 Remark

This exercise is similar to [mt1s, Exercise 1]; our two solutions below imitate the two solutions of the latter exercise.

1.3 First solution

Forget that we fixed n. Recall the following fact ([Math222, Corollary 2.6.4]):

Corollary 1.1. *Let $n \in \mathbb{N}$. Then,* $\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}$.

Also, recall the symmetry property of the binomial coefficients ([Math222, Theorem 1.3.11]):

Theorem 1.2 (Symmetry of the binomial coefficients)**.** *Let $n \in \mathbb{N}$ and $k \in \mathbb{R}$. Then,*

$$\binom{n}{k} = \binom{n}{n-k}.$$

We also recall the following simple fact ([Math222, Proposition 1.3.36]):

Proposition 1.3 (Absorption formula I)**.** *Let $n \in \{1, 2, 3, \ldots\}$ and $m \in \mathbb{R}$. Then,*

$$\binom{m}{n} = \frac{m}{n}\binom{m-1}{n-1}.$$

Now, let $n \in \mathbb{N}$. Then, $2n+1$ is a positive integer. In other words, $2n+1 \in \{1, 2, 3, \ldots\}$. Thus, Proposition 1.3 (applied to $2(2n+1)$ and $2n+1$ instead of m and n) yields

$$\binom{2(2n+1)}{2n+1} = \underbrace{\frac{2(2n+1)}{2n+1}}_{=2}\binom{2(2n+1)-1}{2n+1-1} = 2\binom{2(2n+1)-1}{2n+1-1}$$
$$= 2\binom{4n+1}{2n} \qquad (1)$$

(since $2(2n+1)-1 = 4n+1$ and $2n+1-1 = 2n$).

Each $k \in \mathbb{R}$ satisfies

$$\binom{2n+1}{k} = \binom{2n+1}{2n+1-k}$$

(by Theorem 1.2, applied to $2n+1$ instead of n) and thus

$$\binom{2n+1}{k}^2 = \binom{2n+1}{2n+1-k}^2. \qquad (2)$$

From $n \in \mathbb{N}$, we obtain $0 \leq n \leq 2n+1$. Hence, we can split the sum $\sum_{k=0}^{2n+1}\binom{2n+1}{k}^2$ at $k = n$. We thus obtain

$$\sum_{k=0}^{2n+1}\binom{2n+1}{k}^2 = \sum_{k=0}^{n}\binom{2n+1}{k}^2 + \sum_{k=n+1}^{2n+1}\underbrace{\binom{2n+1}{k}^2}_{\substack{=\binom{2n+1}{2n+1-k}^2 \\ \text{(by (2))}}}$$
$$= \sum_{k=0}^{n}\binom{2n+1}{k}^2 + \sum_{k=n+1}^{2n+1}\binom{2n+1}{2n+1-k}^2$$
$$= \sum_{k=0}^{n}\binom{2n+1}{k}^2 + \sum_{k=0}^{n}\binom{2n+1}{k}^2$$
$$\left(\text{here, we have substituted } k \text{ for } 2n+1-k \text{ in the second sum}\right)$$
$$= 2 \cdot \sum_{k=0}^{n}\binom{2n+1}{k}^2.$$

Comparing this with

$$\sum_{k=0}^{2n+1}\binom{2n+1}{k}^2 = \binom{2(2n+1)}{2n+1} \qquad \left(\text{by Corollary 1.1, applied to } 2n+1 \text{ instead of } n\right),$$

we obtain

$$2 \cdot \sum_{k=0}^{n}\binom{2n+1}{k}^2 = \binom{2(2n+1)}{2n+1} = 2\binom{4n+1}{2n}$$

(by (1)). Dividing both sides of this equality by 2, we find

$$\sum_{k=0}^{n} \binom{2n+1}{k}^2 = \binom{4n+1}{2n}.$$

This solves the exercise.

1.4 Second solution

Forget that we fixed n. Recall the recurrence of the binomial coefficients ([Math222, Theorem 1.3.8]):

Theorem 1.4 (Recurrence of the binomial coefficients)**.** *Let $n \in \mathbb{R}$ and $k \in \mathbb{R}$. Then,*

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}.$$

We also recall the following identity ([Math222, Corollary 2.6.3]):

Corollary 1.5. *Let $x \in \mathbb{R}$ and $y \in \mathbb{N}$. Then,*

$$\sum_{k=0}^{y} \binom{x}{k}\binom{y}{k} = \binom{x+y}{y}.$$

Now, let $n \in \mathbb{N}$. Then,

$$\begin{aligned}
&\sum_{k=0}^{n} \underbrace{\binom{2n+1}{k}^2}_{=\binom{2n+1}{k}\binom{2n+1}{k}} \\
&= \sum_{k=0}^{n} \binom{2n+1}{k} \underbrace{\binom{2n+1}{k}}_{\substack{=\binom{(2n+1)-1}{k-1}+\binom{(2n+1)-1}{k} \\ \text{(by Theorem 1.4, applied to } 2n+1 \text{ instead of } n)}} \\
&= \sum_{k=0}^{n} \binom{2n+1}{k} \left(\binom{(2n+1)-1}{k-1} + \binom{(2n+1)-1}{k} \right) \\
&= \sum_{k=0}^{n} \binom{2n+1}{k} \left(\binom{2n}{k-1} + \binom{2n}{k} \right) \qquad (\text{since } (2n+1)-1 = 2n) \\
&= \sum_{k=0}^{n} \left(\binom{2n+1}{k}\binom{2n}{k-1} + \binom{2n+1}{k}\binom{2n}{k} \right) \\
&= \sum_{k=0}^{n} \binom{2n+1}{k}\binom{2n}{k-1} + \sum_{k=0}^{n} \binom{2n+1}{k}\binom{2n}{k}.
\end{aligned} \tag{3}$$

But each $k \in \mathbb{R}$ satisfies

$$\begin{aligned}
\binom{2n}{k-1} &= \binom{2n}{2n-(k-1)} \\
&\qquad \text{(by Theorem 1.2, applied to } 2n \text{ and } k-1 \text{ instead of } n \text{ and } k) \\
&= \binom{2n}{2n+1-k}
\end{aligned} \tag{4}$$

(since $2n-(k-1)=2n+1-k$) and

$$\binom{2n+1}{k}=\binom{2n+1}{2n+1-k} \qquad \left(\text{by Theorem 1.2, applied to } 2n+1 \text{ instead of } n\right). \tag{5}$$

Now, we can split off the addend for $k=0$ from the sum $\sum_{k=0}^{n}\binom{2n+1}{k}\binom{2n}{k-1}$; we thus find

$$\begin{aligned}
\sum_{k=0}^{n}\binom{2n+1}{k}\binom{2n}{k-1} &= \binom{2n+1}{0}\underbrace{\binom{2n}{0-1}}_{\substack{=0\\ \text{(by the definition of}\\ \text{binomial coefficients,}\\ \text{since } 0-1=-1\notin\mathbb{N})}} + \sum_{k=1}^{n}\binom{2n+1}{k}\binom{2n}{k-1} \\
&= \sum_{k=1}^{n}\underbrace{\binom{2n+1}{k}}_{\substack{=\binom{2n+1}{2n+1-k}\\ \text{(by (5))}}}\underbrace{\binom{2n}{k-1}}_{\substack{=\binom{2n}{2n+1-k}\\ \text{(by (4))}}} \\
&= \sum_{k=1}^{n}\binom{2n+1}{2n+1-k}\binom{2n}{2n+1-k} = \sum_{k=2n+1-n}^{2n+1-1}\binom{2n+1}{k}\binom{2n}{k} \\
&\qquad \left(\text{here, we have substituted } k \text{ for } 2n+1-k \text{ in the sum}\right) \\
&= \sum_{k=n+1}^{2n}\binom{2n+1}{k}\binom{2n}{k}
\end{aligned} \tag{6}$$

(since $2n+1-n=n+1$ and $2n+1-1=2n$). Hence, (3) becomes

$$\begin{aligned}
&\sum_{k=0}^{n}\binom{2n+1}{k}^2 \\
&= \underbrace{\sum_{k=0}^{n}\binom{2n+1}{k}\binom{2n}{k-1}}_{\substack{=\sum_{k=n+1}^{2n}\binom{2n+1}{k}\binom{2n}{k} \\ \text{(by (6))}}} + \sum_{k=0}^{n}\binom{2n+1}{k}\binom{2n}{k} \\
&= \sum_{k=n+1}^{2n}\binom{2n+1}{k}\binom{2n}{k} + \sum_{k=0}^{n}\binom{2n+1}{k}\binom{2n}{k} \\
&= \sum_{k=0}^{n}\binom{2n+1}{k}\binom{2n}{k} + \sum_{k=n+1}^{2n}\binom{2n+1}{k}\binom{2n}{k} \\
&= \sum_{k=0}^{2n}\binom{2n+1}{k}\binom{2n}{k} \\
&\qquad \left(\begin{array}{c}\text{since } \sum_{k=0}^{2n}\binom{2n+1}{k}\binom{2n}{k} = \sum_{k=0}^{n}\binom{2n+1}{k}\binom{2n}{k} + \sum_{k=n+1}^{2n}\binom{2n+1}{k}\binom{2n}{k} \\ \text{(here, we have split the sum } \sum_{k=0}^{2n}\binom{2n+1}{k}\binom{2n}{k} \text{ at } k=n)\end{array}\right) \\
&= \binom{(2n+1)+2n}{2n} \qquad \text{(by Corollary 1.5, applied to } x=2n+1 \text{ and } y=2n) \\
&= \binom{4n+1}{2n} \qquad \left(\text{since } (2n+1)+2n=4n+1\right).
\end{aligned}$$

Thus, the exercise is solved again.

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2 Exercise 2

2.1 Problem

Let $n \in \mathbb{N}$ and $x, y \in \mathbb{R}$. Prove that

$$\sum_{k=1}^{n}\frac{x^k y^{n-k}}{k}\binom{n}{k} = \sum_{i=1}^{n}\frac{\left((x+y)^i - y^i\right)y^{n-i}}{i}.$$

2.2 Remark

This is easily seen to be a generalization of [Math222, Exercise 1.6.4] (indeed, the latter exercise is obtained by setting $x=-1$ and $y=1$). Can you generalize the solution?

