Drexel University, Department of Mathematics

Math 222: Enumerative Combinatorics, Fall 2019: Homework 2

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1 Exercise 1

1.1 Problem

For each $n \in \mathbb{N}$, we define the *n-th harmonic number* H_n by

$$H_n = \frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{n} = \sum_{k=1}^{n} \frac{1}{k}.$$

Prove that

$$H_1 + H_2 + \cdots + H_n = (n+1)(H_{n+1} - 1) \qquad (1)$$

for each $n \in \mathbb{N}$.

1.2 First solution

We shall prove (1) by induction on n:

Induction base: We have $H_{0+1} = H_1 = \frac{1}{1}$ (by the definition of H_1). Thus, $H_{0+1} - 1 = \frac{1}{1} - 1 = 0$.

Comparing

$$H_1 + H_2 + \cdots + H_0 = (\text{empty sum}) = 0$$

with

$$(0+1)\left(\underbrace{H_{0+1}-1}_{=0}\right)=0,$$

we find

$$H_1+H_2+\cdots+H_0=(0+1)\left(H_{0+1}-1\right).$$

In other words, (1) holds for $n=0$. This completes the induction base.

Induction step: Let m be a positive integer. Assume that (1) holds for $n=m-1$. We must prove that (1) holds for $n=m$.

We have assumed that (1) holds for $n=m-1$. In other words,

$$H_1+H_2+\cdots+H_{m-1}=((m-1)+1)\left(H_{(m-1)+1}-1\right).$$

In view of $(m-1)+1=m$, this rewrites as

$$H_1+H_2+\cdots+H_{m-1}=m\left(H_m-1\right). \qquad (2)$$

But the definition of H_m yields

$$H_m=\frac{1}{1}+\frac{1}{2}+\cdots+\frac{1}{m}. \qquad (3)$$

Also, the definition of H_{m+1} yields

$$H_{m+1}=\frac{1}{1}+\frac{1}{2}+\cdots+\frac{1}{m+1}=\underbrace{\left(\frac{1}{1}+\frac{1}{2}+\cdots+\frac{1}{m}\right)}_{\substack{=H_m\\ (\text{by } (3))}}+\frac{1}{m+1}=H_m+\frac{1}{m+1}.$$

Hence,

$$\begin{aligned}(m+1)\left(\underbrace{H_{m+1}}_{=H_m+\frac{1}{m+1}}-1\right)&=(m+1)\left(H_m+\frac{1}{m+1}-1\right)\\ &=(m+1)H_m+\underbrace{(m+1)\cdot\frac{1}{m+1}-(m+1)}_{=1-(m+1)=-m}=(m+1)H_m+(-m)\\ &=(m+1)H_m-m.\end{aligned}$$

Comparing this with

$$\begin{aligned}H_1+H_2+\cdots+H_m&=\underbrace{\left(H_1+H_2+\cdots+H_{m-1}\right)}_{\substack{=m(H_m-1)\\ (\text{by } (2))}}+H_m\\ &=m\left(H_m-1\right)+H_m=mH_m-m+H_m=(m+1)H_m-m,\end{aligned}$$

we obtain

$$H_1+H_2+\cdots+H_m=(m+1)\left(H_{m+1}-1\right).$$

In other words, (1) holds for $n=m$. This completes the induction step. Hence, (1) is proven by induction.

1.3 Second solution

Each $n \in \mathbb{N}$ satisfies

$$H_n = \sum_{k=1}^{n} \frac{1}{k} \qquad (4)$$

(by the definition of H_n).

Now, let $n \in \mathbb{N}$. Then,

$$H_1 + H_2 + \cdots + H_n = \sum_{m=1}^{n} \underbrace{H_m}_{\substack{=\sum_{k=1}^{m} \frac{1}{k} \\ \text{(by (4), applied to } m \text{ instead of } n)}} = \sum_{m=1}^{n} \sum_{k=1}^{m} \frac{1}{k} = \sum_{k=1}^{n} \sum_{m=k}^{n} \frac{1}{k}.$$

Here, the last equality sign is a consequence of one of Fubini's principles for the interchange of summations (namely, [Math222, Corollary 1.6.9]). Thus,

$$H_1 + H_2 + \cdots + H_n = \sum_{k=1}^{n} \underbrace{\sum_{m=k}^{n} \frac{1}{k}}_{\substack{=(n-k+1)\cdot\frac{1}{k} \\ \text{(since this is a sum of } n-k+1 \\ \text{many equal addends)}}} = \sum_{k=1}^{n} (n-k+1) \cdot \frac{1}{k}.$$

Comparing this with

$$\begin{aligned}
(n+1)(H_{n+1} - 1) &= (n+1) \underbrace{H_{n+1}}_{\substack{=\sum_{k=1}^{n+1} \frac{1}{k} \\ \text{(by (4), applied to } n+1 \text{ instead of } n)}} - (n+1) \\
&= (n+1) \sum_{k=1}^{n+1} \frac{1}{k} - \underbrace{(n+1)}_{\substack{=\sum_{k=1}^{n+1} 1 \\ \text{(since } \sum_{k=1}^{n+1} 1 = (n+1)\cdot 1 = n+1)}} = (n+1) \sum_{k=1}^{n+1} \frac{1}{k} - \sum_{k=1}^{n+1} 1 \\
&= \sum_{k=1}^{n+1} \underbrace{\left((n+1) \cdot \frac{1}{k} - 1 \right)}_{=(n-k+1)\cdot\frac{1}{k}} = \sum_{k=1}^{n+1} (n-k+1) \cdot \frac{1}{k} \\
&= \underbrace{(n-(n+1)+1)}_{=0} \cdot \frac{1}{n+1} + \sum_{k=1}^{n} (n-k+1) \cdot \frac{1}{k} \\
&\qquad \left(\text{here, we have split off the addend for } k = n+1 \text{ from the sum}\right) \\
&= \sum_{k=1}^{n} (n-k+1) \cdot \frac{1}{k},
\end{aligned}$$

we obtain $H_1 + H_2 + \cdots + H_n = (n+1)(H_{n+1} - 1)$. This solves the exercise.

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2 Exercise 2

2.1 Problem

Let $n \in \mathbb{N}$. Compute the number of 4-tuples (A, B, C, D) of subsets of $[n]$ satisfying

$$A \cap B = C \cap D.$$

[**Hint:** This is similar to [17f-hw3s, Exercise 1]. It is not necessary to be as detailed as in the solution of part **(a)** of the latter exercise.]

2.2 Solution sketch

We shall say that a 4-tuple (A, B, C, D) of subsets of $[n]$ is *good* if and only if it satisfies $A \cap B = C \cap D$.

We claim the following:

Claim 1: The # of good 4-tuples is 10^n.

Let us first give an informal (but perfectly clear to the experienced reader) proof of this claim, and then formalize it.

Informal proof of Claim 1. A 4-tuple (A, B, C, D) of subsets of $[n]$ is good if and only if it satisfies the following property: Each $i \in [n]$ belongs to

- **either** all four sets A, B, C and D,
- **or** the sets A and C but not B and D,
- **or** the sets A and D but not B and C,
- **or** the sets B and C but not A and D,
- **or** the sets B and D but not A and C,
- **or** the set A but none of the other three sets,
- **or** the set B but none of the other three sets,
- **or** the set C but none of the other three sets,
- **or** the set D but none of the other three sets,
- **or** none of the four sets A, B, C and D.

[1] We shall refer to these 10 possibilities as "Option 1", "Option 2" and so on.

Thus, the following simple algorithm constructs every good 4-tuple (A, B, C, D): For each $i \in [n]$, we decide which of the 10 options listed above the element i should satisfy (i.e.,

[1]Indeed, there is (a priori) a total of 16 options for which of the four sets A, B, C and D the element i belongs to (because i either belongs to A or does not; either belongs to B or does not; either belongs to C or does not; either belongs to D or does not). But out of these 16 options, only the 10 we just listed can occur if (A, B, C, D) is good, since the other 6 would violate the equation $A \cap B = C \cap D$ (since they would either make i belong to $A \cap B$ but not to $C \cap D$, or make i belong to $C \cap D$ but not to $A \cap B$). It is easy to see that, conversely, as long as each i satisfies one of the 10 options listed above, the 4-tuple (A, B, C, D) is good.

whether it satisfies Option 1 or Option 2 etc.). There are 10 choices for it, since these 10 options are mutually exclusive. Thus, in total, there are 10^n good 4-tuples (because we are making this decision once for each of the n elements i of $[n]$). This completes our informal proof of Claim 1. □

Next comes a formalized version of this argument:

Formal proof of Claim 1. Consider a 4-tuple (A, B, C, D) of subsets of $[n]$, and an element $i \in [n]$. This element i either lies in A or does not; it either lies in B or does not; it either lies in C or does not; it either lies in D or does not. Thus, we have a total of 16 possible answers to the question "which of the 4 subsets A, B, C and D does i lie in?". Let us encode these answers as 4-tuples of bits (i.e., of elements of $\{0, 1\}$): Namely, we define

$$\mathbf{w}_{A,B,C,D}(i) = ([i \in A], [i \in B], [i \in C], [i \in D]) \in \{0, 1\}^4$$

(where we are using the Iverson bracket notation). Thus, for example, if i lies in A and D but not in B and not in C, then $\mathbf{w}_{A,B,C,D}(i) = (1, 0, 0, 1)$.

