Next, a result of Euler:

Def. Let $n \in \mathbb{Z}$. Then:

$p_{odd}(n) := (\#$ of partitions of n into odd parts$)$;

$p_{dist}(n) := (\#$ of partitions of n into distinct parts$)$.

Ex.: $p_{odd}(7) = |\{(7), (3,3,1), (3,1,1,1,1), (1,1,1,1,1,1,1), (5,1,1)\}| = 5$;

$p_{dist}(7) = |\{(7), (6,1), (5,2), (4,3), (4,2,1)\}| = 5$.

Thm. 9.2 (Euler). $p_{odd}(n) = p_{dist}(n)$ $\forall n \in \mathbb{N}$.

1st proof. $$\sum_{n \geq 0} p_{odd}(n)\, x^n = \prod_{\substack{k \geq 1; \\ k \text{ odd}}} \frac{1}{1-x^k}$$ (analogous to Thm. 9.1);

$$\sum_{n \geq 0} p_{dist}(n)\, x^n = \prod_{k \geq 1} (1+x^k)$$ (also analogous to Thm. 9.1)

Thus, it remains to prove

(87) $$\prod_{\substack{k \geq 1; \\ k \text{ odd}}} \frac{1}{1-x^k} = \prod_{k \geq 1} (1+x^k).$$

First, prove

(88) $$\frac{1}{1-x} = (1+x)(1+x^2)(1+x^4)(1+x^8) \cdots = \prod_{i \geq 0} (1+x^{2^i}).$$

[1st proof of (88):

$$\text{RHS} = \sum_k x^k \cdot \underbrace{(\#\text{ of ways to write } k \text{ as a sum of distinct powers of } 2)}_{\substack{=1 \\ \text{(by binary representation)}}}$$

$$= \sum_k x^k = \frac{1}{1-x}.$$

2nd proof of (88):

$$\underbrace{(1-x)\cdot(1+x)}(1+x^2)(1+x^4)(1+x^8)\cdots$$

$$= \underbrace{(1-x^2)\cdot(1+x^2)}(1+x^4)(1+x^8)\cdots$$

$$= \underbrace{(1-x^4)\cdot(1+x^4)}(1+x^8)\cdots$$

$$= \underbrace{(1-x^8)\cdot(1+x^8)}\cdots$$

$$\cdots$$

$$= \cdots\cdots = 1.$$

More rigorously: To prove that $(1-x)\cdot RHS = 1$, it suffices to show that $[x^n]((1-x)\cdot RHS) = 0$ $\forall$ positive n. For any given positive n, we need to only perform finitely many steps of the computations above until we get an expression ~~which has none~~ in which x only appears in powers higher than x^n.]

For any odd $k \geq 1$, we can substitute x^k for x in (88); thus we get

$$\frac{1}{1-x^k} = \prod_{i\geq 0}\left(1+x^{k\cdot 2^i}\right).$$

Multiplying these over all odd k, we get

$$\prod_{\substack{k\geq 1;\\ k \text{ odd}}} \frac{1}{1-x^k} = \prod_{\substack{k\geq 1;\\ k \text{ odd}}} \prod_{i\geq 0}\left(1+x^{k\cdot 2^i}\right) = \prod_{m\geq 1}\left(1+x^m\right)$$

(since each $m \geq 1$ can be represented uniquely as $k\cdot 2^i$ with odd $k\geq 1$ and arbitrary $i\geq 0$)

$$= \prod_{k\geq 1}\left(1+x^k\right),$$ so (87) is proven. $\square$

2nd proof (sketch). Construct a bijection

A: {partitions of n into odd parts} $\to$ {partitions of n into distinct parts},

which transforms a partition by repeatedly merging 2 equal ~~parts~~ parts until no more equal parts can be found:

Ex: $(5,5,3,1,1,1) \mapsto (10,3,1,1,1) \mapsto (10,3,2,1)$.

Ex: $(5,3,1,1,1,1) \mapsto (5,3,2,1,1) \mapsto (5,3,2,2) \mapsto (5,4,3)$.

Why this is well-defined: not obvious.

One way to prove this is using the diamond lemma

Another way is by representing each part m of our partition as $k \cdot 2^i$ with odd $k \geq 1$ and arbitrary $i \geq 0$.

stem height

This lets us analyze A in terms of binary representation.