2.3 Solution

Forget that we fixed n, x and y. We recall a few facts. First of all, we recall one version of the Triangular Fubini's principle ([Math222, Corollary 1.6.9]):

Corollary 2.1 (Triangular Fubini's principle II)**.** *Let $n \in \mathbb{N}$. For each pair $(x, y) \in [n] \times [n]$ with $x \leq y$, let $a_{(x,y)}$ be a number. Then,*

$$\sum_{x=1}^{n} \sum_{y=x}^{n} a_{(x,y)} = \sum_{\substack{(x,y) \in [n] \times [n]; \\ x \leq y}} a_{(x,y)} = \sum_{y=1}^{n} \sum_{x=1}^{y} a_{(x,y)}.$$

Let us rewrite Corollary 2.1 by renaming the indices x and y as k and i throughout it (in order to adapt it to how we shall use it in the following solution):

Corollary 2.2 (Triangular Fubini's principle II)**.** *Let $n \in \mathbb{N}$. For each pair $(k, i) \in [n] \times [n]$ with $k \leq i$, let $a_{(k,i)}$ be a number. Then,*

$$\sum_{k=1}^{n} \sum_{i=k}^{n} a_{(k,i)} = \sum_{\substack{(k,i) \in [n] \times [n]; \\ k \leq i}} a_{(k,i)} = \sum_{i=1}^{n} \sum_{k=1}^{i} a_{(k,i)}.$$

Next, we recall the binomial formula ([Math222, Theorem 1.3.24]):

Theorem 2.3 (the binomial formula)**.** *Let $x, y \in \mathbb{R}$. Let $n \in \mathbb{N}$. Then,*

$$(x + y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}.$$

Next, we recall the hockey-stick identity in its first form ([Math222, Theorem 1.3.29]):

Theorem 2.4 ("Hockey-stick identity")**.** *Let $n \in \mathbb{N}$ and $k \in \mathbb{N}$. Then,*

$$\binom{0}{k} + \binom{1}{k} + \binom{2}{k} + \cdots + \binom{n}{k} = \binom{n+1}{k+1}.$$

Now, fix $n \in \mathbb{N}$ and $x, y \in \mathbb{R}$.

Let $k \in [n]$. Then, $1 \leq k \leq n$, so that $k \geq 1$ and thus $k - 1 \in \mathbb{N}$. Note that $k \geq 1$ also entails $k \in \{1, 2, 3, \ldots\}$. Furthermore, $n \geq 1$ (since $1 \leq n$) and thus $n - 1 \in \mathbb{N}$. Hence, Theorem 2.4 (applied to $n - 1$ and $k - 1$ instead of n and k) yields

$$\binom{k-1}{k-1} + \binom{(k-1)+1}{k-1} + \binom{(k-1)+2}{k-1} + \cdots + \binom{n-1}{k-1} = \binom{(n-1)+1}{(k-1)+1} = \binom{n}{k}$$

(since $(n - 1) + 1 = n$ and $(k - 1) + 1 = k$). Hence,

$$\begin{aligned} \binom{n}{k} &= \binom{k-1}{k-1} + \binom{(k-1)+1}{k-1} + \binom{(k-1)+2}{k-1} + \cdots + \binom{n-1}{k-1} \\ &= \binom{k-1}{k-1} + \binom{k}{k-1} + \binom{k+1}{k-1} + \cdots + \binom{n-1}{k-1} = \sum_{i=k}^{n} \binom{i-1}{k-1}. \end{aligned}$$

Multiplying both sides of this equality by $\dfrac{x^k y^{n-k}}{k}$, we obtain

$$\frac{x^k y^{n-k}}{k}\binom{n}{k} = \frac{x^k y^{n-k}}{k}\sum_{i=k}^{n}\binom{i-1}{k-1} = \sum_{i=k}^{n} \underbrace{\frac{x^k y^{n-k}}{k}}_{\substack{=\frac{x^k y^{n-k}}{i}\cdot\frac{i}{k} \\ \text{(since } i\neq 0 \text{ (because } i\geq k\geq 1>0))}} \binom{i-1}{k-1}$$

$$= \sum_{i=k}^{n} \frac{x^k y^{n-k}}{i} \cdot \underbrace{\frac{i}{k}\binom{i-1}{k-1}}_{\substack{=\binom{i}{k} \\ \text{(since Proposition 1.3} \\ \text{(applied to } i \text{ and } k \text{ instead of } m \text{ and } n) \\ \text{yields } \binom{i}{k}=\frac{i}{k}\binom{i-1}{k-1})}}$$

$$= \sum_{i=k}^{n} \frac{x^k y^{n-k}}{i}\binom{i}{k}. \tag{7}$$

Now forget that we fixed k. We thus have proved (7) for each $k \in [n]$. Hence,

$$\sum_{k=1}^{n} \underbrace{\frac{x^k y^{n-k}}{k}\binom{n}{k}}_{\substack{=\sum_{i=k}^{n}\frac{x^k y^{n-k}}{i}\binom{i}{k} \\ \text{(by (7))}}} = \sum_{k=1}^{n}\sum_{i=k}^{n}\frac{x^k y^{n-k}}{i}\binom{i}{k}$$

$$= \sum_{i=1}^{n}\sum_{k=1}^{i}\frac{x^k y^{n-k}}{i}\binom{i}{k}. \tag{8}$$

Here, the last equality sign has been obtained by applying Corollary 2.2 to $a_{(k,i)} = \dfrac{x^k y^{n-k}}{i}\dbinom{i}{k}$.

Now, fix $i \in [n]$. Thus, $1 \leq i \leq n$, so that $n - i \geq 0$ and thus $n - i \in \mathbb{N}$. Hence, y^{n-i} is well-defined. But Theorem 2.3 (applied to i instead of n) yields

$$(x+y)^i = \sum_{k=0}^{i}\binom{i}{k}x^k y^{i-k} = \underbrace{\binom{i}{0}}_{=1}\underbrace{x^0}_{=0}\underbrace{y^{i-0}}_{=y^i} + \sum_{k=1}^{i}\binom{i}{k}x^k y^{i-k}$$

(here, we have split off the addend for $k = 0$ from the sum)

$$= y^i + \sum_{k=1}^{i}\binom{i}{k}x^k y^{i-k}.$$

Subtracting y^i from both sides of this equality, we find

$$(x+y)^i - y^i = \sum_{k=1}^{i}\binom{i}{k}x^k y^{i-k}.$$

Multiplying both sides of this equality by y^{n-i}, we obtain

$$\left((x+y)^i - y^i\right) y^{n-i} = \left(\sum_{k=1}^{i} \binom{i}{k} x^k y^{i-k}\right) y^{n-i} = \sum_{k=1}^{i} \binom{i}{k} x^k \underbrace{y^{i-k} y^{n-i}}_{\substack{=y^{(i-k)+(n-i)}=y^{n-k} \\ \text{(since } (i-k)+(n-i)=n-k)}}$$
$$= \sum_{k=1}^{i} \binom{i}{k} x^k y^{n-k}. \qquad (9)$$

Now, forget that we fixed i. We thus have proven (9) for each $i \in [n]$. Thus, (8) becomes

$$\sum_{k=1}^{n} \frac{x^k y^{n-k}}{k} \binom{n}{k} = \sum_{i=1}^{n} \sum_{k=1}^{i} \underbrace{\frac{x^k y^{n-k}}{i} \binom{i}{k}}_{=\frac{1}{i}\binom{i}{k} x^k y^{n-k}} = \sum_{i=1}^{n} \underbrace{\sum_{k=1}^{i} \frac{1}{i} \binom{i}{k} x^k y^{n-k}}_{=\frac{1}{i}\sum_{k=1}^{i}\binom{i}{k} x^k y^{n-k}} = \sum_{i=1}^{n} \frac{1}{i} \underbrace{\sum_{k=1}^{i} \binom{i}{k} x^k y^{n-k}}_{\substack{=\left((x+y)^i - y^i\right) y^{n-i} \\ \text{(by (9))}}}$$
$$= \sum_{i=1}^{n} \underbrace{\frac{1}{i} \cdot \left((x+y)^i - y^i\right) y^{n-i}}_{=\frac{\left((x+y)^i - y^i\right) y^{n-i}}{i}} = \sum_{i=1}^{n} \frac{\left((x+y)^i - y^i\right) y^{n-i}}{i}.$$

This solves the exercise.

2.4 Remark

Applying the exercise to $x = 1$ and $y = 1$, we obtain

$$\sum_{k=1}^{n} \frac{1^k 1^{n-k}}{k} \binom{n}{k} = \sum_{i=1}^{n} \frac{\left((1+1)^i - 1^i\right) 1^{n-i}}{i}.$$

This rewrites as

$$\sum_{k=1}^{n} \frac{1}{k} \binom{n}{k} = \sum_{i=1}^{n} \frac{\left((1+1)^i - 1^i\right) 1^{n-i}}{i}$$

(because each $k \in [n]$ satisfies $\underbrace{1^k}_{=1} \underbrace{1^{n-k}}_{=1} = 1$). This, in turn, rewrites as

$$\sum_{k=1}^{n} \frac{1}{k} \binom{n}{k} = \sum_{i=1}^{n} \frac{2^i - 1}{i}$$

(since each $i \in [n]$ satisfies $\left((1+1)^i - 1^i\right) \underbrace{1^{n-i}}_{=1} = \Bigl(\underbrace{1+1}_{=2}\Bigr)^i - \underbrace{1^i}_{=1} = 2^i - 1$).

3 Exercise 3

3.1 Problem

Let $m \in \mathbb{N}$. Prove that

$$\sum_{i=0}^{m} (-1)^i \operatorname{sur}(m,i) = (-1)^m .$$

3.2 Solution

Forget that we fixed m. Let us recall two facts from [Math222]. The first of these facts is a simple property of binomial coefficients:

Lemma 3.1. *For any $k \in \mathbb{N}$, we have $\binom{-1}{k} = (-1)^k$.*

Lemma 3.1 is proven in [Math222, Example 1.3.4 **(f)**].

