Drexel University, Department of Mathematics

Math 222: Enumerative Combinatorics, Fall 2019: Homework 1

Darij Grinberg

December 16, 2019

due date: **Monday, 2019-10-07** at the beginning of class,
or before that by email or Blackboard.
Please solve **as many exercises as you can**!

1 Exercise 1

1.1 Problem

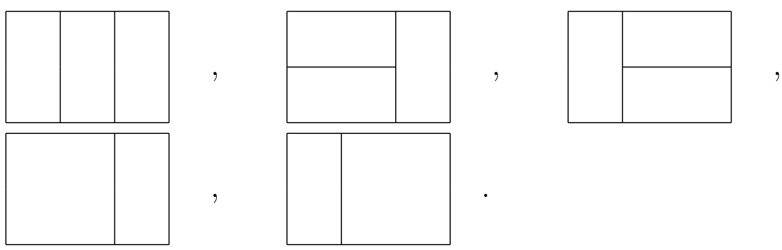
Let us define a slight variation on domino tilings. We shall use the notations of [Math222, §1.1].

A 2×2-*rectangle* will mean a set of the form $\{(i,j), (i,j+1), (i+1,j), (i+1,j+1)\}$ for some $i, j \in \mathbb{Z}$. (Visually, this is just a set of 4 mutually adjacent squares forming a 2×2-rectangle.)

A *pseudomino* will mean a set of squares that is either a domino or a 2×2-rectangle.

If S is a set of squares, then a *pseudomino tiling* of S will mean a set of disjoint pseudominos whose union is S.

For example, here are all five pseudomino tilings of the rectangle $R_{3,2}$:



For any $n \in \mathbb{N}$, we let p_n denote the number of all pseudomino tilings of the rectangle $R_{n,2}$.

[*Example:* We have $p_0 = 1$, $p_1 = 1$, $p_2 = 3$, $p_3 = 5$.]

(a) Find a recursive formula that expresses p_n in terms of p_{n-1} and p_{n-2} when $n \geq 2$.

(b) Prove that $p_n = \dfrac{(-1)^n + 2^{n+1}}{3}$ for each $n \in \mathbb{N}$.

[**Hint:** You don't need to be more detailed than in the proof of [Math222, Proposition 1.1.9].]

1.2 Solution

[...]

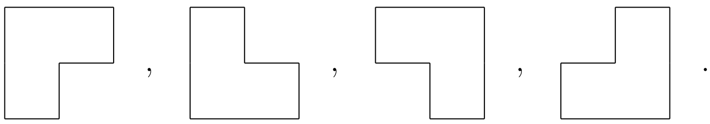
---

2 Exercise 2

2.1 Problem

Again, we shall use the notations of [Math222, §1.1].

An *L-tromino* will mean a set of squares that has one of the four forms



(Formally speaking, it is a set of the form $\{(i,j), (i',j), (i,j')\}$, where $i, j \in \mathbb{Z}$ and $i' \in \{i-1, i+1\}$ and $j' \in \{j-1, j+1\}$.)

If S is a set of squares, then an *L-tromino tiling* of S will mean a set of disjoint L-trominos whose union is S.

For any $n \in \mathbb{N}$, we let L_n denote the number of L-tromino tilings of the rectangle $R_{n,2}$.

We shall use the Iverson bracket notation[1].

Prove that

$$L_n = [3 \mid n] \cdot 2^{n/3} \qquad \text{for each } n \in \mathbb{N}.$$

[**Hint:** Feel free to take inspiration from the solution to [18f-hw1s, Exercise 5].]

---

[1]This means the following:

If $\mathcal{A}$ is any statement (such as "$1+1=2$" or "$1+1=1$" or "there exist infinitely many primes"), then $[\mathcal{A}]$ stands for the number

$$\begin{cases} 1, & \text{if } \mathcal{A} \text{ is true;} \\ 0, & \text{if } \mathcal{A} \text{ is false.} \end{cases}$$

This number belongs to $\{0, 1\}$, and is called the *truth value* of $\mathcal{A}$. For example,

$$[1+1=2] = 1, \qquad [1+1=1] = 0, \qquad [\text{there exist infinitely many primes}] = 1.$$

2.2 Solution

[...]

