Prop. 4.4. For every $\sigma \in S_n$, we have $\ell(\sigma^{-1}) = \ell(\sigma)$.

Proof. The map

$$\begin{aligned} \{\text{inversions of } \sigma\} &\to \{\text{inversions of } \sigma^{-1}\}, \\ (i,j) &\mapsto (\sigma(j), \sigma(i)) \end{aligned}$$

is well-defined and bijective (its inverse map sends (u,v) to $(\sigma^{-1}(v), \sigma^{-1}(u))$). For details: [detnotes, Exercise 5.2 (f)]. $\square$

Prop. 4.5. Let $n \in \mathbb{N}$, $\sigma \in S_n$ and $k \in [n-1]$.

(a) We have

$$\ell(\sigma \circ s_k) = \begin{cases} \ell(\sigma) + 1, & \text{if } \sigma(k) < \sigma(k+1); \\ \ell(\sigma) - 1, & \text{if } \sigma(k) > \sigma(k+1). \end{cases}$$

(b) We have

$$\ell(s_k \circ \sigma) = \begin{cases} \ell(\sigma) + 1, & \text{if } \sigma^{-1}(k) < \sigma^{-1}(k+1); \\ \ell(\sigma) - 1, & \text{if } \sigma^{-1}(k) > \sigma^{-1}(k+1). \end{cases}$$

[Note: $\sigma^{-1}(i)$ is the <u>position</u> in which i appears in the one-line notation of σ.
E.g., if $\sigma = [5,1,2,3,6,4]$, then $\sigma^{-1}(6) = 5$.]

<u>Example:</u> Let $\sigma = [3, 1, 5, 2, 4] \in S_5$ (in one-line notation).

Then:

- the inversions of σ are $(1,2)$, $(1,4)$, $(3,4)$, $(3,5)$;
- the length of σ is 4.

Now, $\sigma \circ s_1 = [1, 3, 5, 2, 4] \in S_5$, and:

- the inversions of $\sigma \circ s_1$ are $(2,4)$, $(3,4)$, $(3,5)$;
- the length of $\sigma \circ s_1$ is 3.

Also, $\sigma \circ s_2 = [3, 5, 1, 2, 4] \in S_5$, and:

- the inversions of $\sigma \circ s_2$ are $(1,3)$, $(1,4)$, $(2,3)$, $(2,4)$, $(2,5)$;
- the length of $\sigma \circ s_2$ is 5.

This illustrates Prop. 4.5 (a).

<u>Proof of Prop. 4.5.</u> (a) The one-line notation of $\sigma \circ s_k$ is obtained from the one-line notation of σ by swapping the k-th and $(k+1)$-st entries. The effect on inversions is:

~~If $\sigma(k) < \sigma(k$~~

- If $\sigma(k) < \sigma(k+1)$, then $\sigma \circ s_k$ has a new inversion $(k, k+1)$.

  If $\sigma(k) > \sigma(k+1)$, then σ had an inversion $(k, k+1)$, which $\sigma \circ s_k$ no longer has.
- Any inversion (i, j) of σ with $(i, j) \neq (k, k+1)$ gives rise to an inversion $(s_k(i), s_k(j))$ of $\sigma \circ s_k$.
- This covers all inversions of $\sigma \circ s_k$.

Thus,
$$\ell(\sigma \circ s_k) = \begin{cases} \ell(\sigma) + 1, & \text{if } \sigma(k) < \sigma(k+1); \\ \ell(\sigma) - 1, & \text{if } \sigma(k) > \sigma(k+1). \end{cases}$$

This proves (a).