The second fact relates binomial coefficients to the numbers $\operatorname{sur}(m,n)$:

Theorem 3.2. *Let $k \in \mathbb{R}$ and $m \in \mathbb{N}$. Then,*

$$k^m = \sum_{i=0}^{m} \operatorname{sur}(m,i) \cdot \binom{k}{i}.$$

This theorem appears in [Math222, §2.6.4]. More precisely: The particular case of Theorem 3.2 when $k \in \mathbb{N}$ is precisely [Math222, Theorem 2.5.1]. In [Math222, §2.6.4], it is shown that the claim of Theorem 3.2 holds not only for $k \in \mathbb{N}$, but more generally for all $k \in \mathbb{R}$.

Now, fix $m \in \mathbb{N}$. Then, Theorem 3.2 (applied to $k = -1$) yields

$$(-1)^m = \sum_{i=0}^{m} \operatorname{sur}(m,i) \cdot \underbrace{\binom{-1}{i}}_{\substack{=(-1)^i \\ \text{(by Lemma 3.1,} \\ \text{applied to } i \\ \text{instead of } k)}} = \sum_{i=0}^{m} \operatorname{sur}(m,i) \cdot (-1)^i = \sum_{i=0}^{m} (-1)^i \operatorname{sur}(m,i) .$$

This solves the exercise.

3.3 Remark

An equivalent version of this exercise also appears in [Sagan19, Theorem 2.2.2]. (What we call $\operatorname{sur}(m,i)$ here corresponds to $i! \cdot S(m,i)$ in the notations of [Sagan19], since $S(m,i)$ stands for the Stirling number $\left\{ \begin{matrix} m \\ i \end{matrix} \right\} = \operatorname{sur}(m,i)/i!$.) The solution given in [Sagan19, Theorem 2.2.2] is combinatorial.

4 Exercise 4

4.1 Problem

Let $n \in \mathbb{N}$.

(a) Prove that

$$\left(\# \text{ of 5-tuples } (a,b,c,d,e) \in [n]^5 \text{ satisfying } a<b<c<d<e\right) = \binom{n}{5}.$$

(b) Find

$$\left(\# \text{ of 5-tuples } (a,b,c,d,e) \in [n]^5 \text{ satisfying } a \leq b < c \leq d < e\right).$$

4.2 Solution sketch

(a) Here is the idea: The 5-tuples $(a,b,c,d,e) \in [n]^5$ satisfying $a<b<c<d<e$ are in bijection with the 5-element subsets of $[n]$ (because any such 5-tuple can be seen as a way of listing the elements of a 5-element subset of $[n]$ in increasing order); thus, there are $\binom{n}{5}$ many of them.

Translated into a more rigorous language, this proof reads as follows:

Each 5-element subset of $[n]$ can be uniquely written in the form $\{a,b,c,d,e\}$ with $a,b,c,d,e \in [n]$ satisfying $a<b<c<d<e$. (This follows easily from [Math222, Proposition 1.4.11].) Thus, the map

$$\left\{(a,b,c,d,e) \in [n]^5 \mid a<b<c<d<e\right\} \to \{\text{5-element subsets of } [n]\},$$
$$(a,b,c,d,e) \mapsto \{a,b,c,d,e\}$$

is a bijection. Hence, the bijection principle yields

$$\left|\left\{(a,b,c,d,e) \in [n]^5 \mid a<b<c<d<e\right\}\right|$$
$$= |\{\text{5-element subsets of } [n]\}|.$$

In other words,

$$\left(\# \text{ of 5-tuples } (a,b,c,d,e) \in [n]^5 \text{ satisfying } a<b<c<d<e\right)$$
$$= (\# \text{ of 5-element subsets of } [n]) = \binom{n}{5}$$

(by the combinatorial interpretation of binomial coefficients, because $[n]$ is an n-element set). This solves part **(a)** of the exercise.

(b) Clearly, the 5-tuples $(a,b,c,d,e) \in [n]^5$ satisfying $a \leq b < c \leq d < e$ are precisely the 5-tuples $(a,b,c,d,e) \in \mathbb{Z}^5$ satisfying $1 \leq a \leq b < c \leq d < e \leq n$. Hence,

$$\left(\# \text{ of 5-tuples } (a,b,c,d,e) \in [n]^5 \text{ satisfying } a \leq b < c \leq d < e\right)$$
$$= \left(\# \text{ of 5-tuples } (a,b,c,d,e) \in \mathbb{Z}^5 \text{ satisfying } 1 \leq a \leq b < c \leq d < e \leq n\right).$$

But if a, b, c, d, e are five integers, then we have the equivalences

$$\begin{aligned} (a \leq b) &\iff (a < b + 1) ; \\ (b < c) &\iff (b + 1 < c + 1) ; \\ (c \leq d) &\iff (c + 1 < d + 2) ; \\ (d < e) &\iff (d + 2 < e + 2) ; \\ (e \leq n) &\iff (e + 2 \leq n + 2) . \end{aligned}$$

Hence, if a, b, c, d, e are five integers, then the chain of inequalities $1 \leq a \leq b < c \leq d < e \leq n$ is equivalent to the chain $1 \leq a < b + 1 < c + 1 < d + 2 < e + 2 \leq n + 2$. Thus,

$$\begin{aligned} & \left(\# \text{ of 5-tuples } (a, b, c, d, e) \in \mathbb{Z}^5 \text{ satisfying } 1 \leq a \leq b < c \leq d < e \leq n\right) \\ & = \left(\# \text{ of 5-tuples } (a, b, c, d, e) \in \mathbb{Z}^5 \text{ satisfying } 1 \leq a < b + 1 < c + 1 < d + 2 < e + 2 \leq n + 2\right) . \end{aligned}$$

But there is a bijection

$$\begin{aligned} & \left\{(a, b, c, d, e) \in \mathbb{Z}^5 \ \mid \ 1 \leq a < b + 1 < c + 1 < d + 2 < e + 2 \leq n + 2\right\} \\ & \to \left\{(a, b, c, d, e) \in \mathbb{Z}^5 \ \mid \ 1 \leq a < b < c < d < e \leq n + 2\right\} , \end{aligned}$$

which sends each 5-tuple (a, b, c, d, e) to $(a, b + 1, c + 1, d + 2, e + 2)$. Hence, the bijection principle yields

$$\begin{aligned} & \left(\# \text{ of 5-tuples } (a, b, c, d, e) \in \mathbb{Z}^5 \text{ satisfying } 1 \leq a < b + 1 < c + 1 < d + 2 < e + 2 \leq n + 2\right) \\ & = \left(\# \text{ of 5-tuples } (a, b, c, d, e) \in \mathbb{Z}^5 \text{ satisfying } 1 \leq a < b < c < d < e \leq n + 2\right) . \end{aligned}$$

Now, combining our above computations, we obtain

$$\begin{aligned} & \left(\# \text{ of 5-tuples } (a, b, c, d, e) \in [n]^5 \text{ satisfying } a \leq b < c \leq d < e\right) \\ & = \left(\# \text{ of 5-tuples } (a, b, c, d, e) \in \mathbb{Z}^5 \text{ satisfying } 1 \leq a \leq b < c \leq d < e \leq n\right) \\ & = \left(\# \text{ of 5-tuples } (a, b, c, d, e) \in \mathbb{Z}^5 \text{ satisfying } 1 \leq a < b + 1 < c + 1 < d + 2 < e + 2 \leq n + 2\right) \\ & = \left(\# \text{ of 5-tuples } (a, b, c, d, e) \in \mathbb{Z}^5 \text{ satisfying } 1 \leq a < b < c < d < e \leq n + 2\right) \\ & = \left(\# \text{ of 5-tuples } (a, b, c, d, e) \in [n+2]^5 \text{ satisfying } a < b < c < d < e\right) \\ & \qquad \left(\begin{array}{c} \text{since the 5-tuples } (a, b, c, d, e) \in \mathbb{Z}^5 \text{ satisfying } 1 \leq a < b < c < d < e \leq n + 2 \\ \text{are precisely the 5-tuples } (a, b, c, d, e) \in [n+2]^5 \text{ satisfying } a < b < c < d < e \end{array}\right) \\ & = \binom{n+2}{5} \qquad \left(\text{by part } \textbf{(a)} \text{ of this exercise, applied to } n + 2 \text{ instead of } n\right) . \end{aligned}$$

This solves part **(b)** of the exercise.