Now, assume that the 4-tuple (A, B, C, D) is good. Then, $\mathbf{w}_{A,B,C,D}(i)$ cannot take certain values. For example, $\mathbf{w}_{A,B,C,D}(i)$ cannot be $(1, 1, 0, 1)$, because in this case, i would be contained in $A \cap B$ (since $[i \in A] = 1$ and $[i \in B] = 1$) but not in $C \cap D$ (since $[i \in C] = 0$), which would contradict the "goodness" condition $A \cap B = C \cap D$. Likewise, there are other values that $\mathbf{w}_{A,B,C,D}(i)$ cannot take. By systematically checking all 16 possible 4-tuples of bits, we can easily see that the set of impossible values of $\mathbf{w}_{A,B,C,D}(i)$ is

$$J := \{(1, 1, 0, 0), (1, 1, 0, 1), (1, 1, 1, 0), (0, 0, 1, 1), (0, 1, 1, 1), (1, 0, 1, 1)\}.$$

Thus, $\mathbf{w}_{A,B,C,D}(i)$ belongs not only to $\{0, 1\}^4$, but to the smaller set $\{0, 1\}^4 \setminus J$. It is easy to see that this smaller set has size $\left|\{0, 1\}^4 \setminus J\right| = 10$.

Now, forget that we fixed i. Thus, we have defined a 4-tuple $\mathbf{w}_{A,B,C,D}(i) \in \{0, 1\}^4 \setminus J$ for each $i \in [n]$ (assuming that (A, B, C, D) is good). In other words, we have defined a map

$$\mathbf{w}_{A,B,C,D} : [n] \to \{0, 1\}^4 \setminus J,$$
$$i \mapsto \mathbf{w}_{A,B,C,D}(i) = ([i \in A], [i \in B], [i \in C], [i \in D]).$$

Note that we can easily reconstruct the 4-tuple (A, B, C, D) from the map $\mathbf{w}_{A,B,C,D}$; indeed,

$$A = \{i \in [n] \mid \text{the 1-st entry of } \mathbf{w}_{A,B,C,D}(i) \text{ is } 1\};$$
$$B = \{i \in [n] \mid \text{the 2-nd entry of } \mathbf{w}_{A,B,C,D}(i) \text{ is } 1\};$$
$$C = \{i \in [n] \mid \text{the 3-rd entry of } \mathbf{w}_{A,B,C,D}(i) \text{ is } 1\};$$
$$D = \{i \in [n] \mid \text{the 4-th entry of } \mathbf{w}_{A,B,C,D}(i) \text{ is } 1\}.$$

Now, forget that we fixed (A, B, C, D). We thus have defined a map $\mathbf{w}_{A,B,C,D} : [n] \to \{0, 1\}^4 \setminus J$ for each good 4-tuple (A, B, C, D). Hence, we can define a map

$$\mathbf{W} : \{\text{good 4-tuples}\} \to \left(\{0, 1\}^4 \setminus J\right)^{[n]},$$
$$(A, B, C, D) \mapsto \mathbf{w}_{A,B,C,D}.$$

(Keep in mind that the notation Y^X, where X and Y are two sets, stands for the set of all maps from X to Y. Thus, the values of this map $\mathbf{W}$ are themselves maps.)

We have previously shown that a good 4-tuple (A, B, C, D) can be reconstructed from the map $\mathbf{w}_{A,B,C,D}$. In other words, the map $\mathbf{W}$ is injective.

Moreover, the map $\mathbf{W}$ is surjective. Indeed, if $\mathbf{f} \in \left(\{0,1\}^4 \setminus J\right)^{[n]}$ is any map, then we can define a 4-tuple (A,B,C,D) of subsets of $[n]$ by setting

$$\begin{aligned} A &= \{i \in [n] \mid \text{the 1-st entry of } \mathbf{f}(i) \text{ is } 1\}; \\ B &= \{i \in [n] \mid \text{the 2-nd entry of } \mathbf{f}(i) \text{ is } 1\}; \\ C &= \{i \in [n] \mid \text{the 3-rd entry of } \mathbf{f}(i) \text{ is } 1\}; \\ D &= \{i \in [n] \mid \text{the 4-th entry of } \mathbf{f}(i) \text{ is } 1\}; \end{aligned}$$

and it is easy to see that this 4-tuple (A,B,C,D) will be good (since $\mathbf{f}(i) \in \{0,1\}^4 \setminus J$ for each $i \in [n]$, which rules out precisely the constellations[2] that would violate $A \cap B = C \cap D$), and furthermore the image of this good 4-tuple (A,B,C,D) under the map $\mathbf{W}$ will be our $\mathbf{f}$.

Thus, we now know that the map $\mathbf{W}$ is injective and surjective. Hence, $\mathbf{W}$ is bijective. Thus, the bijection principle yields

$$\begin{aligned} |\{\text{good 4-tuples}\}| &= \left|\left(\{0,1\}^4 \setminus J\right)^{[n]}\right| = \left|\{0,1\}^4 \setminus J\right|^{|[n]|} \\ &\qquad \left(\text{since } \left|Y^X\right| = |Y|^{|X|} \text{ for any two finite sets } X \text{ and } Y\right) \\ &= 10^n \qquad \left(\text{since } \left|\{0,1\}^4 \setminus J\right| = 10 \text{ and } |[n]| = n\right). \end{aligned}$$

In other words, the # of good 4-tuples is 10^n. This proves Claim 1. □

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3 Exercise 3

3.1 Problem

Let $n \in \mathbb{N}$. A subset S of $[n]$ is said to be *odd-sum* if the sum of the elements of S is odd. How many subsets of $[n]$ are odd-sum?

3.2 First solution sketch

The following solution imitates [Math222, Third proof of Proposition 1.3.28].

Claim 1: We have

$$(\# \text{ of odd-sum subsets of } [n]) = \begin{cases} 0, & \text{if } n = 0; \\ 2^{n-1}, & \text{if } n \neq 0 \end{cases}.$$

[*Proof of Claim 1:* If $n = 0$, then the # of odd-sum subsets of $[n]$ is 0 (since the only subset of $[n]$ is $\varnothing$ in this case, but $\varnothing$ is not odd-sum). Thus, Claim 1 holds when $n = 0$. For the rest of this proof, we shall therefore WLOG assume that $n \neq 0$. Hence, $n \geq 1$, so that $1 \in [n]$.

[2] Exercise to the reader: Make this precise. (Formally speaking, you shouldn't talk about "constellations" but just prove that $A \cap B = C \cap D$ by considering any $i \in [n]$ and showing that $i \in A \cap B$ is equivalent to $i \in C \cap D$.)

Let us say that a subset S of $[n]$ is *even-sum* if the sum of the elements of S is even. Then, each subset of $[n]$ is either even-sum or odd-sum (but not both at the same time). Hence, by the sum rule, we have

$$\begin{aligned}&(\#\text{ of all subsets of }[n])\\&=(\#\text{ of even-sum subsets of }[n])+(\#\text{ of odd-sum subsets of }[n]).\end{aligned}$$

Comparing this with

$$(\#\text{ of all subsets of }[n])=2^n \qquad (\text{by [Math222, Theorem 1.4.1], applied to } S=[n]),$$

we obtain

$$2^n=(\#\text{ of even-sum subsets of }[n])+(\#\text{ of odd-sum subsets of }[n]). \tag{5}$$

