A hyperfactorial divisibility

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long version

Let us define a function $H : \mathbb{N} \to \mathbb{N}$ by

$$H(n) = \prod_{k=0}^{n-1} k! \qquad \text{for every } n \in \mathbb{N}.$$

Our goal is to prove the following theorem:

Theorem 0 (MacMahon). We have

$$H(b+c)\,H(c+a)\,H(a+b) \mid H(a)\,H(b)\,H(c)\,H(a+b+c)$$

for every $a \in \mathbb{N}$, every $b \in \mathbb{N}$ and every $c \in \mathbb{N}$.

Remark: Here, we denote by $\mathbb{N}$ the set $\{0, 1, 2, ...\}$ (and not the set $\{1, 2, 3, ...\}$, as some authors do).

Before we come to the proof, first let us make some definitions:

Notations.

- For any matrix A, we denote by $A\begin{bmatrix} j \\ i \end{bmatrix}$ the entry in the j-th column and the i-th row of A. [This is usually denoted by A_{ij} or by $A_{i,j}$.]

- Let R be a ring. Let $u \in \mathbb{N}$ and $v \in \mathbb{N}$, and let $a_{i,j}$ be an element of R for every $(i,j) \in \{1, 2, ..., u\} \times \{1, 2, ..., v\}$. Then, we denote by $(a_{i,j})_{1 \leq i \leq u}^{1 \leq j \leq v}$ the $u \times v$ matrix $A \in R^{u \times v}$ which satisfies

$$A\begin{bmatrix} j \\ i \end{bmatrix} = a_{i,j} \qquad \text{for every } (i,j) \in \{1, 2, ..., u\} \times \{1, 2, ..., v\}.$$

- Let R be a commutative ring with unity. Let $P \in R[X]$ be a polynomial. Let $j \in \mathbb{N}$. Then, we denote by $\operatorname{coeff}_j P$ the coefficient of the polynomial P before X^j. (In particular, this implies $\operatorname{coeff}_j P = 0$ for every $j > \deg P$.) Thus, for every $P \in R[X]$ and every $d \in \mathbb{N}$ satisfying $\deg P \leq d$, we have

$$P(X) = \sum_{k=0}^{d} \operatorname{coeff}_k(P) \cdot X^k.$$

- If n and m are two integers, then the *binomial coefficient* $\binom{m}{n} \in \mathbb{Q}$ is defined by

$$\binom{m}{n} = \begin{cases} \dfrac{m(m-1)\cdots(m-n+1)}{n!}, & \text{if } n \geq 0; \\ 0, & \text{if } n < 0 \end{cases}.$$

It is well-known that $\binom{m}{n} \in \mathbb{Z}$ for all $n \in \mathbb{Z}$ and $m \in \mathbb{Z}$.

We recall a fact from linear algebra:

Theorem 1 (Vandermonde determinant). Let R be a commutative ring with unity. Let $m \in \mathbb{N}$. Let a_1, a_2, ..., a_m be m elements of R. Then,

$$\det\left(\left(a_i^{j-1}\right)_{1\leq i\leq m}^{1\leq j\leq m}\right) = \prod_{\substack{(i,j)\in\{1,2,...,m\}^2;\\ i>j}} (a_i - a_j).$$

We are more interested in a corollary - and generalization - of this fact:

Theorem 2 (generalized Vandermonde determinant). Let R be a commutative ring with unity. Let $m \in \mathbb{N}$. For every $j \in \{1, 2, ..., m\}$, let $P_j \in R[X]$ be a polynomial such that $\deg(P_j) \leq j-1$. Let a_1, a_2, ..., a_m be m elements of R. Then,

$$\det\left(\left(P_j(a_i)\right)_{1\leq i\leq m}^{1\leq j\leq m}\right) = \left(\prod_{j=1}^{m} \operatorname{coeff}_{j-1}(P_j)\right) \cdot \prod_{\substack{(i,j)\in\{1,2,...,m\}^2;\\ i>j}} (a_i - a_j).$$