(A is called the Glaisher bijection.)

The inverse of A transforms a partition by repeatedly splitting even parts into two equal pieces. $\square$

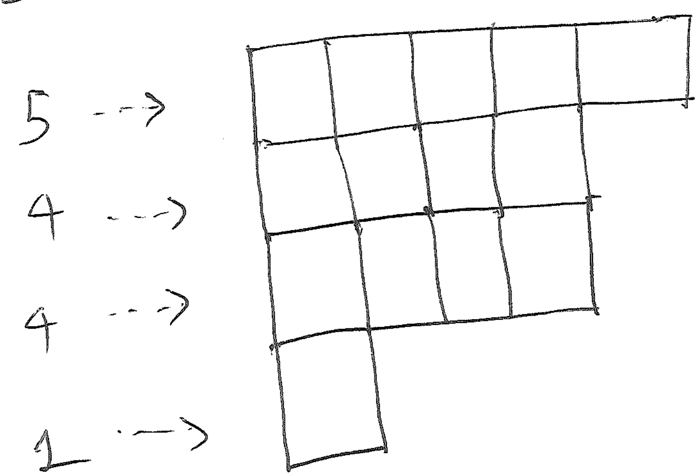
Prop. 9.3. Let $n \in \mathbb{N}$ and $k > 0$. Then,

$p_k(n) = $ (# of partitions of n whose largest part is k).

Proof sketch. Picture proof: e.g., let $k = 4$ and $n = 14$.

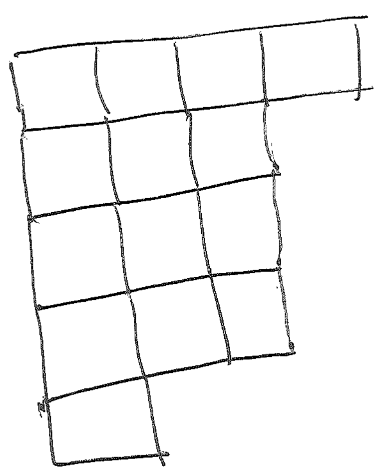
Start with the partition $\lambda = (5, 4, 4, 1)$ of n into k parts.

Draw a table of k left-aligned rows, where the length of each row equals the corresponding part of λ:



This table is called the Young diagram or Ferrers diagram of λ.

Now, flip the table around the $\diagdown$ diagonal:



The lengths of the rows of the resulting table again form a partition of n whose largest part is k. (In our example, this is $(4,3,3,3,1)$.)

This is a bijection (and is called <u>conjugation</u>). $\square$

9.2. Euler's ⬠ # theorem.

<u>Def.</u> For any $k \in \mathbb{Z}$, define $w_k \in \mathbb{N}$ by $w_k = \dfrac{(3k-1)k}{2}$.

This is called a <u>pentagonal number</u>.

<u>Thm. 9.4. (Euler's pentagonal number theorem).</u>

$$\prod_{k=1}^{\infty} (1-x^k) = \sum_{k \in \mathbb{Z}} (-1)^k x^{w_k}.$$

Thus,

$$\prod_{k=1}^{\infty}(1-x^k) = \cdots + x^{26} - x^{15} + x^7 - x^2 + 1 - x + x^5 - x^{12} + x^{22} - x^{35} \pm \cdots$$
$$= 1 - x - x^2 + x^5 + x^7 - x^{12} - x^{15} + x^{22} + x^{26} \pm \cdots$$

Cor. 9.5.

$$p(n) = p(n-1) + p(n-2) - p(n-5) - p(n-7) + p(n-12) + p(n-15) \pm \cdots \quad \forall n > 0.$$
$$= \sum_{k \neq 0} (-1)^{k-1} p(n-\omega_k)$$

Proof of Cor. 9.5 using Thm. 9.4.

$$(90) \qquad \underbrace{\left(\sum_{m \geq 0} p(m) x^m\right)}_{\substack{= \prod_{k \geq 1} \frac{1}{1-x^k} \\ \text{(by Thm. 9.1)}}} \underbrace{\left(\sum_{k \in \mathbb{Z}} (-1)^k x^{\omega_k}\right)}_{\substack{= \prod_{k \geq 1} (1-x^k) \\ \text{(by Thm. 9.4)}}} = 1,$$

