Drexel University, Department of Mathematics

Math 222: Enumerative Combinatorics, Fall 2019: Homework 2

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due date: **Friday, 2019-10-18** at the beginning of class, or before that by email (to darij.grinberg@drexel.edu).

Please solve **5 of the 6 exercises**!

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1 Exercise 1

1.1 Problem

For each $n \in \mathbb{N}$, we define the *n-th harmonic number* H_n by

$$H_n = \frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{n} = \sum_{k=1}^{n} \frac{1}{k}.$$

Prove that

$$H_1 + H_2 + \cdots + H_n = (n+1)(H_{n+1} - 1) \tag{1}$$

for each $n \in \mathbb{N}$.

1.2 Solution

[...]

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2 Exercise 2

2.1 Problem

Let $n \in \mathbb{N}$. Compute the number of 4-tuples (A, B, C, D) of subsets of $[n]$ satisfying

$$A \cap B = C \cap D.$$

[**Hint:** This is similar to [17f-hw3s, Exercise 1]. It is not necessary to be as detailed as in the solution of part **(a)** of the latter exercise.]

2.2 Solution

[...]

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3 Exercise 3

3.1 Problem

Let $n \in \mathbb{N}$. A subset S of $[n]$ is said to be *odd-sum* if the sum of the elements of S is odd. How many subsets of $[n]$ are odd-sum?

3.2 Solution

[...]

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4 Exercise 4

4.1 Problem

Let $n \in \mathbb{N}$. Prove that

$$\sum_{i=0}^{n} 2^i \binom{n-i}{i} = \frac{(-1)^n + 2^{n+1}}{3}. \tag{2}$$

[**Hint:** Remember counting the pseudomino tilings on the previous problem set? Time to count them again! (This is not the only possible solution.)]

4.2 Solution

[...]

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5 Exercise 5

5.1 Problem

Let $n, k \in \mathbb{R}$. Prove that

$$\binom{n}{k+1} \cdot \binom{n-1}{k-1} \cdot \binom{n+1}{k} = \binom{n-1}{k} \cdot \binom{n+1}{k+1} \cdot \binom{n}{k-1}. \qquad (3)$$

[**Hint:** Tempting as it may be to use the $\dfrac{n!}{k!\,(n-k)!}$ formula, keep in mind that it only holds for $n, k \in \mathbb{N}$ with $k \leq n$. When in doubt, go back to the definition of $\dbinom{n}{k}$.]

5.2 Solution

[...]

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6 Exercise 6

6.1 Problem

Fix an $n \in \mathbb{N}$ and an n-element set X.

A *filter basis* (of X) means a nonempty set F of nonempty subsets of X such that for every $A \in F$ and $B \in F$, there exists some $C \in F$ such that $C \subseteq A \cap B$.

For example, if $X = [4]$, then $\{\{1,3\}, \{1,3,4\}, \{1,2,3,4\}\}$ is a filter basis, and so is $\{\{2\}, \{1,2,3\}, \{1,2,4\}, \{2,3,4\}\}$. But $\{\{2,3\}, \{1,3\}, \{1,2,3\}\}$ is not a filter basis (because it contains no $C \subseteq \{2,3\} \cap \{1,3\}$).

Prove the following:

(a) If F is a filter basis, then the intersection of all $A \in F$ does itself belong to F.

(b) The number of all filter bases is

$$\sum_{k=0}^{n-1} \binom{n}{k} 2^{2^k - 1}.$$

6.2 Solution

[...]

References

[Math222] Darij Grinberg, *Enumerative Combinatorics: class notes*, 16 December 2019. `http://www.cip.ifi.lmu.de/~grinberg/t/19fco/n/n.pdf` Also available on the mirror server `http://darijgrinberg.gitlab.io/t/19fco/n/n.pdf`
Caution: The numbering of theorems and formulas in this link might shift when the project gets updated; for a "frozen" version

whose numbering is guaranteed to match that in the citations above, see `https://gitlab.com/darijgrinberg/darijgrinberg.gitlab.io/blob/2dab2743a181d5ba8fc145a661fd274bc37d03be/public/t/19fco/n/n.pdf`

[17f-hw3s] Darij Grinberg, *UMN Fall 2017 Math 4990 homework set #3 with solutions*, `http://www.cip.ifi.lmu.de/~grinberg/t/17f/hw3os.pdf`

[18f-hw1s] Darij Grinberg, *UMN Fall 2018 Math 5705 homework set #1 with solutions*, `http://www.cip.ifi.lmu.de/~grinberg/t/18f/hw1s.pdf`

[hw0s] Darij Grinberg, *Drexel Fall 2019 Math 222 homework set #0 with solutions*, `http://www.cip.ifi.lmu.de/~grinberg/t/19fco/hw0s.pdf`