4.3 Remark

Part **(b)** can also be solved in a different way: A 5-tuple $(a, b, c, d, e) \in [n]^5$ satisfies the chain of inequalities $a \leq b < c \leq d < e$ if and only if it satisfies one of the four chains

$$\begin{aligned} & (a < b < c < d < e) , \qquad (a = b < c < d < e) , \\ & (a < b < c = d < e) , \qquad (a = b < c = d < e) . \end{aligned}$$

Moreover, these four chains are mutually exclusive. Hence, the sum rule yields

$$\begin{aligned}
&\left(\# \text{ of 5-tuples } (a,b,c,d,e) \in [n]^5 \text{ satisfying } a \leq b < c \leq d < e\right) \\
&= \left(\# \text{ of 5-tuples } (a,b,c,d,e) \in [n]^5 \text{ satisfying } a < b < c < d < e\right) \\
&\qquad + \left(\# \text{ of 5-tuples } (a,b,c,d,e) \in [n]^5 \text{ satisfying } a = b < c < d < e\right) \\
&\qquad + \left(\# \text{ of 5-tuples } (a,b,c,d,e) \in [n]^5 \text{ satisfying } a < b < c = d < e\right) \\
&\qquad + \left(\# \text{ of 5-tuples } (a,b,c,d,e) \in [n]^5 \text{ satisfying } a = b < c = d < e\right).
\end{aligned}$$

Now, it remains to compute the four addends on the right hand side of this equality. The first one is easy: By part **(a)** of this exercise, we know that

$$\left(\# \text{ of 5-tuples } (a,b,c,d,e) \in [n]^5 \text{ satisfying } a < b < c < d < e\right) = \binom{n}{5}.$$

As for the second addend, it helps to notice the following: There is a bijection

$$\begin{gathered}
\left\{(a,b,c,d,e) \in [n]^5 \mid a = b < c < d < e\right\} \to \left\{(a,b,c,d) \in [n]^4 \mid a < b < c < d\right\}, \\
(a,b,c,d,e) \mapsto (a,c,d,e)
\end{gathered}$$

(whose inverse map sends each (a,b,c,d) to (a,a,b,c,d)). Thus, the bijection principle yields

$$\begin{aligned}
&\left(\# \text{ of 5-tuples } (a,b,c,d,e) \in [n]^5 \text{ satisfying } a = b < c < d < e\right) \\
&= \left(\# \text{ of 4-tuples } (a,b,c,d) \in [n]^4 \text{ satisfying } a < b < c < d\right) = \binom{n}{4}
\end{aligned}$$

(by the analogue of part **(a)** of this exercise for 4-tuples instead of 5-tuples). Similarly,

$$\left(\# \text{ of 5-tuples } (a,b,c,d,e) \in [n]^5 \text{ satisfying } a < b < c = d < e\right) = \binom{n}{4}$$

and

$$\left(\# \text{ of 5-tuples } (a,b,c,d,e) \in [n]^5 \text{ satisfying } a = b < c = d < e\right) = \binom{n}{3}.$$

Thus, altogether, our above computation becomes

$$\begin{aligned}
&\left(\# \text{ of 5-tuples } (a,b,c,d,e) \in [n]^5 \text{ satisfying } a \leq b < c \leq d < e\right) \\
&= \underbrace{\left(\# \text{ of 5-tuples } (a,b,c,d,e) \in [n]^5 \text{ satisfying } a < b < c < d < e\right)}_{=\binom{n}{5}} \\
&\quad + \underbrace{\left(\# \text{ of 5-tuples } (a,b,c,d,e) \in [n]^5 \text{ satisfying } a = b < c < d < e\right)}_{=\binom{n}{4}} \\
&\quad + \underbrace{\left(\# \text{ of 5-tuples } (a,b,c,d,e) \in [n]^5 \text{ satisfying } a < b < c = d < e\right)}_{=\binom{n}{4}} \\
&\quad + \underbrace{\left(\# \text{ of 5-tuples } (a,b,c,d,e) \in [n]^5 \text{ satisfying } a = b < c = d < e\right)}_{=\binom{n}{3}} \\
&= \underbrace{\binom{n}{5} + \binom{n}{4}}_{\substack{=\binom{n+1}{5} \\ \text{(by the recurrence} \\ \text{of the binomial coefficients)}}} + \underbrace{\binom{n}{4} + \binom{n}{3}}_{\substack{=\binom{n+1}{4} \\ \text{(by the recurrence} \\ \text{of the binomial coefficients)}}} \\
&= \binom{n+1}{5} + \binom{n+1}{4} = \binom{n+2}{5} \qquad \text{(by the recurrence of the binomial coefficients)}.
\end{aligned}$$

This, again, solves part **(b)** of the exercise.

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5 Exercise 5

5.1 Problem

A finite set S of integers is said to be *self-centered* if its size $|S|$ is odd and equals its $(|S|+1)/2$-th smallest element (i.e., its median in the statistical sense).

For example, the sets $\{1,3,5\}$ and $\{2,3,5,6,10\}$ are self-centered, while $\{2,4,6\}$ and $\{2\}$ are not.

(a) Given $n \in \mathbb{N}$ and an odd $k \in \mathbb{N}$, find the # of self-centered k-element subsets of $[n]$. (The result will be a simple explicit formula in terms of binomial coefficients.)

(b) For each $n \in \mathbb{N}$, let a_n be the # of all self-centered subsets of $[n]$. Find the sequence $(a_0, a_1, a_2, \ldots)$ or the sequence $(a_1, a_2, a_3, \ldots)$ in the OEIS. (No explicit sum-less formula is known.)

5.2 Solution sketch

(a) Let $n \in \mathbb{N}$. Let $k \in \mathbb{N}$ be odd. Thus, we can write k in the form $k = 2u + 1$ for some $u \in \mathbb{N}$. Consider this u.

Now, I claim:

> *Claim 1:* Assume that $k \in [n]$. Then, the # of self-centered k-element subsets of $[n]$ is $\dbinom{k-1}{u}\dbinom{n-k}{u}$.

[*Proof of Claim 1:* We will only give an informal proof, since the idea of this argument has already been flogged to death (cf. [Math222, solution to Exercise 1.4.3], [Math222, §2.3, Fourth proof of Theorem 1.3.29] and [Math222, §2.6.5, Second proof of Proposition 2.6.13]).

Let S be a self-centered k-element subset of $[n]$. Then, its size $|S|$ is odd and equals its $(|S|+1)/2$-th smallest element (by the definition of "self-centered"). Since $|S| = k$ (because S is a k-element set), we can rewrite this as follows: The integer k is odd and equals the $(k+1)/2$-th smallest element of S. In other words, the integer k is odd and equals the $(u+1)$-th smallest element of S (since $\left(\underbrace{k}_{=2u+1}+1\right)/2 = ((2u+1)+1)/2 = u+1$). In total, the set S has $2u+1$ elements (since $|S| = k = 2u+1$), and thus can be split into the u smallest elements, the u largest elements and the $(u+1)$-th smallest element. As we know, the latter is k; thus,

- the u smallest elements of S are smaller than k, and thus belong to $\{1, 2, \ldots, k-1\}$;
- the u largest elements of S are larger than k, and thus belong to $\{k+1, k+2, \ldots, n\}$.

Thus, S has the form

$$\begin{aligned} S = \{k\} &\cup (\text{some } u\text{-element subset of } \{1, 2, \ldots, k-1\}) \\ &\cup (\text{some } u\text{-element subset of } \{k+1, k+2, \ldots, n\}). \end{aligned} \tag{10}$$

Forget that we fixed S. We thus have proved that every self-centered k-element subset of $[n]$ can be represented in the form (10). It is moreover clear that this representation is unique (i.e., the two u-element subsets on the right hand side of (10) are uniquely determined by S), and that conversely, every set S of the form (10) is a self-centered k-element subset of $[n]$. Thus, in order to choose a self-centered k-element subset of $[n]$, we only need to choose the following two things (independently):

- some u-element subset of $\{1, 2, \ldots, k-1\}$;
- some u-element subset of $\{k+1, k+2, \ldots, n\}$.

The first of these two things can be chosen in $\dbinom{k-1}{u}$ many ways[1], whereas the second can be chosen in $\dbinom{n-k}{u}$ many ways[2]. Hence, in total, the # of self-centered k-element subsets of $[n]$ is $\dbinom{k-1}{u}\dbinom{n-k}{u}$. This proves Claim 1.]

[1] by the combinatorial interpretation of the binomial coefficients, because $\{1, 2, \ldots, k-1\}$ is a $(k-1)$-element set

[2] by the combinatorial interpretation of the binomial coefficients, because $\{k+1, k+2, \ldots, n\}$ is an $(n-k)$-element set

Can you spot the place where this proof would go wrong if we did not assume that $k \in [n]$? It is well-hidden, but it exists (since Claim 1 would be false for $k > n$).

We have $k - 1 = 2u$ (since $k = 2u + 1$) and $n - \underbrace{k}_{=2u+1} = n - (2u+1) = n - 2u - 1$. Hence, we can restate Claim 1 as follows:

> *Claim 2:* Assume that $k \in [n]$. Then, the # of self-centered k-element subsets of $[n]$ is $\dbinom{2u}{u}\dbinom{n-2u-1}{u}$.

On the other hand, we have $u = (k-1)/2$ (since $k - 1 = 2u$). Hence, we can restate Claim 1 as follows:

> *Claim 3:* Assume that $k \in [n]$. Then, the # of self-centered k-element subsets of $[n]$ is $\dbinom{k-1}{(k-1)/2}\dbinom{n-k}{(k-1)/2}$.

In order to get a complete picture, we need to see what happens if $k \notin [n]$. However, this case is very simple: The size of any self-centered subset S of $[n]$ is an element of S (by the definition of "self-centered") and thus an element of $[n]$ (since S is a subset of $[n]$); thus, a self-centered k-element subset cannot exist unless $k \in [n]$. In other words, if $k \notin [n]$, then the # of self-centered k-element subsets of $[n]$ is 0. Combining this with Claim 1, we obtain the following:

> *Claim 4:* The # of self-centered k-element subsets of $[n]$ is
> $$\begin{cases} \dbinom{k-1}{u}\dbinom{n-k}{u}, & \text{if } k \in [n]; \\ 0, & \text{if } k \notin [n] \end{cases}.$$

(b) The sequence $(a_1, a_2, a_3, \ldots)$ is OEIS sequence A217615.