On the other hand, if we add 1 to an even integer, then we obtain an odd integer. Hence, if S is an even-sum subset of $[n]$ such that $1\notin S$, then $S\cup\{1\}$ is an odd-sum subset of $[n]$. Similarly, if S is an even-sum subset of $[n]$ such that $1\in S$, then $S\setminus\{1\}$ is an odd-sum subset of $[n]$. Thus, the map

$$\begin{aligned}\{\text{even-sum subsets of }[n]\}&\to\{\text{odd-sum subsets of }[n]\},\\ S&\mapsto\begin{cases}S\cup\{1\}, & \text{if } 1\notin S;\\ S\setminus\{1\}, & \text{if } 1\in S\end{cases}\end{aligned}$$

is well-defined. Similarly, the map

$$\begin{aligned}\{\text{odd-sum subsets of }[n]\}&\to\{\text{even-sum subsets of }[n]\},\\ S&\mapsto\begin{cases}S\cup\{1\}, & \text{if } 1\notin S;\\ S\setminus\{1\}, & \text{if } 1\in S\end{cases}\end{aligned}$$

is well-defined. It is straightforward to see that these two maps are mutually inverse, and thus are bijections. Hence, the bijection principle shows that

$$(\#\text{ of even-sum subsets of }[n])=(\#\text{ of odd-sum subsets of }[n]).$$

Thus, (5) becomes

$$\begin{aligned}2^n&=\underbrace{(\#\text{ of even-sum subsets of }[n])}_{=(\#\text{ of odd-sum subsets of }[n])}+(\#\text{ of odd-sum subsets of }[n])\\&=(\#\text{ of odd-sum subsets of }[n])+(\#\text{ of odd-sum subsets of }[n])\\&=2\cdot(\#\text{ of odd-sum subsets of }[n]).\end{aligned}$$

Dividing both sides of this equality by 2, we find $2^n/2=(\#\text{ of odd-sum subsets of }[n])$, so that

$$(\#\text{ of odd-sum subsets of }[n])=2^n/2=2^{n-1}.$$

This proves Claim 1.]

3.3 Second solution sketch

Here is a very rough outline of a different solution.

Again, we WLOG assume that $n \neq 0$, so that $n \geq 1$.

Let E be the set of all even elements of $[n]$, and let O be the set of all odd elements of $[n]$. Then, E and O are disjoint subsets of $[n]$ whose union is $E \cup O = [n]$. Hence, the sum rule yields $|E| + |O| = |[n]| = n$. Moreover, $1 \in [n]$ (since $n \geq 1$), thus $1 \in O$, and therefore $|O| \geq 1$. A subset S of $[n]$ is odd-sum if and only if it contains an odd number of odd elements[3], i.e., if the intersection $S \cap O$ is a set of odd size. Thus, the map

$$\{\text{odd-sum subsets of } [n]\} \to \{\text{subsets of } E\} \times \{\text{subsets of } O \text{ having odd size}\},$$
$$S \mapsto (S \cap E, S \cap O)$$

is a bijection. Hence, by the bijection principle,

$$\begin{aligned}
&(\# \text{ of odd-sum subsets of } [n]) \\
&= |\{\text{subsets of } E\} \times \{\text{subsets of } O \text{ having odd size}\}| \\
&= (\# \text{ of subsets of } E) \cdot (\# \text{ of subsets of } O \text{ having odd size}).
\end{aligned}$$

Now, [Math222, Theorem 1.4.1] yields $(\# \text{ of subsets of } E) = 2^{|E|}$. What is $(\# \text{ of subsets of } O \text{ having odd size})$? Well, the sum rule yields

$$\begin{aligned}
&(\# \text{ of subsets of } O \text{ having odd size}) \\
&= \sum_{\substack{k \in \mathbb{N}; \\ k \text{ is odd}}} \underbrace{(\# \text{ of subsets of } O \text{ having size } k)}_{\substack{= (\# \text{ of } k\text{-element subsets of } O) \\ = \binom{|O|}{k} \\ \text{(by [Math222, Theorem 1.3.12])}}} \\
&\qquad \text{(this is one of those infinite sums with only finitely many nonzero addends)} \\
&= \sum_{\substack{k \in \mathbb{N}; \\ k \text{ is odd}}} \binom{|O|}{k} = \binom{|O|}{1} + \binom{|O|}{3} + \binom{|O|}{5} + \cdots \\
&= 2^{|O|-1} \qquad \text{(by [Math222, Proposition 1.3.34], applied to } |O| \text{ instead of } n).
\end{aligned}$$

Hence,

$$\begin{aligned}
&(\# \text{ of odd-sum subsets of } [n]) \\
&= \underbrace{(\# \text{ of subsets of } E)}_{=2^{|E|}} \cdot \underbrace{(\# \text{ of subsets of } O \text{ having odd size})}_{=2^{|O|-1}} \\
&= 2^{|E|} \cdot 2^{|O|-1} = 2^{|E|+|O|-1} = 2^{n-1} \qquad (\text{since } |E| + |O| = n).
\end{aligned}$$

This solves the exercise again.

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[3]because a sum of integers is odd if and only if it has an odd number of odd addends

4 Exercise 4

4.1 Problem

Let $n \in \mathbb{N}$. Prove that

$$\sum_{i=0}^{n} 2^i \binom{n-i}{i} = \frac{(-1)^n + 2^{n+1}}{3}. \qquad (6)$$

[**Hint:** Remember counting the pseudomino tilings on the previous problem set? Time to count them again! (This is not the only possible solution.)]

4.2 First solution sketch

We WLOG assume that $n > 0$ (since the case $n = 0$ is easily checked by hand).

We shall use the terminology introduced in [Math222, §1.1] for dominos and domino tilings, and we shall use the notion of lacunar sets defined in [Math222, Definition 1.4.2]. We shall furthermore use [hw1s, Exercise 1], and in particular we shall use the notions of "pseudomino" and "pseudomino tiling" defined therein. We let p_n denote the number of all pseudomino tilings of the rectangle $R_{n,2}$. Then, [hw1s, Exercise 1 **(b)**] yields

$$p_n = \frac{(-1)^n + 2^{n+1}}{3}. \qquad (7)$$

A bijection

$$h : \{\text{domino tilings of } R_{n+1,2}\} \to \{\text{lacunar subsets of } [n]\}$$

has been constructed in [Math222, Second proof of Proposition 1.4.9]; it is defined as follows: If T is any domino tiling of $R_{n+1,2}$, then $h(T)$ shall be the set of all $i \in [n+1]$ such that at least one horizontal domino of T starts in column i.

Substituting $n-1$ for n in this construction, we obtain a bijection

$$h' : \{\text{domino tilings of } R_{n,2}\} \to \{\text{lacunar subsets of } [n-1]\}$$

defined as follows: If T is any domino tiling of $R_{n,2}$, then $h'(T)$ shall be the set of all $i \in [n]$ such that at least one horizontal domino of T starts in column i.

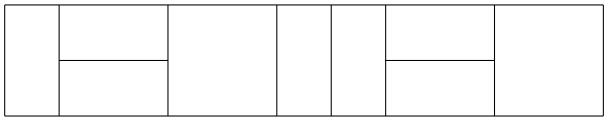
We want to define a bijection similar to h', but with pseudomino tilings instead of domino tilings. The target of this bijection will not be $\{\text{lacunar subsets of } [n-1]\}$ anymore, but rather will be $\{\text{lacunar pairs}\}$, where a *lacunar pair* shall mean a pair (S, T) of two disjoint subsets of $[n-1]$ such that $S \cup T$ is lacunar.

If T is a pseudomino tiling of $R_{n,2}$, then

- we let $h(T)$ be the set of all $i \in [n]$ such that at least one horizontal domino of T starts in column i;
- we let $d(T)$ be the set of all $i \in [n]$ such that at least one 2×2-rectangle of T starts[4] in column i.

[4]The meaning of "starts" here is defined as follows: If $D = \{(i,j), (i,j+1), (i+1,j), (i+1,j+1)\}$ is a 2×2-rectangle, then we say that D *starts* in column i.

For example, if $n = 11$ and

$T =$  ,

then

$$h(T) = \{2, 8\} \qquad \text{and} \qquad d(T) = \{4, 10\}.$$

We now define a map

$$h'' : \{\text{pseudomino tilings of } R_{n,2}\} \to \{\text{lacunar pairs}\}, \\ T \mapsto (h(T), d(T)).$$

This map is well-defined, because if T is a pseudomino tiling of $R_{n,2}$, then the pair $(h(T), d(T))$ is a lacunar pair[5]. Moreover, it is not hard to check that this map h'' is a bijection[6]. Thus, the bijection principle shows that

$$(\# \text{ of pseudomino tilings of } R_{n,2}) = (\# \text{ of lacunar pairs}).$$

But the definition of p_n yields

$$p_n = (\# \text{ of pseudomino tilings of } R_{n,2}) = (\# \text{ of lacunar pairs}). \tag{8}$$

Now, let us count the lacunar pairs. If (S, T) is a lacunar pair, then $S \cup T$ is a lacunar subset of $[n-1]$. Thus, by the sum rule, we have

$$(\# \text{ of lacunar pairs}) = \sum_{\substack{L \text{ is a lacunar} \\ \text{subset of } [n-1]}} (\# \text{ of lacunar pairs } (S, T) \text{ with } S \cup T = L). \tag{9}$$

Now, fix a lacunar subset L of $[n-1]$. How many lacunar pairs (S, T) are there that satisfy $S \cup T = L$?