(b) Applying part (a) to σ^{-1} instead of σ, we get

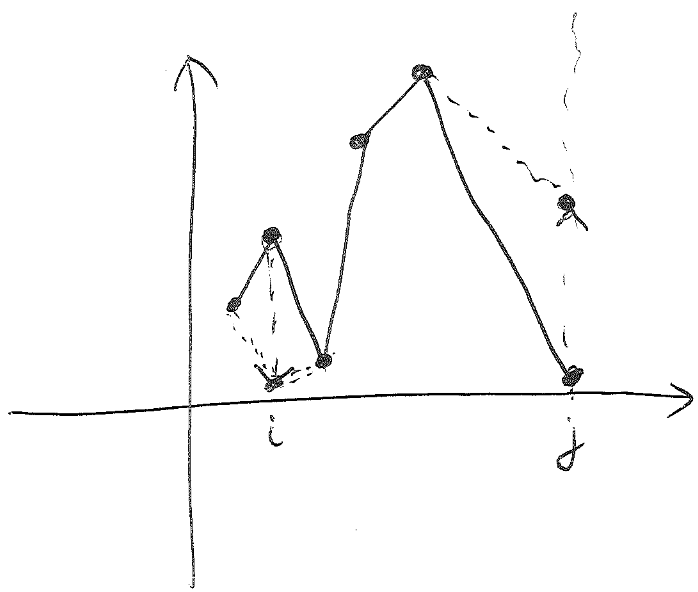
$$(27) \qquad \ell(\sigma^{-1} \circ s_k) = \begin{cases} \ell(\sigma^{-1}) + 1, & \text{if } \sigma^{-1}(k) < \sigma^{-1}(k+1); \\ \ell(\sigma^{-1}) - 1, & \text{if } \sigma^{-1}(k) > \sigma^{-1}(k+1). \end{cases}$$

But Prop. 4.4 yields $\ell(\sigma^{-1}) = \ell(\sigma)$. Also,

$$\ell(\sigma^{-1}\circ \underbrace{s_k}_{=s_k^{-1}}) = \ell(\underbrace{\sigma^{-1}\circ s_k^{-1}}_{=(s_k\circ\sigma)^{-1}}) = \ell((s_k\circ\sigma)^{-1}) \overset{\text{Prop. 4.4}}{=\!=} \ell(s_k\circ\sigma).$$

Thus, (27) transforms into the claim of (b).
(For details, see [detnotes, Exercise 5.2(a)].)

<u>Remark:</u> Let $n\in\mathbb{N}$ and $\sigma\in S_n$. Let $i,j\in[n]$ be such that $i<j$ and $\sigma(i)>\sigma(j)$. Is $\ell(\sigma\circ t_{i,j})<\ell(\sigma)$?



<u>Ex:</u> $\sigma=[1,\underline{6},4,2,\underline{3},5]$,
$i=2,\quad j=5,$
$\sigma\circ t_{i,j}=[1,\underline{3},4,2,\underline{6},5]$.

$\mapsto$ HW #4.

Thm. 4.6. Let $n \in \mathbb{N}$ and $\sigma \in S_n$.

A simple transposition (or, for short, a simple) will mean any of the transpositions $s_1, s_2, \ldots, s_{n-1}$.

(a) We can write σ as a composition of $\ell(\sigma)$ simples.

(b) $\ell(\sigma)$ is the smallest $p \in \mathbb{N}$ such that we can write σ as a ~~permut~~ composition of p simples.

~~Proof~~ [Keep in mind: The composition of 0 simples is id.]

Example: In S_4, we have

$$\underbrace{[4,1,3,2]}_{\text{one-line not.}} = \underbrace{s_2 s_3 s_2}_{= s_3 s_2 s_3} s_1 = s_3 s_2 \underbrace{s_3 s_1}_{= s_1 s_3} = s_3 s_2 s_1 s_3$$

$$= s_2 s_1 s_1 s_3 s_2 s_1 = \ldots\ldots$$

Proof of Thm. 4.6. (a) Induction on $\ell(\sigma)$:

Base: If $\ell(\sigma) = 0$, then $\sigma = \mathrm{id}$, so σ is a composition of 0 simples.

Step: Fix $h \in \mathbb{N}$. Assume (as the IH) that Thm. 4.6 (a) holds for $\ell(\sigma) = h$.

Now, let $\sigma \in S_n$ be such that $\ell(\sigma) = h+1$.

Then, $\ell(\sigma) = h+1 > 0$, so $\sigma \neq \mathrm{id}$.

Hence, $\exists\, k \in [n-1]$ such that $\sigma(k) > \sigma(k+1)$.