Now, the x^n-coefficient on the LHS is

$$\sum_{\substack{m \geq 0; \\ k \in \mathbb{Z}; \\ m + \omega_k = n}} p(m) \cdot (-1)^k = \sum_{\substack{k \in \mathbb{Z}; \\ n - \omega_k \geq 0}} p(n - \omega_k) \cdot (-1)^k$$

$$= \sum_{k \in \mathbb{Z}} p(n - \omega_k) \cdot (-1)^k \qquad \left(\text{since } p(n-\omega_k) = 0 \text{ when } n - \omega_k < 0\right)$$

$$= \sum_{k \in \mathbb{Z}} (-1)^k \, p(n - \omega_k)$$

$$= \underbrace{(-1)^0}_{=1} \underbrace{p(n - \omega_0)}_{\substack{= p(n-0) \\ = p(n)}} + \sum_{\substack{k \in \mathbb{Z}; \\ k \neq 0}} (-1)^k \, p(n - \omega_k)$$

$$= p(n) + \sum_{k \neq 0} (-1)^k \, p(n - \omega_k).$$

But (for $n>0$) the x^n-coefficient on the RHS of (90) is 0.

Comparing yields

$$p(n) + \sum_{k\neq 0} (-1)^k \, p(n-\omega_k) = 0.$$

$\square$

Solve this for $p(n)$.

Rmk. A combinatorial proof of Cor. 9.5 appears in May 2 slides of ~~Sp~~ Spring 2018 Math 4707, §9.5.

9.3. Jacobi's triple product identity

We will prove a stronger result (than Thm. 9.4):

Thm. 9.6 (Jacobi's triple product identity).

$$\prod_{n>0} \left(\left(1+q^{2n-1} z\right)\left(1+q^{2n-1} z^{-1}\right)\left(1-q^{2n}\right) \right) = \sum_{\ell \in \mathbb{Z}} q^{\ell^2} z^{\ell}.$$

(The best way to make this rigorous is to formalize both sides as formal power series in q over the ring of Laurent series in z.

Alternatively, for what we will actually use them for, it suffices to take $q = x^a$ and $z = k \cdot x^b$ for two integers a & b with $a \geq |b|$ and $k \in \mathbb{Q}$. You can check that all of the factors & addends in Thm. 9.6 become proper FPS in x in this case.)

Proof of Thm. 9.4 using Thm. 9.6. Set $q = x^3$ and $z = -x$ in Thm. 9.6. You get

$$\prod_{n>0} \left(\left(1 - x^{(2n-1)3+1}\right) \left(1 - x^{(2n-1)3-1}\right) \left(1 - x^{(2n)3}\right) \right) = \sum_{\ell \in \mathbb{Z}} (-1)^\ell x^{3\ell^2 - \ell}.$$

But the LHS of this equality is simply

$$\prod_{k=1}^{\infty} \left(1 - x^{2k}\right),$$

since each $2k$ (with $k \geq 1$) can be uniquely represented either as $(2n-1)3+1$ or as $(2n-1)3-1$ or as $(2n)3$ for $n>0$. So the equality becomes

$$\prod_{k=1}^{\infty}\left(1-x^{2k}\right) = \sum_{\ell \in \mathbb{Z}} (-1)^{\ell} x^{3\ell^2-\ell} = \sum_{k\in\mathbb{Z}} (-1)^k \underbrace{x^{3k^2-k}}_{=x^{2\omega_k}}$$

(since $3k^2-k = (3k-1)k = 2\omega_k$)

$$= \sum_{k\in\mathbb{Z}} (-1)^k x^{2\omega_k}.$$

"Substituting $x^{1/2}$ for x" in this equality (= using the fact that any two FPSs f & g satisfying $f \circ x^2 = g \circ x^2$ must be equal), we obtain

$$\prod_{k=1}^{\infty}\left(1-x^{k}\right) = \sum_{k\in\mathbb{Z}} (-1)^k x^{\omega_k}.$$