Clearly, if (S, T) is a lacunar pair with $S \cup T = L$, then $S \subseteq S \cup T = L$. Thus, the map

$$\{\text{lacunar pairs } (S, T) \text{ with } S \cup T = L\} \to \{\text{subsets of } L\}, \\ (S, T) \mapsto S$$

is well-defined. On the other hand, if S is any subset of L, then $(S, L \setminus S)$ is a lacunar pair with $S \cup (L \setminus S) = L$. Thus, the map

$$\{\text{subsets of } L\} \to \{\text{lacunar pairs } (S, T) \text{ with } S \cup T = L\}, \\ S \mapsto (S, L \setminus S)$$

is well-defined. It is easy to see that these two maps are mutually inverse[7], and thus are bijections. Hence, the bijection principle yields

$$(\# \text{ of lacunar pairs } (S, T) \text{ with } S \cup T = L) = (\# \text{ of subsets of } L) \\ = 2^{|L|} \tag{10}$$

[5] Check this!

[6] The inverse map sends a lacunar pair (S, T) to the pseudomino tiling of $R_{n,2}$ whose horizontal dominos start in the columns $i \in S$ and whose 2×2-rectangles start in the columns $i \in T$ and whose remaining columns are filled with vertical dominos.

[7] The "hard part" of this is to prove that if (S, T) is a lacunar pair with $S \cup T = L$, then $(S, L \setminus S) = (S, T)$. But even this is trivial: If (S, T) is a lacunar pair with $S \cup T = L$, then $S \cap T = \varnothing$ (since the definition of "lacunar pair" implies that S and T are disjoint), and thus T is the complement of S in L (since $S \cup T = L$), which shows that $T = L \setminus S$, so that $(S, T) = (S, L \setminus S)$.

(by [Math222, Theorem 1.4.1], applied to $|L|$ and L instead of n and S).

Now, forget that we fixed L. We thus have proved (10) for each lacunar subset L of $[n-1]$. Thus, (9) becomes

$$\begin{aligned}
&(\# \text{ of lacunar pairs}) \\
&= \sum_{\substack{L \text{ is a lacunar} \\ \text{subset of } [n-1]}} \underbrace{(\# \text{ of lacunar pairs } (S,T) \text{ with } S \cup T = L)}_{\substack{=2^{|L|} \\ (\text{by } (10))}} \\
&= \sum_{\substack{L \text{ is a lacunar} \\ \text{subset of } [n-1]}} 2^{|L|} \\
&= \sum_{k \in \{0,1,\ldots,n\}} \sum_{\substack{L \text{ is a lacunar} \\ \text{subset of } [n-1]; \\ |L|=k}} \underbrace{2^{|L|}}_{\substack{=2^k \\ (\text{since } |L|=k)}} \\
&\qquad \left(\begin{array}{c} \text{here, we have split the sum } \sum\limits_{\substack{L \text{ is a lacunar} \\ \text{subset of } [n-1]}} 2^{|L|} \text{ according to the value of } |L| \\ \text{(because each subset } L \text{ of } [n-1] \text{ satisfies } |L| \leq |[n-1]| = n-1 \leq n \\ \text{and therefore } |L| \in \{0,1,\ldots,n\} \text{)} \end{array} \right) \\
&= \sum_{k \in \{0,1,\ldots,n\}} \underbrace{\sum_{\substack{L \text{ is a lacunar} \\ \text{subset of } [n-1]; \\ |L|=k}} 2^k}_{=(\# \text{ of lacunar subsets } L \text{ of } [n-1] \text{ such that } |L|=k) \cdot 2^k} \\
&= \sum_{k \in \{0,1,\ldots,n\}} \underbrace{(\# \text{ of lacunar subsets } L \text{ of } [n-1] \text{ such that } |L| = k)}_{\substack{=(\# \text{ of } k\text{-element lacunar subsets of } [n-1]) \\ = \binom{(n-1)+1-k}{k} \\ (\text{by [Math222, Proposition 1.4.10], applied to } n-1 \text{ instead of } n)}} \cdot 2^k \\
&= \underbrace{\sum_{k \in \{0,1,\ldots,n\}}}_{\substack{=\sum_{k=0}^{n} \\ (\text{an equality} \\ \text{of summation signs})}} \underbrace{\binom{(n-1)+1-k}{k}}_{=\binom{n-k}{k}} \cdot 2^k = \sum_{k=0}^{n} \binom{n-k}{k} \cdot 2^k = \sum_{k=0}^{n} 2^k \binom{n-k}{k}.
\end{aligned}$$

Now, (8) becomes

$$p_n = (\# \text{ of lacunar pairs}) = \sum_{k=0}^{n} 2^k \binom{n-k}{k} = \sum_{i=0}^{n} 2^i \binom{n-i}{i}$$

(here, we have renamed the summation index k as i). Comparing this with (7), we obtain

$$\sum_{i=0}^{n} 2^i \binom{n-i}{i} = \frac{(-1)^n + 2^{n+1}}{3}.$$

This solves the exercise.

4.3 Second solution

Here is a purely algebraic solution (similar to [Grinbe15, solution to Exercise 4.4]):

Forget that we fixed n. Set

$$g_n = \sum_{i=0}^{n} 2^i \binom{n-i}{i} \qquad \text{for each } n \in \{-1, 0, 1, \ldots\}. \tag{11}$$

Thus,

$$g_0 = \sum_{i=0}^{0} 2^i \binom{0-i}{i} = \underbrace{2^0}_{=1} \underbrace{\binom{0-0}{0}}_{=1} = 1 \qquad \text{and} \tag{12}$$

$$g_{-1} = \sum_{i=0}^{-1} 2^i \binom{-1-i}{i} = (\text{empty sum}) = 0. \tag{13}$$

On the other hand,

$$\frac{(-1)^0 + 2^{0+1}}{3} = \frac{1+2}{3} = 1 \qquad \text{and} \tag{14}$$

$$\frac{(-1)^{-1} + 2^{-1+1}}{3} = \frac{-1+1}{3} = 0. \tag{15}$$

Comparing (12) with (14), we obtain

$$g_0 = \frac{(-1)^0 + 2^{0+1}}{3}. \tag{16}$$

Comparing (13) with (15), we obtain

$$g_{-1} = \frac{(-1)^{-1} + 2^{-1+1}}{3}. \tag{17}$$

Recall the recurrence of the binomial coefficients:

Theorem 4.1 (Recurrence of the binomial coefficients)**.** *Let $n \in \mathbb{R}$ and $k \in \mathbb{R}$. Then,*

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}.$$

We also recall the following lemma:

Lemma 4.2. *Let $k \in \mathbb{R}$. Then, $\binom{0}{k} = [k = 0]$.*

Here, we are using the Iverson bracket notation.
Also, recall that if $n, k \in \mathbb{R}$ satisfy $k \notin \mathbb{N}$, then

$$\binom{n}{k} = 0. \tag{18}$$

(This is part of the definition of binomial coefficients.)
Now, we claim the following:

Claim 1: We have $g_n = \frac{(-1)^n + 2^{n+1}}{3}$ for each $n \in \{-1, 0, 1, \ldots\}$.

[*Proof of Claim 1:* We shall prove Claim 1 by strong induction on n.

Induction step: Let $m \in \{-1, 0, 1, \ldots\}$. Assume (as the induction hypothesis) that Claim 1 holds for all $n < m$. We must now prove that Claim 1 holds for $n = m$. In other words, we must prove that $g_m = \dfrac{(-1)^m + 2^{m+1}}{3}$. If $m = -1$, then this follows immediately from (17). Hence, for the rest of this proof, we WLOG assume that $m \neq -1$. Combining this with $m \in \{-1, 0, 1, \ldots\}$, we find $m \in \{-1, 0, 1, \ldots\} \setminus \{-1\} = \{0, 1, 2, \ldots\}$.

We must prove that $g_m = \dfrac{(-1)^m + 2^{m+1}}{3}$. If $m = 0$, then this follows immediately from (16). Hence, for the rest of this proof, we WLOG assume that $m \neq 0$. Combining this with $m \in \{0, 1, 2, \ldots\}$, we find $m \in \{0, 1, 2, \ldots\} \setminus \{0\} = \{1, 2, 3, \ldots\}$. Hence, $m-2 \in \{-1, 0, 1, \ldots\}$ and $m - 1 \in \{0, 1, 2, \ldots\} \subseteq \{-1, 0, 1, \ldots\}$ and $m \geq 1$. Also, from $m \neq 0$, we obtain $[m = 0] = 0$.