Both Theorems 1 and 2 can be deduced from the following lemma:

Lemma 3. Let R be a commutative ring with unity. Let $m \in \mathbb{N}$. For every $j \in \{1, 2, ..., m\}$, let $P_j \in R[X]$ be a polynomial such that $\deg(P_j) \leq j-1$. Let a_1, a_2, ..., a_m be m elements of R. Then,

$$\det\left(\left(P_j(a_i)\right)_{1\leq i\leq m}^{1\leq j\leq m}\right) = \left(\prod_{j=1}^{m} \operatorname{coeff}_{j-1}(P_j)\right) \cdot \det\left(\left(a_i^{j-1}\right)_{1\leq i\leq m}^{1\leq j\leq m}\right).$$

Proof of Lemma 3. For every $j \in \{1, 2, ..., m\}$, we have $P_j(X) = \sum\limits_{k=0}^{m-1} \operatorname{coeff}_k(P_j) \cdot X^k$ (since $\deg(P_j) \leq j-1 \leq m-1$, since $j \leq m$). Thus, for every $i \in \{1, 2, ..., m\}$ and $j \in \{1, 2, ..., m\}$, we have

$$P_j(a_i) = \sum_{k=0}^{m-1} \operatorname{coeff}_k(P_j) \cdot a_i^k = \sum_{k=0}^{m-1} a_i^k \cdot \operatorname{coeff}_k(P_j) = \sum_{k=1}^{m} a_i^{k-1} \cdot \operatorname{coeff}_{k-1}(P_j) \quad (1)$$

$$(\text{here we substituted } k-1 \text{ for } k \text{ in the sum}).$$

Hence,

$$\left(P_j(a_i)\right)_{1\leq i\leq m}^{1\leq j\leq m} = \left(a_i^{j-1}\right)_{1\leq i\leq m}^{1\leq j\leq m} \cdot \left(\operatorname{coeff}_{i-1}(P_j)\right)_{1\leq i\leq m}^{1\leq j\leq m} \quad (2)$$

(according to the definition of the product of two matrices)[1].

[1] Here is the *proof of (2)* in more detail: The definition of the product of two matrices yields

$$\left(a_i^{j-1}\right)_{1\leq i\leq m}^{1\leq j\leq m} \cdot \left(\operatorname{coeff}_{i-1}(P_j)\right)_{1\leq i\leq m}^{1\leq j\leq m} = \left(\underbrace{\sum_{k=1}^{m} a_i^{k-1} \cdot \operatorname{coeff}_{k-1}(P_j)}_{\substack{=P_j(a_i)\\ (\text{by } (1))}}\right)_{1\leq i\leq m}^{1\leq j\leq m} = \left(P_j(a_i)\right)_{1\leq i\leq m}^{1\leq j\leq m}.$$

This proves (2).

But the matrix $\left(\operatorname{coeff}_{i-1}(P_j)\right)_{1\leq i\leq m}^{1\leq j\leq m}$ is upper triangular (since $\operatorname{coeff}_{i-1}(P_j)=0$ for every $i\in\{1,2,...,m\}$ and $j\in\{1,2,...,m\}$ satisfying $i>j$ [2]); hence, $\det\left(\left(\operatorname{coeff}_{i-1}(P_j)\right)_{1\leq i\leq m}^{1\leq j\leq m}\right)=\prod_{j=1}^{m}\operatorname{coeff}_{j-1}(P_j)$ (since the determinant of an upper triangular matrix equals the product of its diagonal entries).