(otherwise, we would have $\sigma(1) \leq \sigma(2) \leq \cdots \leq \sigma(n)$
$\Longrightarrow \sigma(1) < \sigma(2) < \cdots < \sigma(n)$
$\Longrightarrow \sigma = \mathrm{id}$.)

Fix such a k. Prop. 4.5 (a) yields

$$\ell(\sigma \circ s_k) = \ell(\sigma) - 1 \quad (\text{since } \sigma(k) > \sigma(k+1))$$
$$= h \quad (\text{since } \ell(\sigma) = h+1).$$

Hence, the IH (applied to $\sigma \circ s_k$ instead of σ) yields that we can write $\sigma \circ s_k$ as a composition of $\ell(\sigma \circ s_k) = h$ simples:

$$\sigma \circ s_k = s_{i_1} \circ s_{i_2} \circ \cdots \circ s_{i_h}.$$

Thus,

$$\sigma = s_{i_1} \circ s_{i_2} \circ \cdots \circ s_{i_h} \circ \underbrace{s_k^{-1}}_{= s_k}$$
$$= s_{i_1} \circ s_{i_2} \circ \cdots \circ s_{i_h} \circ s_k.$$

This shows that we can write σ as a composition of $h+1 = \ell(\sigma)$ simples. Thus, Thm. 4.6 (a) holds for $\ell(\sigma) = h+1$. This completes the inductive proof of Thm. 4.6 (a).

[The idea behind this proof is called "~~bub~~bblesort".]

(b) Prop. 4.5 (a) yields

(28) $$\ell(\sigma \circ s_k) \leq \ell(\sigma) + 1 \qquad \forall \sigma \in S_n \text{ and } k \in [n-1].$$

Thus, $\forall k_1, k_2, \ldots, k_p \in [n-1]$, we have

(29) $$\begin{aligned} \ell(s_{k_1} \circ s_{k_2} \circ \cdots \circ s_{k_p}) &\underset{(28)}{\leq} \ell(s_{k_1} \circ s_{k_2} \circ \cdots \circ s_{k_{p-1}}) + 1 \\ &\underset{(28)}{\leq} \ell(s_{k_1} \circ s_{k_2} \circ \cdots \circ s_{k_{p-2}}) + 2 \\ &\underset{(28)}{\leq} \cdots\cdots \underset{(28)}{\leq} \underbrace{\ell(\mathrm{id})}_{=0} + p = p. \end{aligned}$$

Now, if σ was the composition of $p < \ell(\sigma)$ simples $s_{k_1}, s_{k_2}, \ldots, s_{k_p}$, then (29) would rewrite as $\ell(\sigma) \leq p$, which would contradict $p < \ell(\sigma)$. So (b) is proven.

(For details: [detnotes, Exercise 5.2(g)].) $\square$

Cor. 4.7. Let $n \in \mathbb{N}$.

(a) We have $\ell(\sigma\tau) \equiv \ell(\sigma) + \ell(\tau) \mod 2$ for all $\sigma \in S_n$ and $\tau \in S_n$.

(b) We have $\ell(\sigma\tau) \leq \ell(\sigma) + \ell(\tau)$ for all $\sigma \in S_n$ and $\tau \in S_n$.

(c) If $\sigma = s_{k_1} \circ s_{k_2} \circ \cdots \circ s_{k_q}$, then $q \equiv \ell(\sigma) \mod 2$.

Proof. [detnotes, Exercises 5.2 and 5.3]. $\square$

Prop. 4.8. Let $n \in \mathbb{N}$.