We have $m - 2 \in \{-1, 0, 1, \ldots\}$ and $m - 2 < m$. Thus, Claim 1 holds for $n = m - 2$ (since we assumed that Claim 1 holds for all $n < m$). In other words, we have

$$g_{m-2} = \frac{(-1)^{m-2} + 2^{(m-2)+1}}{3}. \tag{19}$$

We have $m - 1 \in \{-1, 0, 1, \ldots\}$ and $m - 1 < m$. Thus, Claim 1 holds for $n = m - 1$ (since we assumed that Claim 1 holds for all $n < m$). In other words, we have

$$g_{m-1} = \frac{(-1)^{m-1} + 2^{(m-1)+1}}{3}. \tag{20}$$

But the definition of g_{m-2} yields

$$g_{m-2} = \sum_{i=0}^{m-2} 2^i \binom{(m-2) - i}{i}. \tag{21}$$

Likewise, the definition of g_{m-1} yields

$$g_{m-1} = \sum_{i=0}^{m-1} 2^i \binom{(m-1) - i}{i}. \tag{22}$$

Now, the definition of g_m yields

$$g_m = \sum_{i=0}^{m} 2^i \binom{m-i}{i} = 2^m \underbrace{\binom{m-m}{m}}_{\substack{=\binom{0}{m}=[m=0] \\ \text{(by Lemma 4.2,} \\ \text{applied to } k=m)}} + \sum_{i=0}^{m-1} 2^i \binom{m-i}{i}$$

(here, we have split off the addend for $i = m$ from the sum)

$$= 2^m \underbrace{[m=0]}_{=0} + \sum_{i=0}^{m-1} 2^i \binom{m-i}{i} = \sum_{i=0}^{m-1} 2^i \underbrace{\binom{m-i}{i}}_{\substack{=\binom{m-i-1}{i-1}+\binom{m-i-1}{i} \\ \text{(by Theorem 4.1, applied} \\ \text{to } n=m-i \text{ and } k=i)}}$$

$$= \sum_{i=0}^{m-1} \underbrace{2^i \left(\binom{m-i-1}{i-1} + \binom{m-i-1}{i} \right)}_{=2^i \binom{m-i-1}{i-1} + 2^i \binom{m-i-1}{i}} = \sum_{i=0}^{m-1} \left(2^i \binom{m-i-1}{i-1} + 2^i \binom{m-i-1}{i} \right)$$

$$= \sum_{i=0}^{m-1} 2^i \binom{m-i-1}{i-1} + \sum_{i=0}^{m-1} 2^i \binom{m-i-1}{i}. \qquad (23)$$

We shall now massage the two sums on the right hand side of this equality, with the ultimate goal of revealing that the first of them is $2g_{m-2}$ while the second is g_{m-1}.

Let us start with the first sum. We have $m-1 \in \{0,1,2,\ldots\} = \mathbb{N}$ and

$$\sum_{i=0}^{m-1} 2^i \binom{m-i-1}{i-1}$$
$$= 2^0 \underbrace{\binom{m-0-1}{0-1}}_{\substack{=\binom{m-1}{-1}=0 \\ \text{(by (18), applied to } n=m-1 \text{ and } k=-1)}} + \sum_{i=1}^{m-1} 2^i \binom{m-i-1}{i-1}$$
$$\left(\begin{array}{c}\text{here, we have split off the addend for } i=0 \text{ from the sum} \\ \text{(since } 0 \leq m-1 \text{ (because } m \geq 1\text{))}\end{array}\right)$$
$$= \sum_{i=1}^{m-1} 2^i \binom{m-i-1}{i-1} = \sum_{i=0}^{m-2} \underbrace{2^{i+1}}_{=2\cdot 2^i} \underbrace{\binom{m-(i+1)-1}{(i+1)-1}}_{\substack{=\binom{(m-2)-i}{i} \\ \text{(since } m-(i+1)-1=(m-2)-i \\ \text{and } (i+1)-1=i\text{)}}}$$
$$\text{(here, we have substituted } i+1 \text{ for } i \text{ in the sum)}$$
$$= \sum_{i=0}^{m-2} 2\cdot 2^i \binom{(m-2)-i}{i} = 2\cdot \underbrace{\sum_{i=0}^{m-2} 2^i \binom{(m-2)-i}{i}}_{\substack{=g_{m-2} \\ \text{(by (21))}}}$$
$$= 2g_{m-2}. \tag{24}$$

Now, let us take a look at the second sum. We have

$$\sum_{i=0}^{m-1} 2^i \underbrace{\binom{m-i-1}{i}}_{\substack{=\binom{(m-1)-i}{i} \\ \text{(since } m-i-1=(m-1)-i\text{)}}} = \sum_{i=0}^{m-1} 2^i \binom{(m-1)-i}{i} = g_{m-1} \qquad \text{(by (22))}. \tag{25}$$

Now, (23) becomes

$$g_m = \underbrace{\sum_{i=0}^{m} 2^i \binom{m-i-1}{i-1}}_{\substack{=2g_{m-2} \\ \text{(by (24))}}} + \underbrace{\sum_{i=0}^{m} 2^i \binom{m-i-1}{i}}_{\substack{=g_{m-1} \\ \text{(by (25))}}}$$
$$= 2 \underbrace{g_{m-2}}_{\substack{=\frac{(-1)^{m-2}+2^{(m-2)+1}}{3} \\ \text{(by (19))}}} + \underbrace{g_{m-1}}_{\substack{=\frac{(-1)^{m-1}+2^{(m-1)+1}}{3} \\ \text{(by (20))}}}$$
$$= 2 \cdot \frac{(-1)^{m-2}+2^{(m-2)+1}}{3} + \frac{(-1)^{m-1}+2^{(m-1)+1}}{3}$$
$$= \frac{1}{3}\left(2 \cdot \left(\underbrace{(-1)^{m-2}}_{=(-1)^m} + \underbrace{2^{(m-2)+1}}_{=2^{m-1}}\right) + \underbrace{(-1)^{m-1}}_{=-(-1)^m} + \underbrace{2^{(m-1)+1}}_{=2^m}\right)$$
$$= \frac{1}{3}\left(\underbrace{2 \cdot \left((-1)^m + 2^{m-1}\right)}_{=2\cdot(-1)^m+2\cdot 2^{m-1}} - (-1)^m + 2^m\right) = \frac{1}{3}\left(2 \cdot (-1)^m + 2 \cdot 2^{m-1} - (-1)^m + 2^m\right)$$
$$= \frac{1}{3}\left(\underbrace{2 \cdot (-1)^m - (-1)^m}_{=(-1)^m} + \underbrace{2 \cdot 2^{m-1}}_{=2^m} + 2^m\right) = \frac{1}{3}\left((-1)^m + \underbrace{2^m + 2^m}_{=2\cdot 2^m = 2^{m+1}}\right)$$
$$= \frac{1}{3}\left((-1)^m + 2^{m+1}\right) = \frac{(-1)^m + 2^{m+1}}{3}.$$

In other words, Claim 1 holds for $n = m$. This completes the induction step. Thus, the induction proof of Claim 1 is finished.]

Now, let $n \in \mathbb{N}$. Then, $n \in \mathbb{N} \subseteq \{-1, 0, 1, \ldots\}$, so that Claim 1 yields

$$g_n = \frac{(-1)^n + 2^{n+1}}{3}.$$

Comparing this with (11), we obtain

$$\sum_{i=0}^{n} 2^i \binom{n-i}{i} = \frac{(-1)^n + 2^{n+1}}{3}.$$

Thus, the exercise is solved.

---

5 Exercise 5

5.1 Problem

Let $n, k \in \mathbb{R}$. Prove that

$$\binom{n}{k+1} \cdot \binom{n-1}{k-1} \cdot \binom{n+1}{k} = \binom{n-1}{k} \cdot \binom{n+1}{k+1} \cdot \binom{n}{k-1}. \tag{26}$$

[**Hint:** Tempting as it may be to use the $\dfrac{n!}{k!\,(n-k)!}$ formula, keep in mind that it only holds for $n,k\in\mathbb{N}$ with $k\leq n$. When in doubt, go back to the definition of $\dbinom{n}{k}$.]

5.2 Solution

Forget that we fixed n and k. We shall use the following identity:

Proposition 5.1. *Let $n\in\{1,2,3,\ldots\}$ and $m\in\mathbb{R}$. Then,*

$$\binom{m}{n}=\frac{m}{n}\binom{m-1}{n-1}.$$

Proposition 5.1 is the *absorption formula*. A proof of Proposition 5.1 can be found in [Grinbe15, Proposition 3.22][8] or in [Math222, Proposition 1.3.36].