Now, (2) yields

$$\begin{aligned}\det\left(\left(P_j(a_i)\right)_{1\leq i\leq m}^{1\leq j\leq m}\right)&=\det\left(\left(a_i^{j-1}\right)_{1\leq i\leq m}^{1\leq j\leq m}\cdot\left(\operatorname{coeff}_{i-1}(P_j)\right)_{1\leq i\leq m}^{1\leq j\leq m}\right)\\&=\det\left(\left(a_i^{j-1}\right)_{1\leq i\leq m}^{1\leq j\leq m}\right)\cdot\underbrace{\det\left(\left(\operatorname{coeff}_{i-1}(P_j)\right)_{1\leq i\leq m}^{1\leq j\leq m}\right)}_{=\prod_{j=1}^{m}\operatorname{coeff}_{j-1}(P_j)}\\&=\det\left(\left(a_i^{j-1}\right)_{1\leq i\leq m}^{1\leq j\leq m}\right)\cdot\prod_{j=1}^{m}\operatorname{coeff}_{j-1}(P_j)\\&=\left(\prod_{j=1}^{m}\operatorname{coeff}_{j-1}(P_j)\right)\cdot\det\left(\left(a_i^{j-1}\right)_{1\leq i\leq m}^{1\leq j\leq m}\right),\end{aligned}$$

and thus, Lemma 3 is proven.

Proof of Theorem 1. For every $j\in\{1,2,...,m\}$, define a polynomial $P_j\in R[X]$ by $P_j(X)=\prod_{k=1}^{j-1}(X-a_k)$. Then, P_j is a monic polynomial of degree $j-1$ (since P_j is a product of $j-1$ monic polynomials of degree 1 each[3]). In other words, $\deg(P_j)=j-1$ and $\operatorname{coeff}_{j-1}(P_j)=1$ for every $j\in\{1,2,...,m\}$. Obviously, $\deg(P_j)=j-1$ yields $\deg(P_j)\leq j-1$ for every $j\in\{1,2,...,m\}$. Thus, Lemma 3 yields

$$\det\left(\left(P_j(a_i)\right)_{1\leq i\leq m}^{1\leq j\leq m}\right)=\left(\prod_{j=1}^{m}\operatorname{coeff}_{j-1}(P_j)\right)\cdot\det\left(\left(a_i^{j-1}\right)_{1\leq i\leq m}^{1\leq j\leq m}\right).\qquad(3)$$

But the matrix $\left(P_j(a_i)\right)_{1\leq i\leq m}^{1\leq j\leq m}$ is lower triangular (since $P_j(a_i)=0$ for every $i\in\{1,2,...,m\}$ and $j\in\{1,2,...,m\}$ satisfying $i<j$ [4]); hence, $\det\left(\left(P_j(a_i)\right)_{1\leq i\leq m}^{1\leq j\leq m}\right)=\prod_{j=1}^{m}P_j(a_j)$ (since the determinant of a lower triangular matrix equals the product

[2] because $i>j$ yields $i-1>j-1$, thus $i-1>\deg(P_j)$ (since $\deg(P_j)\leq j-1$) and therefore $\operatorname{coeff}_{i-1}(P_j)=0$

[3] because $X-a_k$ is a monic polynomial of degree 1 for every $k\in\{1,2,...,j-1\}$, and we have $P_j(X)=\prod_{k=1}^{j-1}(X-a_k)$

[4] because $i<j$ yields $i\leq j-1$ (since i and j are integers) and thus

$$\begin{aligned}P_j(a_i)&=\prod_{k=1}^{j-1}(a_i-a_k)\qquad\left(\text{since }P_j(X)=\prod_{k=1}^{j-1}(X-a_k)\right)\\&=0\end{aligned}$$

(since the factor of the product $\prod_{k=1}^{j-1}(a_i-a_k)$ for $k=i$ equals $a_i-a_i=0$)

of its diagonal entries). Thus, (3) rewrites as

$$\prod_{j=1}^{m} P_j(a_j) = \left(\prod_{j=1}^{m} \operatorname{coeff}_{j-1}(P_j)\right) \cdot \det\left(\left(a_i^{j-1}\right)_{1\leq i\leq m}^{1\leq j\leq m}\right).$$

Since $\prod_{j=1}^{m} \underbrace{\operatorname{coeff}_{j-1}(P_j)}_{=1} = \prod_{j=1}^{m} 1 = 1$, this simplifies to

$$\prod_{j=1}^{m} P_j(a_j) = \det\left(\left(a_i^{j-1}\right)_{1\leq i\leq m}^{1\leq j\leq m}\right).$$