Proof. Let $n \in \mathbb{N}$. The size of any self-centered subset S of $[n]$ is an element of S (by the definition of "self-centered") and thus an element of $[n]$ (since S is a subset of $[n]$); furthermore, it must be odd (since self-centered sets always have odd size[3]). Hence, the size of any self-centered subset S of $[n]$ is an odd element of $[n]$. Thus, the sum rule yields

$$\begin{aligned} &(\#\text{ of self-centered subsets of } [n]) \\ &= \sum_{\substack{k\in[n];\\ k \text{ is odd}}} \underbrace{(\#\text{ of self-centered subsets of } [n] \text{ having size } k)}_{\substack{=(\#\text{ of self-centered } k\text{-element subsets of } [n]) \\ =\binom{k-1}{(k-1)/2}\binom{n-k}{(k-1)/2} \\ \text{(by Claim 3)}}} \\ &= \sum_{\substack{k\in[n];\\ k \text{ is odd}}} \binom{k-1}{(k-1)/2}\binom{n-k}{(k-1)/2} = \sum_{u=0}^{\lfloor (n-1)/2 \rfloor} \binom{2u}{u}\binom{n-2u-1}{u} \end{aligned}$$

(here, we have substituted $2u + 1$ for k in the sum). Now, the definition of a_n yields

$$a_n = (\#\text{ of self-centered subsets of } [n]) = \sum_{u=0}^{\lfloor (n-1)/2 \rfloor} \binom{2u}{u}\binom{n-2u-1}{u}.$$

[3] by the definition of "self-centered"

This makes it easy to compute $a_0, a_1, a_2, a_3, \ldots$. To wit, we obtain

$$a_0 = 0, \quad a_1 = 1, \quad a_2 = 1, \quad a_3 = 1, \quad a_4 = 3, \quad a_5 = 5,$$
$$a_6 = 7, \quad a_7 = 15, \quad a_8 = 29, \quad a_9 = 49, \quad a_{10} = 95.$$

Entering these values into OEIS, we find nothing. But if we suppress a_0, then we obtain the first entries of OEIS sequence A217615, and one of the comments (“`a(n) is the number of (2k-1)-element subsets of {1, 2, ..., n+1} whose k-th smallest (i.e., k-th largest) element equals 2k-1. - Darij Grinberg, Oct 09 2019`”) convinces us that it is really our sequence $(a_1, a_2, a_3, \ldots)$ (because it describes precisely the # of self-centered subsets of $[n+1]$).

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6 Exercise 6

6.1 Problem

Let $p \in \mathbb{N}$ and $q \in \mathbb{N}$. Prove that

$$\sum_{i=0}^{p} (-1)^i \binom{p}{i} \binom{x-i}{q} = \binom{x-p}{q-p} \qquad \text{for all } x \in \mathbb{R}. \tag{11}$$

6.2 First solution

We shall prove (11) by induction on p:

Induction base: It is straightforward to see that (11) holds for $p = 0$ [4]. This completes the induction base.

Induction step: Let m be a positive integer. Assume that (11) holds for $p = m - 1$. We must prove that (11) holds for $p = m$.

We have assumed that (11) holds for $p = m - 1$. In other words, we have

$$\sum_{i=0}^{m-1} (-1)^i \binom{m-1}{i} \binom{x-i}{q} = \binom{x-(m-1)}{q-(m-1)} \qquad \text{for all } x \in \mathbb{R}. \tag{12}$$

Now, let $x \in \mathbb{R}$. For each $i \in \mathbb{R}$, we have

$$\binom{m}{i} = \binom{m-1}{i-1} + \binom{m-1}{i} \tag{13}$$

[4] *Proof.* For each $x \in \mathbb{R}$, we have

$$\sum_{i=0}^{0} (-1)^i \binom{0}{i} \binom{x-i}{q} = \underbrace{(-1)^0}_{=1} \underbrace{\binom{0}{0}}_{=1} \binom{x-0}{q} = \binom{x-0}{q} = \binom{x-0}{q-0}$$

(since $q = q - 0$). In other words, (11) holds for $p = 0$.

(by Theorem 1.4 (applied to m and i instead of n and k)). Hence,

$$\begin{aligned}
\sum_{i=0}^{m} (-1)^i \underbrace{\binom{m}{i}}_{\substack{=\binom{m-1}{i-1}+\binom{m-1}{i} \\ \text{(by (13))}}} \binom{x-i}{q}
&= \sum_{i=0}^{m} \underbrace{(-1)^i \left(\binom{m-1}{i-1} + \binom{m-1}{i}\right) \binom{x-i}{q}}_{=(-1)^i \binom{m-1}{i-1}\binom{x-i}{q} + (-1)^i \binom{m-1}{i}\binom{x-i}{q}} \\
&= \sum_{i=0}^{m} \left((-1)^i \binom{m-1}{i-1}\binom{x-i}{q} + (-1)^i \binom{m-1}{i}\binom{x-i}{q} \right) \\
&= \sum_{i=0}^{m} (-1)^i \binom{m-1}{i-1}\binom{x-i}{q} + \sum_{i=0}^{m} (-1)^i \binom{m-1}{i}\binom{x-i}{q}. \qquad (14)
\end{aligned}$$

We shall now take a closer look at the two sums on the right hand side of this equality.

The definition of binomial coefficients yields $\binom{m-1}{0-1} = 0$ (since $0-1 = -1 \notin \mathbb{N}$). Also, $m-1 \in \mathbb{N}$ (since m is a positive integer). Thus, Theorem 1.2 (applied to $n = m-1$ and $k = m$) yields

$$\begin{aligned}
\binom{m-1}{m} &= \binom{m-1}{(m-1)-m} = \binom{m-1}{0-1} \qquad \left(\text{since } (m-1)-m = 0-1\right) \\
&= 0.
\end{aligned}$$

Now,

$$\begin{aligned}
&\sum_{i=0}^{m}(-1)^i\binom{m-1}{i-1}\binom{x-i}{q}\\
&=(-1)^0\underbrace{\binom{m-1}{0-1}}_{=0}\binom{x-0}{q}+\sum_{i=1}^{m}(-1)^i\binom{m-1}{i-1}\binom{x-i}{q}\\
&\qquad\qquad\text{(here, we have split off the addend for } i=0 \text{ from the sum)}\\
&=\sum_{i=1}^{m}(-1)^i\binom{m-1}{i-1}\binom{x-i}{q}=\sum_{i=0}^{m-1}\underbrace{(-1)^{i+1}}_{=-(-1)^i}\underbrace{\binom{m-1}{(i+1)-1}}_{\substack{=\binom{m-1}{i}\\ \text{(since } (i+1)-1=i)}}\underbrace{\binom{x-(i+1)}{q}}_{\substack{=\binom{(x-1)-i}{q}\\ \text{(since } x-(i+1)=(x-1)-i)}}\\
&\qquad\qquad\text{(here, we have substituted } i+1 \text{ for } i \text{ in the sum)}\\
&=\sum_{i=0}^{m-1}\left(-(-1)^i\right)\binom{m-1}{i}\binom{(x-1)-i}{q}=-\underbrace{\sum_{i=0}^{m-1}(-1)^i\binom{m-1}{i}\binom{(x-1)-i}{q}}_{\substack{=\binom{(x-1)-(m-1)}{q-(m-1)}\\ \text{(by (12), applied to } x-1 \text{ instead of } x)}}\\
&=-\binom{(x-1)-(m-1)}{q-(m-1)}=-\binom{x-m}{q-(m-1)}
\end{aligned}\tag{15}$$

(since $(x-1)-(m-1)=x-m$). Also,

$$\begin{aligned}
&\sum_{i=0}^{m}(-1)^i\binom{m-1}{i}\binom{x-i}{q}\\
&=(-1)^m\underbrace{\binom{m-1}{m}}_{=0}\binom{x-m}{q}+\sum_{i=0}^{m-1}(-1)^i\binom{m-1}{i}\binom{x-i}{q}\\
&=\sum_{i=0}^{m-1}(-1)^i\binom{m-1}{i}\binom{x-i}{q}=\binom{x-(m-1)}{q-(m-1)}\qquad\text{(by (12))}\\
&=\binom{x-(m-1)-1}{q-(m-1)-1}+\binom{x-(m-1)-1}{q-(m-1)}\\
&\qquad\qquad\text{(by Theorem 1.4, applied to } n=x-(m-1) \text{ and } k=q-(m-1))\\
&=\binom{x-m}{q-m}+\binom{x-m}{q-(m-1)}
\end{aligned}\tag{16}$$

(since $x-(m-1)-1=x-m$ and $q-(m-1)-1=q-m$). Hence, (14) becomes

$$\begin{aligned}
&\sum_{i=0}^{m}(-1)^i\binom{m}{i}\binom{x-i}{q}\\
&=\underbrace{\sum_{i=0}^{m}(-1)^i\binom{m-1}{i-1}\binom{x-i}{q}}_{\substack{=-\binom{x-m}{q-(m-1)}\\ \text{(by (15))}}}+\underbrace{\sum_{i=0}^{m}(-1)^i\binom{m-1}{i}\binom{x-i}{q}}_{\substack{=\binom{x-m}{q-m}+\binom{x-m}{q-(m-1)}\\ \text{(by (16))}}}\\
&=-\binom{x-m}{q-(m-1)}+\binom{x-m}{q-m}+\binom{x-m}{q-(m-1)}=\binom{x-m}{q-m}.
\end{aligned}$$

Now, forget that we fixed x. We thus have proved that

$$\sum_{i=0}^{m}(-1)^i\binom{m}{i}\binom{x-i}{q}=\binom{x-m}{q-m}\qquad\text{for all } x\in\mathbb{R}.$$

In other words, (11) holds for $p=m$. This completes the induction step. Thus, (11) is proved by induction.

This solves the exercise.

6.3 Second solution

The following solution was posted by user "robjohn" in his answer at `https://math.stackexchange.com/questions/2424156/` .

We shall use the Vandermonde convolution formula ([Math222, Theorem 2.6.1]):

Theorem 6.1 (The Vandermonde convolution, or the Chu–Vandermonde identity)**.** *Let $n\in\mathbb{N}$ and $x,y\in\mathbb{R}$. Then,*

$$\binom{x+y}{n}=\sum_{k=0}^{n}\binom{x}{k}\binom{y}{n-k}\tag{17}$$

$$=\sum_{k}\binom{x}{k}\binom{y}{n-k}.\tag{18}$$

Here, the summation sign "$\sum_k$" on the right hand side of (18) means a sum over all $k\in\mathbb{Z}$. (We are thus implicitly claiming that this sum over all $k\in\mathbb{Z}$ is well-defined, i.e., that it has only finitely many nonzero addends.)

We will furthermore use the trinomial revision formula ([Math222, Proposition 1.3.35]):

Proposition 6.2 (Trinomial revision formula)**.** *Let $n,a,b\in\mathbb{R}$. Then,*

$$\binom{n}{a}\binom{a}{b}=\binom{n}{b}\binom{n-b}{a-b}.$$

One last tool we will need is the following fact ([Math222, Proposition 1.3.28]):

Proposition 6.3. *Let $n\in\mathbb{N}$. Then,*

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}=[n=0].\tag{19}$$

(Here, we are using the Iverson bracket notation.)