Also, recall that if $n,k\in\mathbb{R}$ satisfy $k\notin\mathbb{N}$, then

$$\binom{n}{k}=0. \tag{27}$$

(This is part of the definition of binomial coefficients.)

Now, let $n,k\in\mathbb{R}$. We must prove the identity (26). We are in one of the following two cases:

Case 1: We have $k-1\in\mathbb{N}$.

Case 2: We have $k-1\notin\mathbb{N}$.

Let us first consider Case 1. In this case, we have $k-1\in\mathbb{N}$. Hence, $k\in\{1,2,3,\ldots\}$. Thus, Proposition 5.1 (applied to $n+1$ and k instead of m and n) yields

$$\binom{n+1}{k}=\frac{n+1}{k}\binom{(n+1)-1}{k-1}=\frac{n+1}{k}\binom{n}{k-1}$$

(since $(n+1)-1=n$). Also, Proposition 5.1 (applied to n and k instead of m and n) yields

$$\binom{n}{k}=\frac{n}{k}\binom{n-1}{k-1}.$$

Furthermore, from $k\in\{1,2,3,\ldots\}$, we obtain $k+1\in\{2,3,4,\ldots\}\subseteq\{1,2,3,\ldots\}$. Hence, Proposition 5.1 (applied to $n+1$ and $k+1$ instead of m and n) yields

$$\begin{aligned}\binom{n+1}{k+1}&=\frac{n+1}{k+1}\binom{(n+1)-1}{(k+1)-1}\\&=\frac{n+1}{k+1}\underbrace{\binom{n}{k}}_{=\frac{n}{k}\binom{n-1}{k-1}}\qquad\left(\text{since }(n+1)-1=n\text{ and }(k+1)-1=k\right)\\&=\frac{n+1}{k+1}\cdot\frac{n}{k}\binom{n-1}{k-1}.\end{aligned}$$

Also, Proposition 5.1 (applied to n and $k+1$ instead of m and n) yields

$$\binom{n}{k+1}=\frac{n}{k+1}\binom{n-1}{(k+1)-1}=\frac{n}{k+1}\binom{n-1}{k}$$

[8] where it is stated only for $m\in\mathbb{Q}$, but this makes no difference to the proof

(since $(k+1)-1=k$).

Now, comparing

$$\underbrace{\binom{n}{k+1}}_{=\frac{n}{k+1}\binom{n-1}{k}} \cdot \binom{n-1}{k-1} \cdot \underbrace{\binom{n+1}{k}}_{=\frac{n+1}{k}\binom{n}{k-1}}$$
$$= \frac{n}{k+1}\binom{n-1}{k} \cdot \binom{n-1}{k-1} \cdot \frac{n+1}{k}\binom{n}{k-1} = \frac{n(n+1)}{k(k+1)}\binom{n-1}{k} \cdot \binom{n-1}{k-1} \cdot \binom{n}{k-1}$$

with

$$\binom{n-1}{k} \cdot \underbrace{\binom{n+1}{k+1}}_{=\frac{n+1}{k+1}\cdot\frac{n}{k}\binom{n-1}{k-1}} \cdot \binom{n}{k-1}$$
$$= \binom{n-1}{k} \cdot \frac{n+1}{k+1} \cdot \frac{n}{k}\binom{n-1}{k-1} \cdot \binom{n}{k-1} = \frac{n(n+1)}{k(k+1)}\binom{n-1}{k} \cdot \binom{n-1}{k-1} \cdot \binom{n}{k-1},$$

we obtain

$$\binom{n}{k+1} \cdot \binom{n-1}{k-1} \cdot \binom{n+1}{k} = \binom{n-1}{k} \cdot \binom{n+1}{k+1} \cdot \binom{n}{k-1}.$$

Thus, (26) is proven in Case 1.

Let us now consider Case 2. In this case, we have $k-1 \notin \mathbb{N}$. Hence, (27) (applied to $k-1$ instead of k) yields $\binom{n}{k-1} = 0$. Also, (27) (applied to $n-1$ and $k-1$ instead of n and k) yields $\binom{n-1}{k-1} = 0$. Now, comparing

$$\binom{n}{k+1} \cdot \underbrace{\binom{n-1}{k-1}}_{=0} \cdot \binom{n+1}{k} = 0$$

with

$$\binom{n-1}{k} \cdot \binom{n+1}{k+1} \cdot \underbrace{\binom{n}{k-1}}_{=0} = 0,$$

we obtain

$$\binom{n}{k+1} \cdot \binom{n-1}{k-1} \cdot \binom{n+1}{k} = \binom{n-1}{k} \cdot \binom{n+1}{k+1} \cdot \binom{n}{k-1}.$$

Thus, (26) is proven in Case 2.

We have now proven (26) in both Cases 1 and 2. Hence, (26) always holds. This solves the exercise.

5.3 Remark

You don't need to know Proposition 5.1 in order to solve the exercise; it merely helps make the solution slicker. Without Proposition 5.1, you can just apply the definition of binomial coefficients, obtaining (in Case 1) the identities

$$\begin{aligned}
\binom{n}{k+1} &= \frac{n(n-1)(n-2)\cdots(n-k)}{(k+1)!}; \\
\binom{n-1}{k-1} &= \frac{(n-1)(n-2)(n-3)\cdots(n-k+1)}{(k-1)!}; \\
\binom{n+1}{k} &= \frac{(n+1)n(n-1)\cdots(n-k+2)}{k!}; \\
\binom{n-1}{k} &= \frac{(n-1)(n-2)(n-3)\cdots(n-k)}{k!}; \\
\binom{n+1}{k+1} &= \frac{(n+1)n(n-1)\cdots(n-k+1)}{(k+1)!}; \\
\binom{n}{k-1} &= \frac{n(n-1)(n-2)\cdots(n-k+2)}{(k-1)!}.
\end{aligned}$$

Using these identities, (26) rewrites as

$$\begin{aligned}
&\frac{n(n-1)(n-2)\cdots(n-k)}{(k+1)!}\cdot\frac{(n-1)(n-2)(n-3)\cdots(n-k+1)}{(k-1)!} \\
&\qquad\cdot\frac{(n+1)n(n-1)\cdots(n-k+2)}{k!} \\
&= \frac{(n-1)(n-2)(n-3)\cdots(n-k)}{k!}\cdot\frac{(n+1)n(n-1)\cdots(n-k+1)}{(k+1)!} \\
&\qquad\cdot\frac{n(n-1)(n-2)\cdots(n-k+2)}{(k-1)!}.
\end{aligned}$$

But you can convince yourself that the factors on the two sides of this equality are the same (up to order). Thus, the exercise follows.

---

6 Exercise 6

6.1 Problem

Fix an $n \in \mathbb{N}$ and an n-element set X.

A *filter basis* (of X) means a nonempty set F of nonempty subsets of X such that for every $A \in F$ and $B \in F$, there exists some $C \in F$ such that $C \subseteq A \cap B$.

For example, if $X = [4]$, then $\{\{1,3\},\{1,3,4\},\{1,2,3,4\}\}$ is a filter basis, and so is $\{\{2\},\{1,2,3\},\{1,2,4\},\{2,3,4\}\}$. But $\{\{2,3\},\{1,3\},\{1,2,3\}\}$ is not a filter basis (because it contains no $C \subseteq \{2,3\} \cap \{1,3\}$).

Prove the following:

(a) If F is a filter basis, then the intersection of all $A \in F$ does itself belong to F.

(b) The number of all filter bases is

$$\sum_{k=0}^{n-1} \binom{n}{k} 2^{2^k - 1}.$$

6.2 Solution sketch

We shall use the following notation: If Y is any set, then $\mathcal{P}(Y)$ will denote the powerset of Y (that is, the set of all subsets of Y). If the set Y is finite, then we thus have

$$|\mathcal{P}(Y)| = (\# \text{ of subsets of } Y) = 2^{|Y|} \tag{28}$$

(by [Math222, Theorem 1.4.1], applied to Y and $|Y|$ instead of S and n). In particular, $\mathcal{P}(Y)$ is a finite set in this case.

Thus, in particular, $\mathcal{P}(X)$ is a finite set (since X is a finite set).

(a) Let F be a filter basis. Then, F is a set of nonempty subsets of X. Thus, $F \subseteq \mathcal{P}(X)$, so that F is a finite set (since $\mathcal{P}(X)$ is a finite set). Hence, we can write F in the form $F = \{A_1, A_2, \ldots, A_k\}$ for some nonempty subsets $A_1, A_2, \ldots, A_k$ of X (since F is a set of nonempty subsets of X). Consider these $A_1, A_2, \ldots, A_k$. Note that the set $\{A_1, A_2, \ldots, A_k\}$ is nonempty (since $\{A_1, A_2, \ldots, A_k\} = F$ is a filter basis). Thus, $k \neq 0$, so that $k \geq 1$. Note also that $A_1, A_2, \ldots, A_k \in F$ (since $F = \{A_1, A_2, \ldots, A_k\}$).