Thus,

$$\begin{aligned}
\det\left(\left(a_i^{j-1}\right)_{1\leq i\leq m}^{1\leq j\leq m}\right) &= \prod_{j=1}^{m} P_j(a_j) = \underbrace{\prod_{j=1}^{m}}_{=\prod_{j\in\{1,2,\ldots,m\}}} \underbrace{\prod_{k=1}^{j-1}}_{=\prod_{k\in\{1,2,\ldots,j-1\}}=\prod_{k\in\{1,2,\ldots,m\};\ k<j}} (a_j - a_k) \\
&\qquad \left(\text{since } P_j(X) = \prod_{k=1}^{j-1}(X - a_k) \text{ yields } P_j(a_j) = \prod_{k=1}^{j-1}(a_j - a_k)\right) \\
&= \prod_{j\in\{1,2,\ldots,m\}} \prod_{\substack{k\in\{1,2,\ldots,m\};\\ k<j}} (a_j - a_k) = \prod_{\substack{(j,k)\in\{1,2,\ldots,m\}^2;\\ k<j}} (a_j - a_k) \\
&= \prod_{\substack{(i,j)\in\{1,2,\ldots,m\}^2;\\ j<i}} (a_i - a_j) \\
&\qquad (\text{here we renamed } j \text{ and } k \text{ as } i \text{ and } j \text{ in the product}) \\
&= \prod_{\substack{(i,j)\in\{1,2,\ldots,m\}^2;\\ i>j}} (a_i - a_j).
\end{aligned}$$

Hence, Theorem 1 is proven.

Proof of Theorem 2. Lemma 3 yields

$$\begin{aligned}
\det\left(\left(P_j(a_i)\right)_{1\leq i\leq m}^{1\leq j\leq m}\right) &= \left(\prod_{j=1}^{m} \operatorname{coeff}_{j-1}(P_j)\right) \cdot \underbrace{\det\left(\left(a_i^{j-1}\right)_{1\leq i\leq m}^{1\leq j\leq m}\right)}_{\substack{=\prod_{(i,j)\in\{1,2,\ldots,m\}^2;\ i>j}(a_i-a_j)\\ \text{by Theorem 1}}} \\
&= \left(\prod_{j=1}^{m} \operatorname{coeff}_{j-1}(P_j)\right) \cdot \prod_{\substack{(i,j)\in\{1,2,\ldots,m\}^2;\\ i>j}} (a_i - a_j).
\end{aligned}$$

Hence, Theorem 2 is proven.

A consequence of Theorem 2 is the following fact:

Corollary 4. Let R be a commutative ring with unity. Let $m \in \mathbb{N}$. Let $a_1, a_2, ..., a_m$ be m elements of R. Then,

$$\det \left(\left(\prod_{k=1}^{j-1} (a_i - k) \right)_{1 \leq i \leq m}^{1 \leq j \leq m} \right) = \prod_{\substack{(i,j) \in \{1,2,...,m\}^2; \\ i > j}} (a_i - a_j).$$

Proof of Corollary 4. For every $j \in \{1, 2, ..., m\}$, define a polynomial $P_j \in R[X]$ by $P_j(X) = \prod_{k=1}^{j-1} (X - k)$. Then, P_j is a monic polynomial of degree $j - 1$ (since P_j is a product of $j - 1$ monic polynomials of degree 1 each[5]). In other words, $\deg(P_j) = j - 1$ and $\operatorname{coeff}_{j-1}(P_j) = 1$ for every $j \in \{1, 2, ..., m\}$. Obviously, $\deg(P_j) = j - 1$ yields $\deg(P_j) \leq j - 1$ for every $j \in \{1, 2, ..., m\}$. Thus, Theorem 2 yields

$$\det \left((P_j(a_i))_{1 \leq i \leq m}^{1 \leq j \leq m} \right) = \left(\prod_{j=1}^{m} \operatorname{coeff}_{j-1}(P_j) \right) \cdot \prod_{\substack{(i,j) \in \{1,2,...,m\}^2; \\ i > j}} (a_i - a_j).$$

