

Mathematics via Problems, part 1: Algebra by *Arkadiy Skopenkov*, MSRI/AMS 2021;

Mathematics via Problems, part 2: Geometry by *Alexey A. Zaslavsky and Mikhail B. Skopenkov*, MSRI/AMS 2021;

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The mathematical problem book is a literary genre that predates many genres of fiction. In the 8th century AD, Alcuin of York collected 53 logical and arithmetic puzzles “to sharpen the young”. As mathematical competitions spread in the 20th century, the genre expanded, and entire series of problem collections were published, often organized by topic or origin.

The 1950s saw the appearance of books in the Soviet Union that attempted to combine the pleasant with the instructive. Rather than present a hodgepodge of isolated curiosities, the authors carefully selected problems that would lead the reader to rediscover entire fields of mathematics semi-autonomously. Unlike a textbook that spells out each proof, algorithm or example in detail, such a problem book would guide the reader to self-discover important landmarks through a sequence of leading questions. When chosen well and arranged in a good order, these questions would be nowhere as difficult as the original problems that motivated the creation of the theory, but still challenging enough to make for a memorable experience, thus strengthening the learning and motivating the tools and theorems that crystallize out of their solution. Three classic texts following this approach are [DU], [YY] and [G+].

In the West, a similar paradigm had arisen in the 1910s under the name of “Moore method”, nowadays better known as “IBL” (“inquiry-based learning”). However, the two formats differ in how they handle the case of the reader getting stuck on a question. Soviet-style “learning by problem-solving” books include solutions to all problems, which can be consulted after one runs out of ideas. These solutions are typically put in a separate chapter, which often spans more than half of the book. Sometimes, even an intermediate (short) chapter with hints appears, to avoid needlessly “spoiling” an entire problem. In contrast, IBL texts would typically eschew any hints; they are meant to be read in groups, with hurdles being surmounted through discussion and collaboration (and perhaps some guidance from a trainer). A popular example of an IBL text is [Bo].

The two books under review come from the Soviet tradition; they are the first two volumes of a (so far) three-volume series, which intends to cover a variety of mathematical topics by way of problems. However, unlike many other Russian texts, they don’t always give complete solutions. The first one actually does so on the rarest occasions, while leaving most problems with mere hints, answers or nothing at all; in this, it bears more resemblance to American IBL texts.

Both Soviet-style and IBL-style problem books have two audiences: **(a)** self-learners who are reading the books on their own with no outside help, **(b)** learners working in groups and under guidance of a trainer, and **(c)** lecturers and trainers using the books as raw materials. It is easiest for a book to be useful to

audience **(c)**, as such readers often have prior familiarity with the material and thus are unlikely to get stuck or confused. Audience **(b)** can rely on a trainer to smoothen the rough edges of such a book when there are any. Audience **(a)**, however, can be significantly hindered by pedagogical flaws of the text, such as out-of-order problems, missing hints or lacking guidance on which problems are crucial and which can be skipped.

Part 1: Algebra.

The first book is concerned with *algebra* in the most classical sense: the solution of polynomial equations (usually in one variable, over \mathbb{Z} , \mathbb{Q} , \mathbb{R} or \mathbb{C} or sometimes modulo n). It starts at the level of elementary olympiad problems, and ends in a rather advanced Galois-theoretical jungle. While necessarily introducing some algebraic concepts such as number fields and their extensions, it eschews basic abstractions such as groups, often in favor of examples and ad-hoc constructions.

Chapters 1-2 are a crash course in elementary number theory, leading from divisibility all the way to quadratic reciprocity in just 30 pages. Negative numbers are treated inconsistently, and some important results are left unproven. Hints are few and far between, and not always for the hardest or most important results (e.g., all general claims about primitive roots are left as exercises with no hints). The order of the material also makes for a bumpy ride; in particular, prime numbers are treated before remainders and gcds, which needlessly complicates some problems. With the exception of a few nonstandard exercises and the rather unique (but not overly interesting) Section 1.7, all of the material here can be found in classical textbooks such as [N+], [A+], [Bu], [Ta] or [DD], which are less hurried and can be read in a semi-IBL fashion if so desired by treating the theorems as problems. It would perhaps be good and topical to include some properties of Dedekind sums, which are elementary but rarely appear in elementary literature.