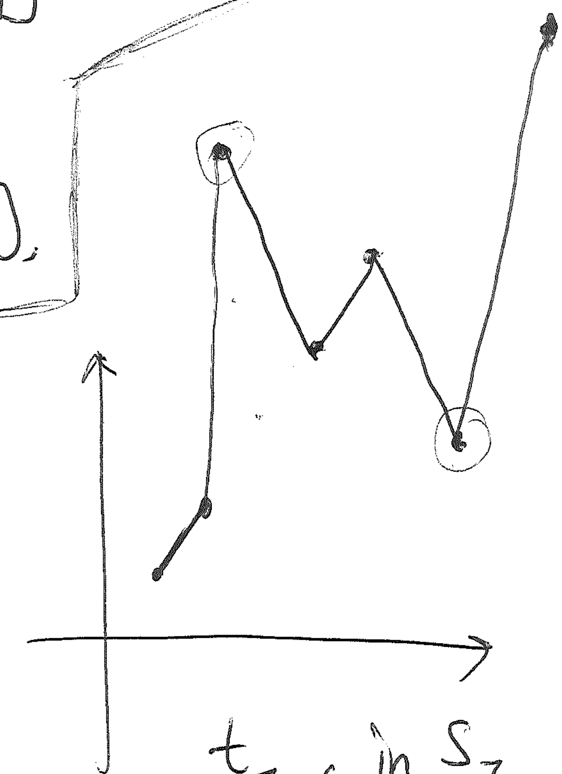
(a) We have $\ell(s_k) = 1$ for any $k \in [n-1]$.

(b) We have $\ell(t_{i,j}) = 2|i-j| - 1$ for any distinct $i, j \in [n]$.

(c) We have $\ell(\operatorname{cyc}_{i,i+1,\ldots,i+k-1}) = k-1 \ \forall i, k$.

(d) We have $\ell(\operatorname{cyc}_{i_1,i_2,\ldots,i_k}) \geq k-1$ for any distinct $i_1, i_2, \ldots, i_k \in [n]$.

(e) We have $\ell(\mathrm{id}) = 0$ and $\ell(w_0) = \binom{n}{2}$.



$t_{3,6}$ in S_7

Proof. (a) follows from (b).
(b) is [detnotes, Exercise 5.10].
(c), (d) are [detnotes, Exercise 5.16].
(e) is trivial. $\square$

Rmk. For a given k and n, how many $\sigma \in S_n$ have $\ell(\sigma) = k$?

- The # of $\sigma \in S_n$ having $\ell(\sigma) = 0$ is 1 (namely, just $\sigma = \mathrm{id}$).
- The # of — // — 1 is $n-1$ (namely, just $\sigma = s_i$ with $i \in [n-1]$).
- The # of $\sigma \in S_n$ having $\ell(\sigma) = 2$ is $n(n+1)/2$.

(Proof: Such σ have the form $\sigma = s_i s_j$ for $i \neq j$ (by ~~for $\sigma = s_i s_j$ with $i \neq j$~~ Thm. 4.6).
If $i > j+1$, we can rewrite them as $\sigma = s_j s_i$.
So WLOG assume $i \leq j+1$.
This gives $\sum_{j=1}^{n} j = n(n+1)/2$ options for (i, j).
These all yield different permutations σ.)

What about the general case?

There is no explicit formula, but there is a generating function:

Prop. 4.9. Let $n \in \mathbb{N}$. Then,

$$\sum_{w \in S_n} x^{\ell(w)} = \prod_{i=1}^{n-1} \left(1 + x + x^2 + \dots + x^i\right)$$

$$= (1+x) \cdot (1+x+x^2) \cdot (1+x+x^2+x^3) \cdot \ldots \cdot (1+x+x^2+\dots+x^{n-1}).$$

□

Proof. [detnotes, §5.8].

Def. Let $n \in \mathbb{N}$, $\sigma \in S_n$ and $k \in [n-1]$.

We say that k is a *descent* of σ if $\sigma(k) > \sigma(k+1)$.

The *descent set* of σ, called $\operatorname{Des} \sigma$, is the set of all descents of σ.

Exercise: Fix $n \geq 4$.

(a) How many $\sigma \in S_n$ have 0 descents?

(b) ——— // ——— 1 descent?

(c) ——— // ——— $n-1$ descents?