Now, let us return to the exercise. Let $x \in \mathbb{R}$. Then,

$$\begin{aligned}
&\sum_{i=0}^{p} (-1)^i \underbrace{\binom{p}{i}}_{\substack{=\binom{p}{p-i} \\ \text{(by Theorem 1.2,} \\ \text{applied to } n=p \text{ and } k=i)}} \underbrace{\binom{x-i}{q}}_{\substack{=\binom{(p-i)+(x-p)}{q} \\ \text{(since } x-i=(p-i)+(x-p))}} \\
&= \sum_{i=0}^{p} (-1)^i \binom{p}{p-i} \underbrace{\binom{(p-i)+(x-p)}{q}}_{\substack{=\sum_{k=0}^{q} \binom{p-i}{k}\binom{x-p}{q-k} \\ \text{(by (17), applied to } p-i,\ x-p \text{ and } q \\ \text{instead of } x,\ y \text{ and } n)}} \\
&= \sum_{i=0}^{p} (-1)^i \binom{p}{p-i} \sum_{k=0}^{q} \binom{p-i}{k}\binom{x-p}{q-k} \\
&= \underbrace{\sum_{i=0}^{p}\sum_{k=0}^{q}}_{=\sum_{k=0}^{q}\sum_{i=0}^{p}} (-1)^i \underbrace{\binom{p}{p-i}\binom{p-i}{k}}_{\substack{=\binom{p}{k}\binom{p-k}{(p-i)-k} \\ \text{(by Proposition 6.2,} \\ \text{applied to } n=p,\ a=p-i \text{ and } b=k)}} \binom{x-p}{q-k} \\
&= \sum_{k=0}^{q}\sum_{i=0}^{p} (-1)^i \binom{p}{k}\binom{p-k}{(p-i)-k}\binom{x-p}{q-k} \\
&= \sum_{k=0}^{q} \binom{p}{k}\binom{x-p}{q-k} \sum_{i=0}^{p} (-1)^i \binom{p-k}{(p-i)-k}. \qquad (20)
\end{aligned}$$

Now, we claim that

$$\sum_{i=0}^{p} (-1)^i \binom{p-k}{(p-i)-k} = [p=k] \qquad (21)$$

for each $k \in \mathbb{N}$.

[*Proof of* (21)*:* Let $k \in \mathbb{N}$. We must prove the equality (21). We are in one of the following two cases:

Case 1: We have $p < k$.

Case 2: We have $p \geq k$.

Let us first consider Case 1. In this case, we have $p < k$. Hence, $p - k < 0$. Thus, each $i \in \{0, 1, \ldots, p\}$ satisfies $(p-i)-k \notin \mathbb{N}$ (since $(p-i)-k = p-k-\underbrace{i}_{\geq 0} \leq p-k < 0$) and therefore

$$\binom{p-k}{(p-i)-k} = 0 \qquad (22)$$

(by the definition of binomial coefficients). Hence,

$$\sum_{i=0}^{p} (-1)^i \underbrace{\binom{p-k}{(p-i)-k}}_{\substack{=0 \\ \text{(by (22))}}} = \sum_{i=0}^{p} (-1)^i 0 = 0.$$

Comparing this with

$$[p = k] = 0 \qquad (\text{since } p \neq k \text{ (because } p < k)),$$

we obtain

$$\sum_{i=0}^{p} (-1)^i \binom{p-k}{(p-i)-k} = [p = k].$$

Hence, (21) is proved in Case 1.

Let us next consider Case 2. In this case, we have $p \geq k$. Hence, $p - k \geq 0$, so that $p - k \in \mathbb{N}$. Also, $p - k \leq p$ (since $k \in \mathbb{N}$). Now,

$$\begin{aligned}
&\sum_{i=0}^{p} (-1)^i \underbrace{\binom{p-k}{(p-i)-k}}_{\substack{=\binom{p-k}{p-k-i} \\ (\text{since } (p-i)-k=p-k-i)}} \\
&= \sum_{i=0}^{p} (-1)^i \binom{p-k}{p-k-i} \\
&= \sum_{i=0}^{p-k} (-1)^i \underbrace{\binom{p-k}{p-k-i}}_{\substack{=\binom{p-k}{(p-k)-(p-k-i)} \\ (\text{by Theorem 1.2,} \\ \text{applied to } p-k \text{ and } p-k-i \text{ instead of } n \text{ and } k)}} + \sum_{i=p-k+1}^{p} (-1)^i \underbrace{\binom{p-k}{p-k-i}}_{\substack{=0 \\ (\text{by the definition of binomial} \\ \text{coefficients, since } p-k-i\notin\mathbb{N} \\ (\text{because } i\geq p-k+1>p-k \\ \text{and thus } p-k-i<0))}} \\
&\qquad (\text{here, we have split the sum at } i = p-k, \text{ since } 0 \leq p-k \leq p) \\
&= \sum_{i=0}^{p-k} (-1)^i \underbrace{\binom{p-k}{(p-k)-(p-k-i)}}_{\substack{=\binom{p-k}{i} \\ (\text{since } (p-k)-(p-k-i)=i)}} + \underbrace{\sum_{i=p-k+1}^{p} (-1)^i 0}_{=0} = \sum_{i=0}^{p-k} (-1)^i \binom{p-k}{i} \\
&= [p-k = 0] \qquad (\text{by Proposition 6.3, applied to } n = p-k) \\
&= [p = k] \qquad (\text{since the statement "} p-k=0 \text{" is equivalent to "} p=k \text{"}).
\end{aligned}$$

Thus, (21) is proven in Case 2.

We have now proved (21) in each of the two Cases 1 and 2. Thus, (21) always holds.]

Now, (20) becomes

$$\begin{aligned}
\sum_{i=0}^{p} (-1)^i \binom{p}{i}\binom{x-i}{q} &= \underbrace{\sum_{k=0}^{q}}_{=\sum_{k\in\{0,1,\ldots,q\}}} \binom{p}{k}\binom{x-p}{q-k} \underbrace{\sum_{i=0}^{p} (-1)^i \binom{p-k}{(p-i)-k}}_{\substack{=[p=k] \\ (\text{by (21)})}} \\
&= \sum_{k\in\{0,1,\ldots,q\}} \binom{p}{k}\binom{x-p}{q-k} [p = k]. \qquad (23)
\end{aligned}$$

Now, we shall distinguish between two cases:

Case 1: We have $p \leq q$.

Case 2: We have $p > q$.

Let us first consider Case 1. In this case, we have $p \leq q$. Hence, $p \in \{0, 1, \ldots, q\}$ (since $p \in \mathbb{N}$). Also, Theorem 1.2 (applied to $n = p$ and $k = p$) yields $\dbinom{p}{p} = \dbinom{p}{p-p} = \dbinom{p}{0} = 1$. Now, (23) becomes

$$\begin{aligned}
&\sum_{i=0}^{p} (-1)^i \binom{p}{i} \binom{x-i}{q} \\
&= \sum_{k \in \{0,1,\ldots,q\}} \binom{p}{k} \binom{x-p}{q-k} [p = k] \\
&= \underbrace{\binom{p}{p}}_{=1} \binom{x-p}{q-p} \underbrace{[p = p]}_{\substack{=1 \\ (\text{since } p=p)}} + \sum_{\substack{k \in \{0,1,\ldots,q\}; \\ k \neq p}} \binom{p}{k} \binom{x-p}{q-k} \underbrace{[p = k]}_{\substack{=0 \\ (\text{since } p \neq k \\ (\text{because } k \neq p))}} \\
&\qquad \left(\begin{array}{c} \text{here, we have split off the addend for } k = p \text{ from the sum,} \\ \text{since } p \in \{0, 1, \ldots, q\} \end{array} \right) \\
&= \binom{x-p}{q-p} + \underbrace{\sum_{\substack{k \in \{0,1,\ldots,q\}; \\ k \neq p}} \binom{p}{k} \binom{x-p}{q-k} 0}_{=0} = \binom{x-p}{q-p}.
\end{aligned}$$

Hence, the exercise is solved in Case 1.

Let us now consider Case 2. In this case, we have $p > q$. Hence, $q < p$. Therefore, $q-p < 0$, so that $q-p \notin \mathbb{N}$ and thus $\dbinom{x-p}{q-p} = 0$ (by the definition of binomial coefficients). On the other hand, each $k \in \{0, 1, \ldots, q\}$ satisfies $k \leq q < p$ and thus $k \neq p$ and thus $p \neq k$ and therefore

$$[p = k] = 0. \tag{24}$$

Now, (23) becomes

$$\begin{aligned}
\sum_{i=0}^{p} (-1)^i \binom{p}{i} \binom{x-i}{q} &= \sum_{k \in \{0,1,\ldots,q\}} \binom{p}{k} \binom{x-p}{q-k} \underbrace{[p = k]}_{\substack{=0 \\ (\text{by } (24))}} \\
&= \sum_{k \in \{0,1,\ldots,q\}} \binom{p}{k} \binom{x-p}{q-k} 0 = 0 = \binom{x-p}{q-p}
\end{aligned}$$

(since $\dbinom{x-p}{q-p} = 0$). Hence, the exercise is solved in Case 2.

We have now solved the exercise in both Cases 1 and 2. Thus, the exercise is solved.

6.4 Third solution (sketched)

The following solution identifies the exercise as a "mutated version" of the Chu–Vandermonde identity (except that we also need the polynomial identity trick, because without it we only obtain a particular case of the exercise).