---

3 Exercise 3

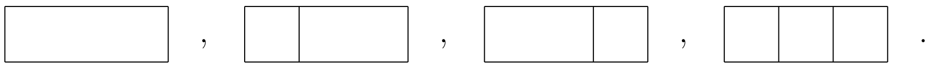
3.1 Problem

Again, we shall use the notations of [Math222, §1.1].

A *horimino* shall mean a rectangle of height 1 and positive width (i.e., formally speaking, a set of the form $\{(i,1), (i+1,1), \ldots, (j,1)\}$ for some integers $i \leq j$).

If S is a set of squares, then a *horimino tiling* of S will mean a set of disjoint horiminos whose union is S.

For example, here are all four horimino tilings of the rectangle $R_{3,1}$:



Let $n \in \mathbb{N}$. Find a simple expression for the number of all horimino tilings of the rectangle $R_{n,1}$.

[**Hint:** Make sure your answer works for $n = 0$ (you might need to handle this case separately).]

3.2 Solution

[...]

---

4 Exercise 4

4.1 Problem

Let $n \in \mathbb{N}$. Prove that

$$0^2 \cdot 1! + 1^2 \cdot 2! + 2^2 \cdot 3! + \cdots + (n-1)^2 \cdot n! = (n-2) \cdot (n+1)! + 2.$$

4.2 Solution

[...]

---

5 Exercise 5

5.1 Problem

Let $(u_0, u_1, u_2, \ldots)$ be a sequence of real numbers such that every $n \geq 1$ satisfies

$$u_n = nu_{n-1} + (-1)^n . \tag{1}$$

Prove that $u_n = (n-1)(u_{n-1} + u_{n-2})$ for each $n \geq 2$.

5.2 Remark

This shows that the recurrence $D_n = nD_{n-1} + (-1)^n$ for the derangement numbers implies the recurrence $D_n = (n-1)(D_{n-1} + D_{n-2})$.

5.3 Solution

[...]

---

6 Exercise 6

6.1 Problem

Let n and m be positive integers.

An n-tuple $(i_1, i_2, \ldots, i_n) \in \{0, 1, \ldots, m\}^n$ is said to be *even* if the sum $i_1 + i_2 + \cdots + i_n$ is even. (For example, the 4-tuple $(1, 0, 4, 1)$ is even, whereas $(1, 0, 3, 1)$ is not.)

(a) Find a formula for the number of all even n-tuples $(i_1, i_2, \ldots, i_n) \in \{0, 1, \ldots, m\}^n$ when m is odd.

(b) Find a formula for the number of all even n-tuples $(i_1, i_2, \ldots, i_n) \in \{0, 1, \ldots, m\}^n$ when m is even.

[**Hint:** Particular cases of this exercise (for $m = 1, 2, 3$) were done in [hw0s, Exercise 3] and [18f-hw1s, Exercises 2 and 1]. Can you generalize some of that reasoning?]

6.2 Solution

[...]

References

[Math222] Darij Grinberg, *Enumerative Combinatorics: class notes*, 16 December 2019. `http://www.cip.ifi.lmu.de/~grinberg/t/19fco/n/n.pdf` Also available on the mirror server `http://darijgrinberg.gitlab.io/t/19fco/n/n.pdf`
Caution: The numbering of theorems and formulas in this link might shift when the project gets updated; for a "frozen" version whose numbering is guaranteed to match that in the citations above,

see https://gitlab.com/darijgrinberg/darijgrinberg.gitlab.io/blob/2dab2743a181d5ba8fc145a661fd274bc37d03be/public/t/19fco/n/n.pdf

[18f-hw1s] Darij Grinberg, *UMN Fall 2018 Math 5705 homework set #1 with solutions*, http://www.cip.ifi.lmu.de/~grinberg/t/18f/hw1s.pdf

[hw0s] Darij Grinberg, *Drexel Fall 2019 Math 222 homework set #0 with solutions*, http://www.cip.ifi.lmu.de/~grinberg/t/19fco/hw0s.pdf