$\square$

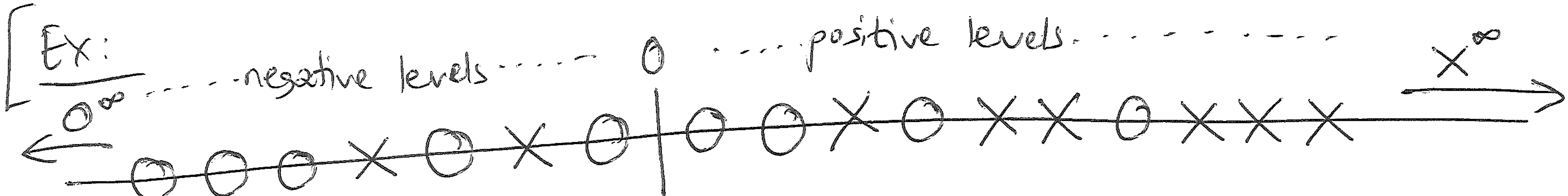
Proof of Thm. 9.6. This comes from Cameron's AC-notes, going back to Borcherds:

A level means a number of the form $p+\frac{1}{2}$ with $p \in \mathbb{Z}$.

A state is a set of levels which contains

- all but finitely many negative levels, and
- only finitely many positive levels.

[Ex:



Legend: O = a level contained in the state = "electrons";
X = — // — not contained — // — = "holes".]

For any state S, we define

- the energy of S to be $\sum_{\substack{p>0;\\ p\in S}} p - \sum_{\substack{p<0;\\ p\notin S}} p \in \{\frac{0}{2},\frac{1}{2},\frac{2}{2},\dots\}$;

- the particle number of S to be

($\#$ of $p>0$ such that $p\in S$) $-$ ($\#$ of $p<0$ such that $p\notin S$)

$\in \mathbb{Z}$.

[In the example above:

the energy is $\frac{1}{2}+\frac{3}{2}+\frac{7}{2}+\frac{13}{2}-\frac{-3}{2}-\frac{-7}{2}$;

the particle number is $4-2=2$.]

We want to prove Thm. 9.6. Rewrite it by replacing q by $q^{1/2}$:

$$\prod_{n>0}\left(\left(1+q^{n-1/2}z\right)\left(1+q^{n-1/2}z^{-1}\right)\left(1-q^n\right)\right)=\sum_{\ell\in\mathbb{Z}} q^{\ell^2/2}z^{\ell}.$$

Moving the $(1-q^n)$'s to the RHS, we rewrite this as

$$\prod_{n>0} \left(\left(1+q^{n-1/2} z\right)\left(1+q^{n-1/2} z^{-1}\right) \right)$$

$$(93) \qquad = \left(\sum_{\ell \in \mathbb{Z}} q^{\ell^2/2} z^{\ell} \right) \left(\prod_{n>0} \left(1-q^n\right)^{-1} \right).$$

~~Claim 1: For all Let $m \in \frac{1}{2}\mathbb{Z}$~~

Let $\frac{1}{2}\mathbb{N} := \left\{ \frac{0}{2}, \frac{1}{2}, \frac{2}{2}, \frac{3}{2}, \ldots \right\}$.

Claim 1: Let ~~$m \in \mathbb{N}$ and $\ell \in$~~ $m \in \frac{1}{2}\mathbb{N}$ and $\ell \in \mathbb{Z}$.

(a) The coefficient of $q^m z^{\ell}$ on the LHS of (93) is the # of states with energy m & particle number ℓ.

(b) Same for the RHS.

Once Claim 1 will be proven, (93) will follow, & thus Thm. 9.6 will be proven.

Proof of Claim 1:

(a) LHS

$$= \prod_{n>0} \left(\left(1+q^{n-1/2} z\right) \left(1+q^{n-1/2} z^{-1}\right) \right)$$

$$\cancel{= \prod_{m \text{ is a level}} \left(1+q^{|m|} z\right)}$$

$$= \left(\prod_{p \text{ is a positive level}} \left(1+q^{p} z\right) \right) \cdot \left(\prod_{p \text{ is a negative level}} \left(1+q^{-p} z^{-1}\right) \right).$$

If we expand this product, we get a sum over all states S; the addend is $q^{\text{energy}(S)} \cdot z^{\text{particle number}(S)}$.

Claim 1 (a) follows.