We shall use the upper negation formula ([Math222, Proposition 1.3.7]):

Proposition 6.4 (Upper negation formula)**.** *Let $n \in \mathbb{R}$ and $k \in \mathbb{Z}$. Then,*

$$\binom{-n}{k} = (-1)^k \binom{n+k-1}{k}.$$

We will further use the following variant of the Chu–Vandermonde identity:

Corollary 6.5. *Let $n \in \mathbb{R}$, $x \in \mathbb{N}$ and $y \in \mathbb{R}$. Then,*

$$\sum_{i=0}^{x} \binom{x}{i} \binom{y}{n-i} = \binom{x+y}{n}. \qquad (25)$$

Proof of Corollary 6.5 (sketched). We must prove (25). If $n \notin \mathbb{N}$, then each $i \in \{0, 1, \ldots, x\}$ satisfies $n - i \notin \mathbb{N}$ and thus $\binom{y}{n-i} = 0$ (by the definition of binomial coefficients). Hence, if $n \notin \mathbb{N}$, then the left hand side of (25) equals 0. So does the right hand side (if $n \notin \mathbb{N}$). Hence, if $n \notin \mathbb{N}$, then (25) is proven. Thus, for the rest of this proof, we WLOG assume that $n \in \mathbb{N}$. Hence, (18) yields

$$\begin{aligned}
\binom{x+y}{n} &= \sum_{k} \binom{x}{k} \binom{y}{n-k} = \sum_{\substack{k \in \mathbb{Z};\\ k \leq x}} \binom{x}{k} \binom{y}{n-k} + \sum_{\substack{k \in \mathbb{Z};\\ k > x}} \underbrace{\binom{x}{k}}_{\substack{=0 \\ (\text{since } x \in \mathbb{N} \\ \text{and } k > x)}} \binom{y}{n-k} \\
&= \sum_{\substack{k \in \mathbb{Z};\\ k \leq x}} \binom{x}{k} \binom{y}{n-k} = \underbrace{\sum_{\substack{k \in \mathbb{Z};\\ k \leq x;\\ k \geq 0}}}_{= \sum_{k=0}^{x}} \binom{x}{k} \binom{y}{n-k} + \sum_{\substack{k \in \mathbb{Z};\\ k \leq x;\\ k < 0}} \underbrace{\binom{x}{k}}_{\substack{=0 \\ (\text{since } k < 0)}} \binom{y}{n-k} \\
&= \sum_{k=0}^{x} \binom{x}{k} \binom{y}{n-k} = \sum_{i=0}^{x} \binom{x}{i} \binom{y}{n-i}
\end{aligned}$$

(here, we have renamed the summation index k as i). This proves (25). Thus, Corollary 6.5 is proved. □

Finally, we shall use the polynomial identity trick in the following form ([Math222, Corollary 2.6.9]):

Corollary 6.6. *If a polynomial P has infinitely many roots, then P is the zero polynomial.*

Now, let us first prove the particular case of the exercise in which $x \in \{p, p+1, p+2, \ldots\}$ (that is, x is an integer $\geq p$):

Claim 1: Let $x \in \{p, p+1, p+2, \ldots\}$. Then,

$$\sum_{i=0}^{p} (-1)^i \binom{p}{i} \binom{x-i}{q} = \binom{x-p}{q-p}.$$

[*Proof of Claim 1:* We have $x \geq p$ (since $x \in \{p, p+1, p+2, \ldots\}$) and $x \in \mathbb{Z}$ (for the same reason). For each $i \in \{0, 1, \ldots, p\}$, we have $i \leq p$ and thus $p \geq i$ and therefore

$x-i\in\mathbb{N}$ (since $x\geq p\geq i$ and $x\in\mathbb{Z}$). Hence, for each $i\in\{0,1,\ldots,p\}$, we have

$$\begin{aligned}\dbinom{x-i}{q}&=\dbinom{x-i}{x-i-q}\qquad\text{(by Theorem 1.2, applied to } n=x-i \text{ and } k=q\text{)}\\&=\dbinom{-(i-x)}{x-i-q}\qquad\text{(since } x-i=-(i-x)\text{)}\\&=(-1)^{x-i-q}\dbinom{i-x+x-i-q-1}{x-i-q}\\&\qquad\text{(by Proposition 6.4, applied to } n=i-x \text{ and } k=x-i-q\text{)}\\&=(-1)^{x-i-q}\dbinom{-q-1}{x-i-q}\qquad\text{(since } i-x+x-i-q-1=-q-1\text{)}.\end{aligned}$$

Thus,

$$\begin{aligned}&\sum_{i=0}^{p}(-1)^i\dbinom{p}{i}\underbrace{\dbinom{x-i}{q}}_{=(-1)^{x-i-q}\binom{-q-1}{x-i-q}}\\&=\sum_{i=0}^{p}(-1)^i\dbinom{p}{i}(-1)^{x-i-q}\dbinom{-q-1}{x-i-q}=\sum_{i=0}^{p}\underbrace{(-1)^i(-1)^{x-i-q}}_{=(-1)^{i+(x-i-q)}=(-1)^{x-q}}\dbinom{p}{i}\underbrace{\dbinom{-q-1}{x-i-q}}_{\substack{=\binom{-q-1}{x-q-i}\\ \text{(since } x-i-q=x-q-i\text{)}}}\\&=\sum_{i=0}^{p}(-1)^{x-q}\dbinom{p}{i}\dbinom{-q-1}{x-q-i}=(-1)^{x-q}\underbrace{\sum_{i=0}^{p}\dbinom{p}{i}\dbinom{-q-1}{x-q-i}}_{\substack{=\binom{p+(-q-1)}{x-q}\\ \text{(by Corollary 6.5,}\\ \text{applied to } x-q,\ p \text{ and } -q-1 \text{ instead of } n,\ x \text{ and } y\text{)}}}\\&=(-1)^{x-q}\dbinom{p+(-q-1)}{x-q}=(-1)^{x-q}\dbinom{p-q-1}{x-q}\end{aligned}\tag{26}$$

(since $p+(-q-1)=p-q-1$).

But $x\in\{p,p+1,p+2,\ldots\}$ and thus $x-p\in\mathbb{N}$. Hence, Theorem 1.2 (applied to $n=x-p$ and $k=q-p$) yields

$$\begin{aligned}\dbinom{x-p}{q-p}&=\dbinom{x-p}{(x-p)-(q-p)}=\dbinom{x-p}{x-q}\qquad\text{(since } (x-p)-(q-p)=x-q\text{)}\\&=\dbinom{-(p-x)}{x-q}\qquad\text{(since } x-p=-(p-x)\text{)}\\&=(-1)^{x-q}\dbinom{p-x+x-q-1}{x-q}\\&\qquad\text{(by Proposition 6.4, applied to } n=p-x \text{ and } k=x-q\text{)}\\&=(-1)^{x-q}\dbinom{p-q-1}{x-q}\qquad\text{(since } p-x+x-q-1=p-q-1\text{)}\end{aligned}$$

Comparing this with (26), we obtain

$$\sum_{i=0}^{p} (-1)^i \binom{p}{i} \binom{x-i}{q} = \binom{x-p}{q-p}.$$

This proves Claim 1.]

Having proven Claim 1, we can now solve the exercise by an easy application of the polynomial identity trick:

Define the polynomial

$$P = \sum_{i=0}^{p} (-1)^i \binom{p}{i} \binom{X-i}{q} - \binom{X-p}{q-p}$$

(in 1 variable X, with real coefficients).[5] Then, each $x \in \{p, p+1, p+2, \ldots\}$ satisfies

$$P(x) = \underbrace{\sum_{i=0}^{p} (-1)^i \binom{p}{i} \binom{x-i}{q}}_{\substack{=\binom{x-p}{q-p} \\ \text{(by Claim 1)}}} - \binom{x-p}{q-p} = \binom{x-p}{q-p} - \binom{x-p}{q-p} = 0.$$

In other words, each $x \in \{p, p+1, p+2, \ldots\}$ is a root of P (by the definition of "root"). Hence, the polynomial P has infinitely many roots (since there are infinitely many $x \in \{p, p+1, p+2, \ldots\}$). Hence, Corollary 6.6 yields that P is the zero polynomial. In other words, $P = 0$. Hence, for each $x \in \mathbb{R}$, we have $P(x) = 0$ and thus

$$0 = P(x) = \sum_{i=0}^{p} (-1)^i \binom{p}{i} \binom{x-i}{q} - \binom{x-p}{q-p}$$

and thus

$$\sum_{i=0}^{p} (-1)^i \binom{p}{i} \binom{x-i}{q} = \binom{x-p}{q-p}.$$

This solves the exercise.

6.5 Fourth solution (sketched)

The exercise can also be seen as a consequence of two basic facts in the theory of finite differences. Let us introduce just enough of this theory to solve the exercise.

Forget that we fixed p and q.

Let $\mathbb{R}^*$ be the set of all finite lists of real numbers; in other words, let $\mathbb{R}^* = \mathbb{R}^0 \cup \mathbb{R}^1 \cup \mathbb{R}^2 \cup \mathbb{R}^3 \cup \cdots$. Define a map $\Delta : \mathbb{R}^* \to \mathbb{R}^*$ by setting

$$\Delta(a_1, a_2, \ldots, a_n) = (a_2 - a_1, a_3 - a_2, \ldots, a_n - a_{n-1}) \qquad \text{for all } (a_1, a_2, \ldots, a_n) \in \mathbb{R}^*.$$

(Thus, in particular, Δ sends the empty list () to the empty list ().)

This map Δ is commonly called the *difference operator*, since it takes any list of numbers to the list of differences between the consecutive entries of the former list.

[5] This is a well-defined polynomial, because we know that $\binom{F}{n}$ is a well-defined polynomial whenever F is a polynomial and $n \in \mathbb{N}$.

Example 6.7. We have

$$\Delta(1,1,1) = (0,0); \qquad \Delta(1,1,1,1,1) = (0,0,0,0);$$
$$\Delta(1,3,5,7) = (2,2,2); \qquad \Delta(1,4,9,16) = (3,5,7).$$

The first basic fact that we shall use about finite differences is the following:

Proposition 6.8. *Let $p \in \mathbb{N}$, $x \in \mathbb{R}$ and $q \in \mathbb{R}$. Then,*

$$\Delta\left(\binom{x+0}{q}, \binom{x+1}{q}, \ldots, \binom{x+p}{q}\right) = \left(\binom{x+0}{q-1}, \binom{x+1}{q-1}, \ldots, \binom{x+(p-1)}{q-1}\right).$$

In words, this proposition is saying that if we apply the map Δ to a list of binomial coefficients with the same "denominator" q and with their numerators increasing by 1 at each step, then we get a similar list, but without the last entry and with denominator $q-1$ instead of q.