Chapter 3 is devoted to polynomials and complex numbers. Section 3.2, which surveys various methods for solving cubic and quartic equations, is a highlight; it derives Cardano's and Ferrari's formulas elegantly and elementarily (and gives good hints for the problems, unlike many other parts of the book).¹ The other sections are a mixed bag: They too include some interesting material, marred by difficulty spikes and cryptic hints. (For example, the hint to problem 3.6.3 is good, but uses the notion of "lexicographic order" without definition.) Section 3.9 takes a side trip into discrete geometry, using some of the prior results to refute Borsuk's conjecture on sets in n -dimensional space. This is deep, very technical and far from the topic of this book; I am not sure if it is treated any better here than in [AZ1].

¹A particularly nice trick is the use of the identity $x^3 + y^3 + z^3 - 3xyz = (x + y + z)(x^2 + y^2 + z^2 - yz - zx - xy)$ for solving the cubic equation $x^3 + px + q = 0$. Indeed, this identity shows that $x^3 + y^3 + z^3 - 3xyz = 0$ whenever x, y, z are three numbers satisfying $x + y + z = 0$. Thus, $x = -u - v$ is a root of the polynomial $x^3 + px + q$ when $p = 3uv$ and $q = u^3 + v^3$; hence, it only remains to write the known coefficients p and q as $3uv$ and $u^3 + v^3$, which is easy. Cardano's formula thus follows.

Chapter 4, about permutations, is particularly unfortunate. In Section 4.3 (which, to my knowledge, is not used in the rest of the book), the fundamentals of enumeration under group action are presented (culminating in the not-really-Burnside lemma with applications), in a language that eschews the notion of a group (even of a permutation group!) in favor of confusing ad-hoc visualizations involving carousels, trains and cockroaches. In the condensed form they are presented here (with telegraphic hints and – surprisingly for this subject! – not a single picture), they are almost incomprehensible. Had I not known the proof of Burnside’s lemma (4.3.10) beforehand, I would not have understood the “train station” metaphor used to motivate it here.² How is this approach any better than [Gu, §§6.1–6.2] or [Bo, §6]? And why assume that a reader of an algebra book is afraid of formulas? Sections 4.1 and 4.2, while written better, contain nothing that I could not find in standard texts on combinatorics, except for the (nice) problem 4.2.6.

Chapter 5 is a short introduction to olympiad-level inequalities: Jensen, AM–GM, Cauchy–Schwarz, Hölder and their ilk. It does not go deep but covers the basics, well enough for whetting the reader’s appetite. Much more can be found in books like [AS], [Le], [St], [Ci], [Hu].

Chapters 6 and 7 represent a slight detour towards and into some basic analysis, which is used to enumerate real roots of a polynomial. The selection here is eclectic and includes finite differences, infinite sums, Sturm’s method and some infinitary combinatorics. Much of this is not readily found in problem books, but once again the reader is left wanting for hints or solutions. Particularly unique is Section 6.6, which proves the transcendence of the Liouville and Mahler numbers. The proofs are elementary and combinatorial, although the presentation feels less clear than it could be.

Chapter 8 is the culmination of the book. It is concerned with questions of solving polynomial equations (over \mathbb{Q}) using radicals, real radicals and compass-and-ruler constructions. In particular, most of the classical (im)possibilities of Galois theory are proved here: $\sqrt[3]{2}$ and the regular 9-gon are not constructible with compass and ruler, but regular p -gons for $p = 2^{2^n} + 1$ prime are (Gauss); some quintics are not solvable in radicals (Abel–Ruffini); a cubic with three real and no rational roots is not solvable in real radicals (“casus irreducibilis”). The author takes a rather concrete approach, eschewing groups and Galois theory in favor of resolvents, symmetric polynomials and some low-level linear algebra. I am rather delighted to see these proofs taking center stage in a book, after the stepmotherly treatment they tend to receive in modern texts on algebra, which usually name-check them to motivate Galois theory but rarely stoop down to prove them, let alone constructively! (Stewart’s [Sw], at least, constructs the