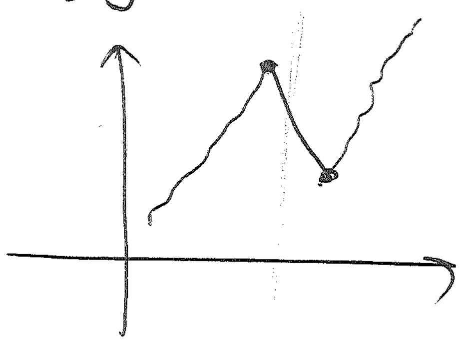
(d) ——— // ——— satisfy $1 \in \operatorname{Des} \sigma$? (i.e., $\sigma(1) > \sigma(2)$)

(e) ——— // ——— satisfy $1, 2 \in \operatorname{Des} \sigma$? (i.e., $\sigma(1) > \sigma(2) > \sigma(3)$)

(f) ——— // ——— satisfy $1, 3 \in \operatorname{Des} \sigma$? (i.e., $\sigma(1) > \sigma(2)$ and $\sigma(3) > \sigma(4)$)

Answer: (a) Only 1, namely id.

~~(b) Exactly $n-1$, namely s_k for $k \in [n-1]$.~~



(d) The answer is $\frac{n!}{2}$.

First proof: The map

$$\{\sigma \in S_n \mid \sigma(1) > \sigma(2)\} \to \{\sigma \in S_n \mid \sigma(1) < \sigma(2)\},$$
$$\sigma \mapsto \sigma \circ s_1$$

is bijective. So each of the 2 sets is half as large as S_n.

Second proof: To construct $\sigma \in S_n$ satisfying $\sigma(1) > \sigma(2)$, proceed as follows:

- Choose the set $\{\sigma(1), \sigma(2)\}$.
  There are $\binom{n}{2}$ options.
  Thus, $\sigma(1)$ and $\sigma(2)$ are already chosen (as $\sigma(1) > \sigma(2)$).
- Choose $\sigma(3), \sigma(4), \ldots, \sigma(n)$.
  There are $(n-2)!$ options.

$\Rightarrow$ The total # is $\binom{n}{2} \cdot (n-2)! = \frac{n!}{2!} = \frac{n!}{2}$.

(e) $\frac{n!}{3!}$.

(f) $\frac{n!}{2! \cdot 2!} = \frac{n!}{4}$.

(See Spring 2018 Math 4707 Mt 1 §0.2 for details.)

(b) First of all, ~~given~~ fix $i \in [n-1]$. The # of $\sigma \in S_n$ satisfying $\underbrace{\operatorname{Des}\sigma = \{i\}}$ is what?

$\Longleftrightarrow \sigma(1) < \sigma(2) < \cdots < \sigma(i) > \sigma(i+1) < \sigma(i+2) < \cdots < \sigma(n)$.

We have

$$(\# \text{ of } \sigma \in S_n \text{ satisfying } \sigma(1) < \sigma(2) < \cdots < \sigma(i) \text{ and } \sigma(i+1) < \sigma(i+2) < \cdots < \sigma(n))$$

$$= \binom{n}{i} \quad (\text{since it is enough to choose } \{\sigma(1), \sigma(2), \ldots, \sigma(i)\}).$$

All but one of these σ's satisfy $\sigma(i) > \sigma(i+1)$.

Thus,

$$(\# \text{ of } \sigma \in S_n \text{ satisfying } \sigma(1) < \sigma(2) < \cdots < \sigma(i) > \sigma(i+1) < \sigma(i+2) < \cdots < \sigma(n))$$

$$= \binom{n}{i} - 1.$$

~~Now,~~ In other words,

$$(\# \text{ of } \sigma \in S_n \text{ satisfying } \operatorname{Des} \sigma = \{i\}) = \binom{n}{i} - 1.$$

Summing this over all $i \in [n-1]$, we get

$$(\# \text{ of } \sigma \in S_n \text{ having exactly } 1 \text{ descent})$$

$$= \sum_{i=1}^{n-1} \left(\binom{n}{i} - 1 \right) = \underbrace{\sum_{i=1}^{n-1} \binom{n}{i}}_{=2^n - 2} - (n-1) = 2^n - (n+1).$$

(c) Only 1 permutation $\sigma \in S_n$ has $n-1$ descents, namely w_0.

4.4. Signs

Def. Let $n \in \mathbb{N}$. The sign of a permutation $\sigma \in S_n$ is $(-1)^{\ell(\sigma)}$. It is called $(-1)^\sigma$ or $\operatorname{sgn}(\sigma)$ or $\operatorname{sign}(\sigma)$ or $\varepsilon(\sigma)$.