Proof of Proposition 6.8 (sketched). Because of how Δ was defined, this boils down to checking that

$$\binom{x+(k+1)}{q} - \binom{x+k}{q} = \binom{x+k}{q-1} \qquad \text{for each } k \in \{0,1,\ldots,p-1\}.$$

But this follows easily from the recurrence of the binomial coefficients (more precisely, from Theorem 1.4, applied to $x+(k+1)$ and q instead of n and k). □

Now, as usual, we set $\Delta^k = \underbrace{\Delta \circ \Delta \circ \cdots \circ \Delta}_{k \text{ times}}$ for each $k \in \mathbb{N}$. Thus, Δ^k is simply the map that applies Δ a total of k times. (In particular, $\Delta^0 = \operatorname{id}$ and $\Delta^1 = \Delta$.)

Example 6.9. We have

$$\underbrace{\Delta^0}_{=\operatorname{id}}(1,4,9,16) = (1,4,9,16);$$
$$\underbrace{\Delta^1}_{=\Delta}(1,4,9,16) = \Delta(1,4,9,16) = (3,5,7),$$
$$\underbrace{\Delta^2}_{=\Delta\circ\Delta}(1,4,9,16) = \Delta\left(\underbrace{\Delta(1,4,9,16)}_{=(3,5,7)}\right) = \Delta(3,5,7) = (2,2),$$
$$\underbrace{\Delta^3}_{=\Delta\circ\Delta\circ\Delta}(1,4,9,16) = \Delta\left(\underbrace{\Delta(\Delta(1,4,9,16))}_{=(2,2)}\right) = \Delta(2,2) = (0),$$
$$\underbrace{\Delta^4}_{=\Delta\circ\Delta\circ\Delta\circ\Delta}(1,4,9,16) = \Delta\left(\underbrace{\Delta(\Delta(\Delta(1,4,9,16)))}_{=(0)}\right) = \Delta(0) = ().$$

Thus, $\Delta^k(1,4,9,16) = ()$ for all $k \geq 4$ (since $\Delta() = ()$).

From Proposition 6.8, we can easily obtain the following:

Corollary 6.10. *Let $p \in \mathbb{N}$, $k \in \mathbb{N}$, $x \in \mathbb{R}$ and $q \in \mathbb{R}$. Then,*

$$\Delta^k\left(\binom{x+0}{q}, \binom{x+1}{q}, \ldots, \binom{x+p}{q}\right) = \left(\binom{x+0}{q-k}, \binom{x+1}{q-k}, \ldots, \binom{x+(p-k)}{q-k}\right).$$

Proof of Corollary 6.10. Induction on k. The induction step uses Proposition 6.8. □

The second fact about finite differences that we will need is the following explicit description of how Δ^k acts on a tuple:

Proposition 6.11. *If $k \in \mathbb{N}$, $n \in \mathbb{N}$, $(a_1, a_2, \ldots, a_n) \in \mathbb{R}^n$ and $p \in \{1, 2, \ldots, n-k\}$, then the p-th entry of the list $\Delta^k (a_1, a_2, \ldots, a_n)$ is*

$$\sum_{i=0}^{k} (-1)^{k-i} \binom{k}{i} a_{p+i}.$$

For a proof of Proposition 6.11, see my math.stackexchange post

`https://math.stackexchange.com/a/1379518/`

(where it appears as Theorem 2).

We shall use the following variant of Proposition 6.11:

Corollary 6.12. *If $p \in \mathbb{N}$, $n \in \mathbb{N}$, $(a_1, a_2, \ldots, a_n) \in \mathbb{R}^n$ and $r \in \{1, 2, \ldots, n-p\}$, then the r-th entry of the list $\Delta^p (a_1, a_2, \ldots, a_n)$ is*

$$\sum_{i=0}^{p} (-1)^{p-i} \binom{p}{i} a_{r+i}.$$

Corollary 6.12 is just Proposition 6.11, with k and p renamed as p and r.

Now, let us solve the exercise. Let $p \in \mathbb{N}$, $q \in \mathbb{N}$ and $x \in \mathbb{R}$. Corollary 6.10 (applied to p and $x - p$ instead of k and x) yields

$$\begin{aligned} &\Delta^p \left(\binom{(x-p)+0}{q}, \binom{(x-p)+1}{q}, \ldots, \binom{(x-p)+p}{q} \right) \\ &= \left(\binom{(x-p)+0}{q-p}, \binom{(x-p)+1}{q-p}, \ldots, \binom{(x-p)+(p-p)}{q-p} \right). \end{aligned} \tag{27}$$

For each $i \in \mathbb{Z}$, define a real a_i by $a_i = \binom{(x-p)+i-1}{q}$. Then,

$$(a_1, a_2, \ldots, a_{p+1}) = \left(\binom{(x-p)+0}{q}, \binom{(x-p)+1}{q}, \ldots, \binom{(x-p)+p}{q} \right).$$

In view of this, we can rewrite (27) as

$$\begin{aligned} &\Delta^p (a_1, a_2, \ldots, a_{p+1}) \\ &= \left(\binom{(x-p)+0}{q-p}, \binom{(x-p)+1}{q-p}, \ldots, \binom{(x-p)+(p-p)}{q-p} \right). \end{aligned} \tag{28}$$

Hence,

$$(\text{the 1-st entry of the list } \Delta^p (a_1, a_2, \ldots, a_{p+1})) = \binom{(x-p)+0}{q-p} = \binom{x-p}{q-p}. \tag{29}$$

Corollary 6.12 (applied to $n = p+1$ and $r = 1$) yields that the 1-st entry of the list $\Delta^p (a_1, a_2, \ldots, a_{p+1})$ is

$$\sum_{i=0}^{p} (-1)^{p-i} \binom{p}{i} \underbrace{\binom{\overbrace{(x-p)+(1+i)}^{=a_{1+i}}-1}{q}}_{\text{(by the definition of } a_{1+i}\text{)}}$$

$$= \sum_{i=0}^{p} (-1)^{p-i} \underbrace{\binom{p}{i}}_{\substack{=\binom{p}{p-i} \\ \text{(by Theorem 1.2,} \\ \text{applied to } n=p \text{ and } k=i)}} \underbrace{\binom{(x-p)+(1+i)-1}{q}}_{\substack{=\binom{x-(p-i)}{q} \\ \text{(since } (x-p)+(1+i)-1=x-(p-i))}}$$

$$= \sum_{i=0}^{p} (-1)^{p-i} \binom{p}{p-i} \binom{x-(p-i)}{q} = \sum_{i=0}^{p} (-1)^{i} \binom{p}{i} \binom{x-i}{q}$$

(here, we have substituted i for $p-i$ in the sum). Thus,

$$(\text{the 1-st entry of the list } \Delta^p (a_1, a_2, \ldots, a_{p+1})) = \sum_{i=0}^{p} (-1)^{i} \binom{p}{i} \binom{x-i}{q}.$$

Comparing this with (29), we obtain

$$\sum_{i=0}^{p} (-1)^{i} \binom{p}{i} \binom{x-i}{q} = \binom{x-p}{q-p}.$$

This solves the exercise.

6.6 Remark

Various other arguments can be used. For example, instead of proving Claim 1 in our third solution above using the Chu–Vandermonde identity, we could have proved it using the Principle of Inclusion and Exclusion[6].

A cheap (but occasionally useful) generalization of the exercise can be obtained by allowing q to range over $\mathbb{R}$ instead of $\mathbb{N}$:

Theorem 6.13. *Let $p \in \mathbb{N}$ and $q \in \mathbb{R}$. Prove that*

$$\sum_{i=0}^{p} (-1)^{i} \binom{p}{i} \binom{x-i}{q} = \binom{x-p}{q-p} \qquad \text{for all } x \in \mathbb{R}. \tag{30}$$

[6]**Hint:** How many q-element subsets of $[x]$ contain all of $1, 2, \ldots, p$? On the one hand, the answer is $\binom{x-p}{q-p}$; on the other hand, it can be obtained via the Principle of Inclusion and Exclusion (by counting how many q-element subsets of $[x]$ fail to contain some of these elements). Comparing the results will yield Claim 1.

For details, see the answer by Brian M. Scott at `https://math.stackexchange.com/questions/1536015` . Or see [Galvin17, proof of Identity 17.1] for the proof of Claim 1 with p, q and x renamed as n, j and N (slightly restated and with an unnecessary $x \geq q$ condition).

Proof of Theorem 6.13. If $q \in \mathbb{N}$, then this follows immediately from (11). Hence, for the rest of this proof, we can WLOG assume that we don't have $q \in \mathbb{N}$. Assume this. Thus, $q \notin \mathbb{N}$.

Let $x \in \mathbb{R}$. If we had $q - p \in \mathbb{N}$, then we would have $q = \underbrace{(q-p)}_{\in \mathbb{N}} + \underbrace{p}_{\in \mathbb{N}} \in \mathbb{N}$, which would contradict $q \notin \mathbb{N}$. Hence, we cannot have $q - p \in \mathbb{N}$. Thus, $q - p \notin \mathbb{N}$. Hence, the definition of binomial coefficients yields $\dbinom{x-p}{q-p} = 0$. Comparing this with

$$\sum_{i=0}^{p} (-1)^i \dbinom{p}{i} \underbrace{\dbinom{x-i}{q}}_{\substack{=0 \\ \text{(by the definition of} \\ \text{binomial coefficients,} \\ \text{since } q \notin \mathbb{N})}} = \sum_{i=0}^{p} (-1)^i \dbinom{p}{i} 0 = 0,$$

we obtain $\sum\limits_{i=0}^{p} (-1)^i \dbinom{p}{i} \dbinom{x-i}{q} = \dbinom{x-p}{q-p}$. This proves Theorem 6.13. □

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