We have assumed that F is a filter basis. Hence, F is nonempty and has the property that for every $A \in F$ and $B \in F$,

$$\text{there exists some } C \in F \text{ such that } C \subseteq A \cap B. \tag{29}$$

Now, we claim the following:

Claim 1: For each $i \in [k]$, there exists some $C_i \in F$ such that

$$C_i \subseteq A_1 \cap A_2 \cap \cdots \cap A_i.$$

[*Proof of Claim 1:* We shall prove Claim 1 by induction on i:

Induction base: We have $A_1 \in F$ (since $A_1, A_2, \ldots, A_k \in F$). Thus, there exists some $C_1 \in F$ such that $C_1 \subseteq A_1$ (namely, $C_1 = A_1$ does the trick). In other words, Claim 1 holds for $i = 1$. This completes the induction base.

Induction step: Let $j \in [k]$ be such that $j > 1$. Assume that Claim 1 holds for $i = j-1$. We must prove that Claim 1 holds for $i = j$.

We have assumed that Claim 1 holds for $i = j - 1$. In other words, there exists some $C_{j-1} \in F$ such that $C_{j-1} \subseteq A_1 \cap A_2 \cap \cdots \cap A_{j-1}$. Consider this C_{j-1}. Recall that $A_1, A_2, \ldots, A_k \in F$. Hence, $A_j \in F$. Thus, (29) (applied to $A = C_{j-1}$ and $B = A_j$) shows that there exists some $C \in F$ such that $C \subseteq C_{j-1} \cap A_j$. Consider this C. Thus,

$$C \subseteq \underbrace{C_{j-1}}_{\subseteq A_1 \cap A_2 \cap \cdots \cap A_{j-1}} \cap A_j \subseteq (A_1 \cap A_2 \cap \cdots \cap A_{j-1}) \cap A_j = A_1 \cap A_2 \cap \cdots \cap A_j.$$

Hence, there exists some $C_j \in F$ such that $C_j \subseteq A_1 \cap A_2 \cap \cdots \cap A_j$ (namely, $C_j = C$). In other words, Claim 1 holds for $i = j$. This completes the induction step. Thus, Claim 1 is proven by induction.]

Now, recall that $k \geq 1$, so that $k \in [k]$. Hence, Claim 1 (applied to $i = k$) shows that there exists some $C_k \in F$ such that $C_k \subseteq A_1 \cap A_2 \cap \cdots \cap A_k$. Consider this C_k. Now,

$C_k \in F = \{A_1, A_2, \ldots, A_k\}$. In other words, $C_k = A_j$ for some $j \in [k]$. Consider this j. Combining $A_1 \cap A_2 \cap \cdots \cap A_k \subseteq A_j = C_k$ with $C_k \subseteq A_1 \cap A_2 \cap \cdots \cap A_k$, we obtain $A_1 \cap A_2 \cap \cdots \cap A_k = C_k \in F$.

But $F = \{A_1, A_2, \ldots, A_k\}$. Hence, the intersection of all $A \in F$ is $A_1 \cap A_2 \cap \cdots \cap A_k$, and thus does itself belong to F (since $A_1 \cap A_2 \cap \cdots \cap A_k \in F$). This solves part **(a)** of the exercise.

(b) A bit of terminology will come useful: If F is any filter basis, then the *core* of F is defined to be the intersection of all $A \in F$. This core does itself belong to F (by part **(a)** of the exercise). In other words,

$$\text{if } K \text{ is the core of a filter basis } F \text{, then } K \in F. \tag{30}$$

Now, instead of counting all filter bases right away, let us count only all filter bases with a given core:

Claim 2: Let K be a nonempty subset of X. Then,

$$(\# \text{ of filter bases with core } K) = 2^{2^{n-|K|}-1}.$$

We won't prove this right away, since we can make our job a little bit easier with some more terminology (and with two more auxiliary claims that we will prove before returning to prove Claim 2).

Previously, we have defined

$$\mathcal{P}(Y) = \{\text{all subsets of } Y\} \qquad \text{for any set } Y.$$

Now, let us introduce a subtler notation: If Y and Z are any two sets, then we define

$$\mathcal{P}(Y, Z) = \{\text{all sets } S \text{ such that } Z \subseteq S \subseteq Y\}.$$

This is the set of all sets "lying between" Z and Y (that is, the set of all sets S satisfying $Z \subseteq S \subseteq Y$). For example,

$$\mathcal{P}(\{1,2,3,4\}, \{1,3\}) = \{\{1,3\}, \{1,2,3\}, \{1,3,4\}, \{1,2,3,4\}\};$$
$$\mathcal{P}(\{1,2,3,4\}, \{1,2,3\}) = \{\{1,2,3\}, \{1,2,3,4\}\};$$
$$\mathcal{P}(\{1,2,3,4\}, \{1,2,3,4\}) = \{\{1,2,3,4\}\}.$$

We will only use the notation $\mathcal{P}(Y, Z)$ in the case when $Z \subseteq Y$, since otherwise $\mathcal{P}(Y, Z) = \varnothing$. In this case, it is easy to compute the size of $\mathcal{P}(Y, Z)$:

Claim 3: Let Y be a finite set. Let Z be a subset of Y. Then,

$$|\mathcal{P}(Y, Z)| = 2^{|Y \setminus Z|}.$$

[*Proof of Claim 3:* Here is the idea: The elements of $\mathcal{P}(Y, Z)$ are the subsets S of Y that contain Z as a subset. To choose such an S, we only need to decide which elements of $Y \setminus Z$ go into S (since the elements of Z are already forced to go into S); and this can be done in $2^{|Y \setminus Z|}$ many ways (since we have 2 choices for each of the $|Y \setminus Z|$ many elements of $Y \setminus Z$). Hence, $|\mathcal{P}(Y, Z)| = 2^{|Y \setminus Z|}$.

A formal version of this argument looks as follows: The maps

$$\mathcal{P}(Y,Z) \to \mathcal{P}(Y \setminus Z), \\ S \mapsto S \setminus Z$$

and

$$\mathcal{P}(Y \setminus Z) \to \mathcal{P}(Y,Z), \\ T \mapsto T \cup Z$$

are easily seen to be well-defined and mutually inverse; hence, they are bijections. Thus, the bijection principle yields $|\mathcal{P}(Y,Z)| = |\mathcal{P}(Y \setminus Z)| = 2^{|Y \setminus Z|}$ (by (28), applied to $Y \setminus Z$ instead of Y). This proves Claim 3.]

Claim 4: Let K be a nonempty subset of X. Then,

$$\{\text{filter bases with core } K\} = \mathcal{P}(\mathcal{P}(X,K), \{K\}).$$

Before we prove Claim 4, let us spell out what it says without the symbols: "Let K be a nonempty subset of X. Then, the filter bases with core K are precisely the sets lying between $\{K\}$ and the set of all sets lying between K and X.". Or, to make it more intuitive: "Let K be a nonempty subset of X. Then, a filter basis with core K will consist of sets lying between K and X, and will always contain K. Conversely, any set consisting of sets lying between K and X is a filter basis with core K as long as it contains K.".

[*Proof of Claim 4:* We shall first prove that

$$\{\text{filter bases with core } K\} \subseteq \mathcal{P}(\mathcal{P}(X,K), \{K\}). \tag{31}$$

Indeed, let $F \in \{\text{filter bases with core } K\}$. We shall show that $F \in \mathcal{P}(\mathcal{P}(X,K), \{K\})$.

Indeed, F is a filter basis with core K (since $F \in \{\text{filter bases with core } K\}$). Thus, $K \in F$ (by (30)). Hence, $\{K\} \subseteq F$. Moreover, F is a set of subsets of X (since F is a filter basis); thus, each $A \in F$ is a subset of X. But K is the core of F, that is, the intersection of all $A \in F$ (by the definition of a core). Therefore, each $A \in F$ satisfies $K \subseteq A$ and thus $K \subseteq A \subseteq X$ (since A is a subset of X). In other words, each $A \in F$ belongs to $\mathcal{P}(X,K)$ (since $K \subseteq A \subseteq X$ means precisely that $A \in \mathcal{P}(X,K)$ (by the definition of $\mathcal{P}(X,K)$)). In other words, $F \subseteq \mathcal{P}(X,K)$. Hence, $\{K\} \subseteq F \subseteq \mathcal{P}(X,K)$. In other words, $F \in \mathcal{P}(\mathcal{P}(X,K), \{K\})$ (by the definition of $\mathcal{P}(\mathcal{P}(X,K), \{K\})$).