Since $P_j(a_i) = \prod_{k=1}^{j-1} (a_i - k)$ for every $i \in \{1, 2, ..., m\}$ and $j \in \{1, 2, ..., m\}$ (since $P_j(X) = \prod_{k=1}^{j-1} (X - k)$), this rewrites as

$$\det \left(\left(\prod_{k=1}^{j-1} (a_i - k) \right)_{1 \leq i \leq m}^{1 \leq j \leq m} \right) = \left(\prod_{j=1}^{m} \operatorname{coeff}_{j-1}(P_j) \right) \cdot \prod_{\substack{(i,j) \in \{1,2,...,m\}^2; \\ i > j}} (a_i - a_j).$$

Since $\prod_{j=1}^{m} \underbrace{\operatorname{coeff}_{j-1}(P_j)}_{=1} = \prod_{j=1}^{m} 1 = 1$, this simplifies to

$$\det \left(\left(\prod_{k=1}^{j-1} (a_i - k) \right)_{1 \leq i \leq m}^{1 \leq j \leq m} \right) = \prod_{\substack{(i,j) \in \{1,2,...,m\}^2; \\ i > j}} (a_i - a_j).$$

Hence, Corollary 4 is proven.

We shall need the following simple lemma:

Lemma 5. Let $m \in \mathbb{N}$. Then,

$$\prod_{\substack{(i,j) \in \{1,2,...,m\}^2; \\ i > j}} (i - j) = H(m).$$

[5] because $X - k$ is a monic polynomial of degree 1 for every $k \in \{1, 2, ..., j - 1\}$, and we have $P_j(X) = \prod_{k=1}^{j-1} (X - k)$

Proof of Lemma 5. We have

$$\prod_{\substack{(i,j)\in\{1,2,\ldots,m\}^2;\\ i>j}} (i-j) = \underbrace{\prod_{\substack{(i,j)\in\{0,1,\ldots,m-1\}^2;\\ i+1>j+1}}}_{\substack{=\prod_{\substack{(i,j)\in\{0,1,\ldots,m-1\}^2;\\ i>j}}\\ \text{(since } i+1>j+1 \text{ is equivalent to } i>j\text{)}}} \left(\underbrace{(i+1)-(j+1)}_{=i-j}\right)$$

(here we substituted $i+1$ and $j+1$ for i and j in the product)

$$= \underbrace{\prod_{\substack{(i,j)\in\{0,1,\ldots,m-1\}^2;\\ i>j}}}_{=\prod_{i\in\{0,1,\ldots,m-1\}}\prod_{\substack{j\in\{0,1,\ldots,m-1\};\\ i>j}}} (i-j)$$

$$= \prod_{i\in\{0,1,\ldots,m-1\}} \underbrace{\prod_{\substack{j\in\{0,1,\ldots,m-1\};\\ i>j}}}_{\substack{=\prod_{\substack{j\in\mathbb{N};\\ j\le m-1 \text{ and } i>j}}\\ \text{(since } j\in\{0,1,\ldots,m-1\} \text{ is equivalent}\\ \text{to } (j\in\mathbb{N} \text{ and } j\le m-1))}} (i-j)$$

$$= \prod_{i\in\{0,1,\ldots,m-1\}} \underbrace{\prod_{\substack{j\in\mathbb{N};\\ j\le m-1 \text{ and } i>j}}}_{\substack{=\prod_{\substack{j\in\mathbb{N};\\ i>j}}\\ \text{(since the assertion}\\ (j\le m-1 \text{ and } i>j) \text{ is equivalent to } (i>j)\\ \text{(because if } i>j, \text{ then } j\le m-1 \text{ (since}\\ i\in\{0,1,\ldots,m-1\} \text{ yields } i\le m-1)))}} (i-j)$$

$$= \prod_{i\in\{0,1,\ldots,m-1\}} \underbrace{\prod_{\substack{j\in\mathbb{N};\\ i>j}}}_{=\prod_{\substack{j\in\mathbb{N};\\ j<i}}=\prod_{j=0}^{i-1}} (i-j) = \prod_{i\in\{0,1,\ldots,m-1\}} \prod_{j=0}^{i-1} (i-j) = \prod_{i\in\{0,1,\ldots,m-1\}} \underbrace{\prod_{j=1}^{i} j}_{=i!}$$

