Drexel University, Department of Mathematics

Math 222: Enumerative Combinatorics, Fall 2019: Homework 3

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December 16, 2019

due date: **Monday, 2019-11-04** at the beginning of class,
or before that through Blackboard.

Please solve **5 of the 6 exercises**!

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1 Exercise 1

1.1 Problem

Let $n \in \mathbb{N}$. Prove that

$$\sum_{k=0}^{n} \binom{2n+1}{k}^2 = \binom{4n+1}{2n}.$$

1.2 Solution

[...]

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2 Exercise 2

2.1 Problem

Let $n \in \mathbb{N}$ and $x, y \in \mathbb{R}$. Prove that

$$\sum_{k=1}^{n} \frac{x^k y^{n-k}}{k} \binom{n}{k} = \sum_{i=1}^{n} \frac{\left((x+y)^i - y^i\right) y^{n-i}}{i}.$$

2.2 Remark

This is easily seen to be a generalization of [Math222, Exercise 1.6.4] (indeed, the latter exercise is obtained by setting $x = -1$ and $y = 1$). Can you generalize the solution?

2.3 Solution

[...]

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3 Exercise 3

3.1 Problem

Let $m \in \mathbb{N}$. Prove that

$$\sum_{i=0}^{m} (-1)^i \operatorname{sur}(m, i) = (-1)^m.$$

3.2 Solution

[...]

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4 Exercise 4

4.1 Problem

Let $n \in \mathbb{N}$.

(a) Prove that

$$\left(\# \text{ of 5-tuples } (a, b, c, d, e) \in [n]^5 \text{ satisfying } a < b < c < d < e\right) = \binom{n}{5}.$$

(b) Find

$$\left(\# \text{ of 5-tuples } (a, b, c, d, e) \in [n]^5 \text{ satisfying } a \leq b < c \leq d < e\right).$$

4.2 Solution

[...]

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5 Exercise 5

5.1 Problem

A finite set S of integers is said to be *self-centered* if its size $|S|$ is odd and equals its $(|S|+1)/2$-th smallest element (i.e., its median in the statistical sense).

For example, the sets $\{1,3,5\}$ and $\{2,3,5,6,10\}$ are self-centered, while $\{2,4,6\}$ and $\{2\}$ are not.

- **(a)** Given $n \in \mathbb{N}$ and an odd $k \in \mathbb{N}$, find the # of self-centered k-element subsets of $[n]$. (The result will be a simple explicit formula in terms of binomial coefficients.)
- **(b)** For each $n \in \mathbb{N}$, let a_n be the # of all self-centered subsets of $[n]$. Find the sequence $(a_0, a_1, a_2, \ldots)$ or the sequence $(a_1, a_2, a_3, \ldots)$ in the OEIS. (No explicit sum-less formula is known.)

5.2 Solution

[...]

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6 Exercise 6

6.1 Problem

Let $p \in \mathbb{N}$ and $q \in \mathbb{N}$. Prove that

$$\sum_{i=0}^{p} (-1)^i \binom{p}{i} \binom{x-i}{q} = \binom{x-p}{q-p} \qquad \text{for all } x \in \mathbb{R}.$$

6.2 Solution

[...]

References

[Math222] Darij Grinberg, *Enumerative Combinatorics: class notes*, 16 December 2019. `http://www.cip.ifi.lmu.de/~grinberg/t/19fco/n/n.pdf` Also available on the mirror server `http://darijgrinberg.gitlab.io/t/19fco/n/n.pdf`
Caution: The numbering of theorems and formulas in this link might shift when the project gets updated; for a "frozen" version

whose numbering is guaranteed to match that in the citations above, see https://gitlab.com/darijgrinberg/darijgrinberg.gitlab.io/blob/2dab2743a181d5ba8fc145a661fd274bc37d03be/public/t/19fco/n/n.pdf