²For those equally confused: A “train” is an n -tuple of colors. The cyclic group C_n acts on such n -tuples by cyclically rotating the entries. A “station” is an orbit of this action. The “period” of a train is the smallest $k > 0$ such that the k -th power of the generator fixes the train. A “passenger” is a pair of a train and an element of $\{0, 1, \dots, k - 1\}$, modelling an element of the cyclic group.

17-gon.)

The highlight here is Section 8.2, which covers the “positive part” of the theory: Gauss’s famous *Disquisitiones* result that regular n -gons are constructible for all positive integers n whose Euler totient $\varphi(n)$ is a power of 2; and a related result that $e^{2\pi i/p}$ can be expressed through $(p-1)$ -st roots for any prime p . The proofs are less than a page long (§8.2.E) and quite nice, yet surprisingly missing from all texts I have seen except for [Sw, Theorem 20.13 and proof of Theorem 21.3] (which is much less concrete and elementary).

The remaining 40 pages of the book (§8.3 and §8.4) address the “negative part” of the theory (non-constructibility and non-expressibility). Here, unfortunately, all the weaknesses of the present book manifest themselves to their fullest. Abstraction and complexity rise significantly, as the author discusses symmetry classes of polynomials and normal field extensions, while still dutifully avoiding the word “group”. The advantage of this quasi-concreteness is not clear, as the results being proved are mostly negative. Hints and proofs are not always given, and when they are, they often cite later problems. A lot of apocryphal material is included (not just Abel–Ruffini and Gauss, but also criteria for real radicals or expressibility with a single radical) that is not easily found in textbooks, but the road is long and highly bumpy. I am afraid that few readers will find the destination worth the trip.

What can thus be made of this book? It is clearly not a soulless write-only rehashing of already well-exposed material to lengthen the author’s CV; there is a lot here that is not easily found elsewhere, and several good pedagogical choices can be seen in the selection of problems (along with some questionable ones in their ordering). Olympiad trainers – the above-mentioned audience (c) – will likely find good material to work with in the book. Yet the sparsity of hints and solutions, which are often insufficient for the non-expert reader, limits its usefulness for audience (a) and perhaps even for (b). For instance, how much can you get out of the hint to problem 3.9.1?³ And this is a first problem in its section, seemingly a warm-up exercise; if it was clearly marked as a bonus problem, then the reader could at least skip it. Audience (b) will at least have a trainer to tell which problems are important.

Not only in this aspect does the book read like a draft rushed into print. It also teems with typos, ambiguous wording (“any” is a dangerous word) and occasionally more serious gaps. Laudably, the author is tracking errata on his website (https://old.mccme.ru/circles/oim/algebra_eng.pdf). A second edition with fewer errors, more hints and a better ordering will greatly improve the usability of the book. If this lengthens the text, some digressions (such as infinite sums and the Burnside lemma) could be removed to focus on its main topic (polynomials and their roots). Sections 3.2 and 8.2 are the treasures hidden in this book, and it is a pity to see them buried under so much debris.

³Something closer to an actual hint can be found in https://stanford.edu/~sfh/discrete_geo3.pdf.

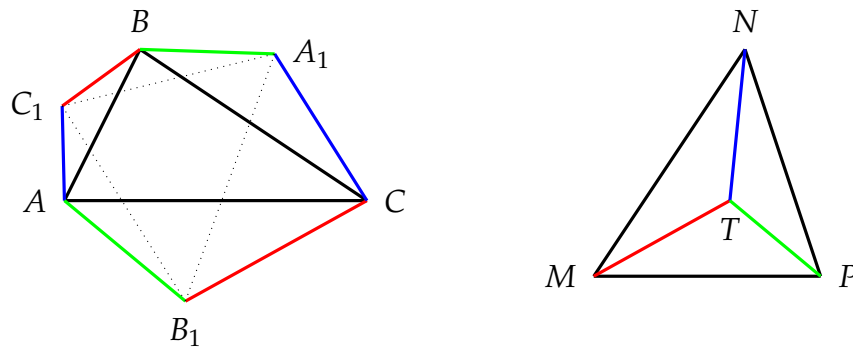


Figure 1: The generalized Napoleon theorem. Some edges have been colored as a visual aid (corresponding sides of similar triangles have the same color).