(here we substituted $i-j$ for j in the second product)

$$= \underbrace{\prod_{i\in\{0,1,\ldots,m-1\}}}_{=\prod_{i=0}^{m-1}} i! = \prod_{i=0}^{m-1} i! = \prod_{k=0}^{m-1} k! \qquad \text{(here we renamed } i \text{ as } k \text{ in the product)}$$

$$= H(m).$$

Hence, Lemma 5 is proven.

Now let us prove Theorem 0: Let $a\in\mathbb{N}$, let $b\in\mathbb{N}$ and let $c\in\mathbb{N}$.

We have

$$H(a+b+c)=\prod_{k=0}^{a+b+c-1}k!=\left(\underbrace{\prod_{k=0}^{a+b-1}k!}_{=H(a+b)}\right)\cdot\prod_{k=a+b}^{a+b+c-1}k!=H(a+b)\cdot\prod_{k=a+b}^{a+b+c-1}k!$$

$$=H(a+b)\cdot\prod_{i=1}^{c}(a+b+i-1)!$$

$$(\text{here we substituted } a+b+i-1 \text{ for } k \text{ in the product}),\qquad(4)$$

$$H(b+c)=\prod_{k=0}^{b+c-1}k!=\left(\underbrace{\prod_{k=0}^{b-1}k!}_{=H(b)}\right)\cdot\prod_{k=b}^{b+c-1}k!=H(b)\cdot\prod_{k=b}^{b+c-1}k!=H(b)\cdot\prod_{i=1}^{c}(b+i-1)!$$

$$(\text{here we substituted } b+i-1 \text{ for } k \text{ in the product}),\qquad(5)$$

$$H(c+a)=\prod_{k=0}^{c+a-1}k!=\left(\underbrace{\prod_{k=0}^{a-1}k!}_{=H(a)}\right)\cdot\prod_{k=a}^{c+a-1}k!=H(a)\cdot\prod_{k=a}^{c+a-1}k!=H(a)\cdot\prod_{i=1}^{c}(a+i-1)!$$

$$(\text{here we substituted } a+i-1 \text{ for } k \text{ in the product}).\qquad(6)$$

Next, we show a lemma:

Lemma 6. For every $i\in\mathbb{N}$ and $j\in\mathbb{N}$ satisfying $i\geq 1$ and $j\geq 1$, we have

$$\binom{a+b+i-1}{a+i-j}=\frac{(a+b+i-1)!}{(a+i-1)!\cdot(b+j-1)!}\cdot\prod_{k=1}^{j-1}(a+i-k).$$

Proof of Lemma 6. Let $i\in\mathbb{N}$ and $j\in\mathbb{N}$ be such that $i\geq 1$ and $j\geq 1$. One of the following two cases must hold:

Case 1: We have $a+i-j\geq 0$.

Case 2: We have $a+i-j<0$.

In Case 1, we have

$$(a+i-1)!=\prod_{k=1}^{a+i-1}k=\left(\underbrace{\prod_{k=1}^{a+i-j}k}_{=(a+i-j)!}\right)\cdot\prod_{k=a+i-j+1}^{a+i-1}k=(a+i-j)!\cdot\prod_{k=a+i-j+1}^{a+i-1}k$$