Thm. 4.10. Let $n \in \mathbb{N}$.

(a) $(-1)^{\mathrm{id}} = 1$.

(b) $(-1)^{t_{i,j}} = -1$.

(c) $(-1)^{\mathrm{cyc}_{i_1, i_2, \ldots, i_k}} = (-1)^{k-1}$ $\forall$ distinct $i_1, i_2, \ldots, i_k \in [n]$.

(d) $(-1)^{\sigma\tau} = (-1)^\sigma (-1)^\tau$ $\forall \sigma, \tau \in S_n$.

(e) $(-1)^{\sigma^{-1}} = (-1)^\sigma$ $\forall \sigma \in S_n$. (The LHS is to be read as $(-1)^{(\sigma^{-1})}$.)

(f) $(-1)^{\sigma_1 \sigma_2 \cdots \sigma_p} = (-1)^{\sigma_1} (-1)^{\sigma_2} \cdots (-1)^{\sigma_p}$ $\forall \sigma_1, \sigma_2, \ldots, \sigma_p \in S_n$.

(g) $(-1)^{\sigma\tau\sigma^{-1}} = (-1)^\tau$ $\forall \sigma, \tau \in S_n$.

(h) $(-1)^{\sigma} = \prod_{1\leq i<j\leq n} \frac{\sigma(i)-\sigma(j)}{i-j}$. $\forall \sigma\in S_n$.

(i) If $x_1, x_2, \ldots, x_n$ are any n numbers, and $\sigma\in S_n$, then

$$\prod_{1\leq i<j\leq n} (x_{\sigma(i)} - x_{\sigma(j)}) = (-1)^{\sigma} \prod_{1\leq i<j\leq n} (x_i - x_j).$$

Proof. (a) $(-1)^{id} = (-1)^{\ell(id)} = (-1)^0 = 1$.

(b) Prop. 4.8 (b) yields that $\ell(t_{i,j}) = 2|i-j|-1$ is odd.

(d) $(-1)^{\sigma\tau} = (-1)^{\ell(\sigma\tau)} = (-1)^{\ell(\sigma)+\ell(\tau)}$ (by Cor. 4.7 (a))

$$= \underbrace{(-1)^{\ell(\sigma)}}_{=(-1)^{\sigma}} \cdot \underbrace{(-1)^{\ell(\tau)}}_{=(-1)^{\tau}} = (-1)^{\sigma}\cdot(-1)^{\tau}.$$

(e) $(-1)^{\sigma^{-1}} = (-1)^{\ell(\sigma^{-1})} = (-1)^{\ell(\sigma)}$ (by Prop. 4.4)

$= (-1)^{\sigma}$.

(f) Use induction on p and parts (a) & (d).

(g) $(-1)^{\sigma\tau\sigma^{-1}} \overset{(f)}{=\!=} (-1)^{\sigma} (-1)^{\tau} \underbrace{(-1)^{\sigma^{-1}}}_{\overset{(e)}{=} (-1)^{\sigma}}$

$$= \underbrace{\left(\left(-1\right)^{\sigma}\right)^2}_{\substack{=(\pm 1)^2 \\ =1}} (-1)^{\tau} = (-1)^{\tau}.$$

(c) Prop. 4.3 (a) says $\operatorname{cyc}_{i_1, i_2, \ldots, i_k} = t_{i_1, i_2} t_{i_2, i_3} \cdots t_{i_{k-1}, i_k}$.

Hence,

$$(-1)^{\operatorname{cyc}_{i_1, i_2, \ldots, i_k}} = (-1)^{t_{i_1, i_2} t_{i_2, i_3} \cdots t_{i_{k-1}, i_k}}$$

$$\overset{\text{part (f)}}{=\!=} (-1)^{t_{i_1, i_2}} (-1)^{t_{i_2, i_3}} \cdots (-1)^{t_{i_{k-1}, i_k}}$$

$$\overset{\text{part (b)}}{=\!=} \underbrace{(-1) \cdot (-1) \cdot \cdots \cdot (-1)}_{k-1 \text{ factors}} = (-1)^{k-1}.$$

(h), (i): see [detnotes, Exercise 5.13]. $\square$