Part 2: Geometry.

The second book focuses on Euclidean geometry – mostly planar, and mostly rather elementary, which does not mean “simple” or “well-known”. It is therefore close kin to [Pr], [Sh] and (to some extent) [Ch], although it expects a somewhat higher level of sophistication from the reader; this makes it a good sequel to [Pr].

Chapter 1 deals with the geometry of the triangle. While far from comprehensive (only 38 pages!), it collects a fair amount of interesting and beautiful problems, often (not always) with complete solutions. Topics such as Simson lines and mixtilinear incircles (here called “semi-inscribed circles”) are presented through sequences of well-chosen problems, arranged in a pedagogically reasonable order to allow each to be solved using the ones before (although occasionally, the more basic problems from a later section get used as well).

The last four sections of Chapter 1 are my favorite parts of this book, covering the mid-arc points (Section 1.7), the mixtilinear incircles (1.8), the generalized Napoleon theorem (1.9) and isogonal conjugation (1.10). The generalized Napoleon theorem, in particular, is a gem of elementary geometry that is far less known than it deserves. It goes back to Rigby [Ri, Theorem 3.1] and has recently been re-exposed in [Be] and [Eg], but its best statement is as follows:

Theorem 0.1 (Rigby 1988). Let ABC and MNP be two triangles, and T any point in the plane. Let A_1, B_1, C_1 be three points in the plane such that $\triangle ABC_1 \sim \triangle NMT$ and $\triangle BCA_1 \sim \triangle PNT$ and $\triangle CAB_1 \sim \triangle MPT$. Here, the symbol “ \sim ” means “directly similar” (i.e., similar and having the same orientation), and we understand a triangle to be an ordered triple of its vertices (so $\triangle ABC$ is not similar to $\triangle BAC$).

Then, $\triangle A_1B_1C_1 \sim \triangle MNP$. (See Figure 1.)

There are 6 degrees of freedom in this theorem (up to similitude), which is an unusually high number for triangle geometry; thus, its broad applicability

is not unexpected, though hardly unwelcome! Napoleon's classical theorem is obtained by taking $\triangle MNP$ equilateral and T to be its center, but several much less obvious choices yield other interesting results. In particular, when M, N, P, T are collinear, Theorem 0.1 degenerates to Menelaos's theorem. A recent result [KB, Theorem 3] by Kiss and Bíró also follows easily (exercise!) from Theorem 0.1:

Corollary 0.2 (Kiss, Bíró 2021). Let ABC be a triangle with orthocenter H and circumcenter O . Let $D = AH \cap BO$ and $E = BH \cap CO$ and $F = CH \cap AO$. Then, $\triangle DEF$ is inversely similar to $\triangle ABC$ (that is, similar with opposite orientation), and the points H, D, E, F lie on one circle.

Eight further applications are shown in the book under review. To complete the kaleidoscope, I would add that the theorem can be viewed as a geometric avatar of divided symmetrization⁴.

There is much more to like in Chapter 1 than can fit in this review; the book makes use of its space sparingly and wisely here. Problems 1.1.7 (a concurrency with many degrees of freedom) and 1.10.17 (Pascal's theorem proved using isogonal conjugation) are two of my favorites.