$$=(a+i-j)!\cdot\prod_{k=1}^{j-1}(a+i-k)$$

$$(\text{here we substituted } a+i-k \text{ for } k \text{ in the product}),$$

so that

$$\frac{(a+i-1)!}{(a+i-j)!} = \prod_{k=1}^{j-1} (a+i-k) \,. \tag{7}$$

Now,

$$\begin{aligned}
\binom{a+b+i-1}{a+i-j} &= \frac{(a+b+i-1)!}{(a+i-j)! \cdot ((a+b+i-1)-(a+i-j))!} = \frac{(a+b+i-1)!}{(a+i-j)! \cdot (b+j-1)!} \\
&\qquad \left(\text{since } (a+b+i-1)-(a+i-j) = b+j-1\right) \\
&= \frac{(a+b+i-1)!}{(a+i-1)! \cdot (b+j-1)!} \cdot \frac{(a+i-1)!}{(a+i-j)!} \\
&= \frac{(a+b+i-1)!}{(a+i-1)! \cdot (b+j-1)!} \cdot \prod_{k=1}^{j-1} (a+i-k) \qquad (\text{by } (7))\,.
\end{aligned}$$

Hence, Lemma 6 holds in Case 1.

In Case 2, we have

$$\begin{aligned}
\prod_{k=1}^{j-1} (a+i-k) &= \left(\prod_{k=1}^{a+i-1} (a+i-k)\right) \cdot \left(\underbrace{\prod_{k=a+i}^{a+i} (a+i-k)}_{=a+i-(a+i)=0}\right) \cdot \prod_{k=a+i+1}^{j-1} (a+i-k) \\
&\qquad (\text{since } a+i-j<0 \text{ yields } a+i<j) \\
&= 0,
\end{aligned}$$

so that

$$\begin{aligned}
\binom{a+b+i-1}{a+i-j} &= 0 \qquad (\text{since } a+i-j<0) \\
&= \frac{(a+b+i-1)!}{(a+i-1)! \cdot (b+j-1)!} \cdot \underbrace{0}_{=\prod_{k=1}^{j-1}(a+i-k)} \\
&= \frac{(a+b+i-1)!}{(a+i-1)! \cdot (b+j-1)!} \cdot \prod_{k=1}^{j-1} (a+i-k)\,.
\end{aligned}$$

Hence, Lemma 6 holds in Case 2.

Hence, in both cases, Lemma 6 holds. Thus, Lemma 6 always holds, and this completes the proof of Lemma 6.

Another trivial lemma:

Lemma 7. Let R be a commutative ring with unity. Let $u \in \mathbb{N}$ and $v \in \mathbb{N}$, and let $a_{i,j}$ be an element of R for every $(i,j) \in \{1,2,...,u\} \times \{1,2,...,v\}$.

(a) Let α_1, α_2, ..., α_u be u elements of R. Then,

$$\left(\begin{cases} \alpha_i, & \text{if } j=i; \\ 0, & \text{if } j \neq i \end{cases}\right)_{1\leq i\leq u}^{1\leq j\leq u} \cdot \left(a_{i,j}\right)_{1\leq i\leq u}^{1\leq j\leq v} = \left(\alpha_i a_{i,j}\right)_{1\leq i\leq u}^{1\leq j\leq v}.$$

(b) Let β_1, β_2, ..., β_v be v elements of R. Then,

$$\left(a_{i,j}\right)_{1\leq i\leq u}^{1\leq j\leq v} \cdot \left(\begin{cases} \beta_i, & \text{if } j = i; \\ 0, & \text{if } j \neq i \end{cases} \right)_{1\leq i\leq v}^{1\leq j\leq v} = \left(a_{i,j}\beta_j\right)_{1\leq i\leq u}^{1\leq j\leq v}.$$

(c) Let α_1, α_2, ..., α_u be u elements of R. Let β_1, β_2, ..., β_v be v elements of R. Then,

$$\left(\begin{cases} \alpha_i, & \text{if } j = i; \\ 0, & \text{if } j \neq i \end{cases} \right)_{1\leq i\leq u}^{1\leq j\leq u} \cdot \left(a_{i,j}\right)_{1\leq i\leq u}^{1\leq j\leq v} \cdot \left(\begin{cases} \beta_i, & \text{if } j = i; \\ 0, & \text{if } j \neq i \end{cases} \right)_{1\leq i\leq v}^{1\leq j\leq v} = \left(\alpha_i a_{i,j}\beta_j\right)_{1\leq i\leq u}^{1\leq j\leq v}.$$