Forget that we fixed F. We thus have shown that $F \in \mathcal{P}(\mathcal{P}(X,K), \{K\})$ for each $F \in \{\text{filter bases with core } K\}$. This proves (31).

On the other hand, let us prove that

$$\mathcal{P}(\mathcal{P}(X,K), \{K\}) \subseteq \{\text{filter bases with core } K\}. \tag{32}$$

Indeed, let $G \in \mathcal{P}(\mathcal{P}(X,K), \{K\})$. We shall prove that $G \in \{\text{filter bases with core } K\}$.

From $G \in \mathcal{P}(\mathcal{P}(X,K), \{K\})$, we obtain $\{K\} \subseteq G \subseteq \mathcal{P}(X,K)$ (by the definition of $\mathcal{P}(\mathcal{P}(X,K), \{K\})$). Thus, $K \in \{K\} \subseteq G$. Moreover, each element A of G belongs to $\mathcal{P}(X,K)$ (since $G \subseteq \mathcal{P}(X,K)$), and thus satisfies $K \subseteq A \subseteq X$ (by the definition of $\mathcal{P}(X,K)$). Thus, each $A \in G$ is a nonempty subset of X (indeed, it is a subset of X because $A \subseteq X$, and it is nonempty because $K \subseteq A$ for the nonempty set K). Thus, G is a set of nonempty subsets of X. Furthermore, G itself is nonempty, since $K \in G$. Finally, for every $A \in G$ and $B \in G$, we have $K \subseteq A$ (since $A \in G \subseteq \mathcal{P}(X,K)$ entails that $K \subseteq A \subseteq X$)

and $K \subseteq B$ (similarly) and therefore $K \subseteq A \cap B$. Hence, for every $A \in G$ and $B \in G$, there exists some $C \in G$ such that $C \subseteq A \cap B$ (namely, $C = K$).

Thus, G is a nonempty set of nonempty subsets of X such that for every $A \in G$ and $B \in G$, there exists some $C \in G$ such that $C \subseteq A \cap B$. In other words, G is a filter basis (by the definition of a filter basis).

Now, let L be the core of G. Thus, L is the intersection of all $A \in G$ (by the definition of a core). Hence, $L \subseteq A$ for each $A \in G$. Applying this to $A = K$, we obtain $L \subseteq K$ (since $K \in G$). Conversely, we can easily see that $K \subseteq L$ as follows: Since L is the core of the filter basis G, we have $L \in G$ (by (30), applied to G and L instead of F and K). Hence, $L \in G \subseteq \mathcal{P}(X, K)$, so that $K \subseteq L \subseteq X$ (by the definition of $\mathcal{P}(X, K)$), and thus in particular $K \subseteq L$. Combining $L \subseteq K$ with $K \subseteq L$, we obtain $L = K$. In other words, the core of G is K (since L is the core of G). Hence, G is a filter basis with core K. In other words, $G \in \{\text{filter bases with core } K\}$.

Forget that we fixed G. We thus have shown that $G \in \{\text{filter bases with core } K\}$ for each $G \in \mathcal{P}(\mathcal{P}(X, K), \{K\})$. This proves (32).

We have now proved the two relations (31) and (32). Combining them, we obtain

$$\{\text{filter bases with core } K\} = \mathcal{P}(\mathcal{P}(X, K), \{K\}).$$

Thus, Claim 4 is proven.]

Claim 2 is now easy:

[*Proof of Claim 2:* We know that K is a subset of X. Thus,

$$|X \setminus K| = \underbrace{|X|}_{\substack{=n \\ \text{(since } X \text{ is an } n\text{-element set)}}} - |K| = n - |K|$$

and

$$\begin{aligned} |\mathcal{P}(X, K)| &= 2^{|X \setminus K|} && \text{(by Claim 3, applied to } Y = X \text{ and } Z = K\text{)} \\ &= 2^{n-|K|} && (\text{since } |X \setminus K| = n - |K|). \end{aligned}$$

But $K \subseteq K \subseteq X$ and thus $K \in \mathcal{P}(X, K)$ (by the definition of $\mathcal{P}(X, K)$). Hence, $\{K\}$ is a subset of $\mathcal{P}(X, K)$. Thus,

$$|\mathcal{P}(X, K) \setminus \{K\}| = \underbrace{|\mathcal{P}(X, K)|}_{=2^{n-|K|}} - \underbrace{|\{K\}|}_{=1} = 2^{n-|K|} - 1.$$

Now,

$$\begin{aligned} &(\# \text{ of filter bases with core } K) \\ &= \left| \underbrace{\{\text{filter bases with core } K\}}_{\substack{=\mathcal{P}(\mathcal{P}(X,K),\{K\}) \\ \text{(by Claim 4)}}} \right| = |\mathcal{P}(\mathcal{P}(X, K), \{K\})| \\ &= 2^{|\mathcal{P}(X,K) \setminus \{K\}|} && \text{(by Claim 3, applied to } Y = \mathcal{P}(X, K) \text{ and } Z = \{K\}\text{)} \\ &= 2^{2^{n-|K|}-1} && \left(\text{since } |\mathcal{P}(X, K) \setminus \{K\}| = 2^{n-|K|} - 1\right). \end{aligned}$$

This proves Claim 2.]

At last, we can solve the actual problem:

If F is any filter basis, then the core of F does itself belong to F (as we have already seen), and thus is a nonempty subset of X (since F is a set of nonempty subsets of X). Hence, the sum rule shows that

$$\begin{aligned}
&(\# \text{ of filter bases}) \\
&= \sum_{\substack{K \text{ is a nonempty} \\ \text{subset of } X}} \underbrace{(\# \text{ of filter bases with core } K)}_{\substack{=2^{2^{n-|K|}-1} \\ \text{(by Claim 2)}}} = \sum_{\substack{K \text{ is a nonempty} \\ \text{subset of } X}} 2^{2^{n-|K|}-1} \\
&= \underbrace{\sum_{k\in\{1,2,\ldots,n\}}}_{=\sum_{k=1}^{n}} \sum_{\substack{K \text{ is a nonempty} \\ \text{subset of } X; \\ |K|=k}} \underbrace{2^{2^{n-|K|}-1}}_{\substack{=2^{2^{n-k}-1} \\ \text{(since } |K|=k)}} \\
&\qquad \left(\begin{array}{c} \text{here, we have split the sum according to the value of } |K|, \\ \text{because if } K \text{ is a nonempty subset of } X, \text{ then } |K| \in \{1,2,\ldots,n\} \\ \text{(since } X \text{ is an } n\text{-element set)} \end{array}\right) \\
&= \sum_{k=1}^{n} \underbrace{\sum_{\substack{K \text{ is a nonempty} \\ \text{subset of } X; \\ |K|=k}} 2^{2^{n-k}-1}}_{=(\# \text{ of nonempty subsets } K \text{ of } X \text{ satisfying } |K|=k)\cdot 2^{2^{n-k}-1}} \\
&= \sum_{k=1}^{n} \underbrace{(\# \text{ of nonempty subsets } K \text{ of } X \text{ satisfying } |K|=k)}_{\substack{=(\# \text{ of nonempty } k\text{-element subsets of } X) \\ =(\# \text{ of } k\text{-element subsets of } X) \\ \text{(since every } k\text{-element subset of } X \text{ is nonempty} \\ \text{(because } k\geq 1>0))}} \cdot 2^{2^{n-k}-1} \\
&= \sum_{k=1}^{n} \underbrace{(\# \text{ of } k\text{-element subsets of } X)}_{\substack{=\binom{n}{k} \\ \text{(by [Math222, Theorem 1.3.12], since } X \text{ is an } n\text{-element set)}}} \cdot 2^{2^{n-k}-1} \\
&= \sum_{k=1}^{n} \underbrace{\binom{n}{k}}_{\substack{=\binom{n}{n-k} \\ \text{(by [Math222, Theorem 1.3.11])}}} 2^{2^{n-k}-1} \\
&= \sum_{k=1}^{n} \binom{n}{n-k} 2^{2^{n-k}-1} = \sum_{k=0}^{n-1} \binom{n}{k} 2^{2^{k}-1}
\end{aligned}$$

(here, we have substituted k for $n-k$ in the sum). This solves part **(b)** of the exercise.

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whose numbering is guaranteed to match that in the citations above, see https://gitlab.com/darijgrinberg/darijgrinberg.gitlab.io/blob/2dab2743a181d5ba8fc145a661fd274bc37d03be/public/t/19fco/n/n.pdf

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