(b) RHS

$$= \left(\sum_{l \in \mathbb{Z}} q^{l^2/2} z^l \right) \underbrace{\left(\prod_{n>0} (1-q^n)^{-1} \right)}$$

$$\overline{\overline{\text{Thm. 9.1}}} \sum_{n \geq 0} p(n) q^n$$

$$= \sum_{\substack{\lambda \text{ is a} \\ \text{partition}}} q^{|\lambda|}$$

(where $|\lambda| := \lambda_1 + \lambda_2 + \dots + \lambda_k$ for any partition $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k)$)

$$= \left(\sum_{l \in \mathbb{Z}} q^{l^2/2} z^l \right) \sum_{\substack{\lambda \text{ is a} \\ \text{partition}}} q^{|\lambda|}$$

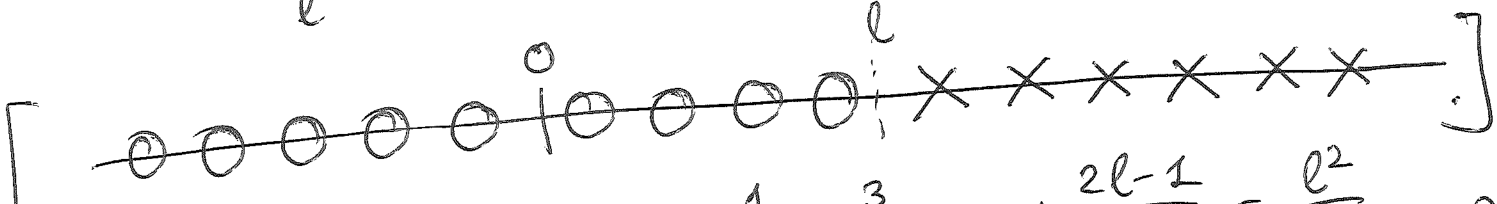
$$= \sum_{l \in \mathbb{Z}} \sum_{\substack{\lambda \text{ is a} \\ \text{partition}}} q^{l^2/2 + |\lambda|} z^l .$$

Thus, in order to prove Claim 1 (b), we need to find a bijection

$$\Phi : \{\text{partitions } \lambda \text{ with } \ell^2/2 + |\lambda| = m\}$$
$$\to \{\text{states with energy } m \text{ \& particle number } \ell\}$$

for fixed m, ℓ.

We define the state G_ℓ ("the ℓ-ground state") by
$$G_\ell = \{\text{all levels} < \ell\}.$$



The energy of G_ℓ is $\frac{1}{2} + \frac{3}{2} + \dots + \frac{2\ell-1}{2} = \frac{\ell^2}{2}$, and its particle number is ℓ.

If S is a state, and if $p \in S$, and if ~~$q \in \mathbb{N}$ is a~~ q is a positive integer such that $p+q \notin S$, we let

~~jump $p \to p+q$~~ ~~jump$_q(S$~~ $\text{jump}_{p,q}(S) := (S \setminus \{p\}) \cup \{p+q\}$.

We say that $\text{jump}_{p,q}(S)$ is obtained from S by letting the electron ~~at~~ at level p jump q steps (to the right).
Note that $\text{jump}_{p,q}(S)$ has the ~~part~~ same particle number as S, whereas its energy is q higher than that of S.
So a jumping particle raises the energy but keeps the particle number unchanged.
For any partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$, we define the state $E_{\ell,\lambda}$ (called an "excited state") to be the state

$$\text{jump}_{\ell-k+\frac{1}{2},\lambda_k}\left(\cdots \left(\text{jump}_{\ell-2+\frac{1}{2},\lambda_2}\left(\text{jump}_{\ell-1+\frac{1}{2},\lambda_1}(G_\ell) \right) \right) \cdots \right)$$

$$= \{ \text{all levels} < \ell - k \} \cup \{ \ell - i + \tfrac{1}{2} + \lambda_i \mid i \in [k] \}.$$

This state $E_{\ell,\lambda}$ has energy $\frac{\ell^2}{2} + |\lambda|$ and particle number ℓ. Furthermore, every state with particle number ℓ can be written as $E_{\ell,\lambda}$ for a unique partition λ.

$\Longrightarrow$ We get a bijection

$$\Phi : \{\text{partitions } \lambda \text{ with } \ell^2/2 + |\lambda| = m\} \to \{\text{states with energy } m \text{ \& particle number } \ell\},$$

$$\lambda \mapsto E_{\ell,\lambda}.$$

This completes the proof of Claim 1 (b).]

So Thm. 9.6 is proven. $\square$

Reading recommendation:

[Hirschhorn: "Partial Fractions and Four Classical Theorems in Number Theory", AMM 107 (2000), pp. 260–264].