The remaining chapters are often shorter and less generous about solutions. Some sections contain complete solutions to almost all problems; others give just a few bare hints or not even that; there appears to be no logic behind this other than the predilections of the different authors. Nevertheless, there is much to learn here if one is not completionist and is willing to skip some unduly challenging problems. Chapter 2 highlights the properties of circles (radical axes, Ptolemy's and Casey's theorems⁵). Chapter 3 discusses several types of geometric transformations and their applications. Chapter 4 introduces centers-of-mass (i.e., weighted averages of vectors), cross-ratios and polarity. Complex numbers are briefly discussed in Chapter 5, probably too briefly for a first encounter but

⁴To explain this, let us prove Theorem 0.1 using complex numbers: Identify the points $A, B, C, M, N, P, T, A_1, B_1, C_1$ with complex numbers $a, b, c, m, n, p, t, a_1, b_1, c_1$, and assume WLOG that $t = 0$ (that is, place the origin at T). Then, $\triangle ABC_1 \sim \triangle NMT$ yields $(a - c_1) / (b - c_1) = (n - t) / (m - t) = n/m$ (since $t = 0$), so that $c_1 = \frac{ma - nb}{m - n}$, and similarly $a_1 = \frac{nb - pc}{n - p}$ and $b_1 = \frac{pc - ma}{p - m}$. Our goal is to prove that $\triangle A_1B_1C_1 \sim \triangle MNP$, that is, $(a_1 - b_1) / (b_1 - c_1) = (m - n) / (n - p)$, or, equivalently, $(a_1 - b_1) / (m - n) = (b_1 - c_1) / (n - p)$. But besides being a straightforward and short computation, this can be interpreted in terms of forward divided differences ([Fl, §1–§2]): Let f be a polynomial sending m, n, p to ma, nb, pc . Then, $a_1 = \frac{nb - pc}{n - p} = \frac{f(n) - f(p)}{n - p} = f[n, p]$ (this would be $[n, p]f$ in the notations of [Fl], but we put the f up front) and similarly $b_1 = f[p, m]$. Hence, $(a_1 - b_1) / (m - n) = (f[n, p] - f[p, m]) / (m - n) = f[n, p, m]$. Similarly, $(b_1 - c_1) / (n - p) = f[p, m, n]$. So our claim becomes $f[n, p, m] = f[p, m, n]$, which is a particular case of the symmetry of divided differences.

⁵Casey's theorem (Problem 2.6.12) is given a stepmotherly treatment (the proof of sufficiency is far from sufficient). However, this is typical of geometry textbooks.

useful as a supplement (e.g.) to [Ch, Chapter 6]. Geometric constructions and loci get their moments in Chapter 6.

Chapter 7 makes a daring foray into solid and n -dimensional geometry, with a slant towards combinatorial questions (tilings and coverings) and volumes. While this certainly adds variety to the book, I am bewildered by the thought of ever confronting students (even the most talented ones) with this material in this form. Already the introductory sections pose serious challenges if one is not blessed with high visuospatial skills or wants to prove the answers rigorously. In the following sections, every other problem introduces some new concept (and some of them – such as "surface area" and "semi-regular body" – without definition). Even worse, barely any hints are given in this chapter, unlike the rest of the book; this is reminiscent of book 1 at its worst.

Chapter 8 returns to regular programming – and quality – with a series of miscellaneous topics: geometric optimization, areas, conics and curvilinear triangles. The treatment of conics here is nowhere as comprehensive as in [AZ2], but it serves as a good appetizer.

All in all, apart from Chapter 7, the book is well-written, thought-out and finished, even if more hints would be welcome in some of the sections. Errors are rare: I have spotted only one incorrect problem (2.3.5) and (outside of Chapter 7) only one undefined concept (the "excircles" – also called "escribed" or "exscribed" circles – of a triangle). Here, too, the authors keep track of known errors (<https://users.mccme.ru/mskopenkov/skopenkov-pdf/mb126-erratum.pdf>). With the minor caveats above, I can recommend this book to students and teachers alike (audiences **(a)**, **(b)**, **(c)**), provided that it is not their first encounter with the subject, as it is not meant to be introductory. But there is no lack of introductory texts on Euclidean geometry in various forms and styles that can get the reader to the level of experience needed for this present book.

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