(d) Let α_1, α_2, ..., α_u be u elements of R. Let β_1, β_2, ..., β_v be v elements of R. If $u = v$, then

$$\det\left(\left(\alpha_i a_{i,j}\beta_j\right)_{1\leq i\leq u}^{1\leq j\leq v}\right) = \left(\prod_{i=1}^{u} \alpha_i\right) \cdot \left(\prod_{i=1}^{v} \beta_i\right) \cdot \det\left(\left(a_{i,j}\right)_{1\leq i\leq u}^{1\leq j\leq v}\right).$$

Proof of Lemma 7. **(a)** For every $i \in \{1, 2, ..., u\}$ and $j \in \{1, 2, ..., v\}$, we have

$$\begin{aligned}
&\underbrace{\sum_{\ell=1}^{u}}_{=\sum_{\ell\in\{1,2,...,u\}}} \begin{cases} \alpha_i, & \text{if } \ell = i; \\ 0, & \text{if } \ell \neq i \end{cases} \cdot a_{\ell,j} = \sum_{\ell\in\{1,2,...,u\}} \begin{cases} \alpha_i, & \text{if } \ell = i; \\ 0, & \text{if } \ell \neq i \end{cases} \cdot a_{\ell,j} \\
&= \sum_{\substack{\ell\in\{1,2,...,u\};\\ \ell=i}} \underbrace{\begin{cases} \alpha_i, & \text{if } \ell = i; \\ 0, & \text{if } \ell \neq i \end{cases}}_{=\alpha_i,\text{ since } \ell=i} \cdot a_{\ell,j} + \sum_{\substack{\ell\in\{1,2,...,u\};\\ \ell\neq i}} \underbrace{\begin{cases} \alpha_i, & \text{if } \ell = i; \\ 0, & \text{if } \ell \neq i \end{cases}}_{=0,\text{ since } \ell\neq i} \cdot a_{\ell,j} \\
&= \sum_{\substack{\ell\in\{1,2,...,u\};\\ \ell=i}} \alpha_i a_{\ell,j} + \underbrace{\sum_{\substack{\ell\in\{1,2,...,u\};\\ \ell\neq i}} 0 \cdot a_{\ell,j}}_{=0} = \sum_{\substack{\ell\in\{1,2,...,u\};\\ \ell=i}} \alpha_i a_{\ell,j} = \sum_{\ell\in\{i\}} \alpha_i a_{\ell,j} \\
&\qquad \left(\text{since } i \in \{1, 2, ..., u\} \text{ yields } \{\ell \in \{1, 2, ..., u\} \mid \ell = i\} = \{i\}\right) \\
&= \alpha_i a_{i,j}.
\end{aligned}$$

Thus,

$$\left(\begin{cases} \alpha_i, & \text{if } j = i; \\ 0, & \text{if } j \neq i \end{cases} \right)_{1\leq i\leq u}^{1\leq j\leq u} \cdot \left(a_{i,j}\right)_{1\leq i\leq u}^{1\leq j\leq v} = \left(\sum_{\ell=1}^{u} \begin{cases} \alpha_i, & \text{if } \ell = i; \\ 0, & \text{if } \ell \neq i \end{cases} \cdot a_{\ell,j} \right)_{1\leq i\leq u}^{1\leq j\leq v} = \left(\alpha_i a_{i,j}\right)_{1\leq i\leq u}^{1\leq j\leq v},$$

and thus, Lemma 7 **(a)** is proven.

(b) For every $i \in \{1,2,...,u\}$ and $j \in \{1,2,...,v\}$, we have

$$\begin{aligned}
&\underbrace{\sum_{\ell=1}^{v}}_{=\sum_{\ell\in\{1,2,...,v\}}} a_{i,\ell} \cdot \begin{cases} \beta_\ell, & \text{if } j=\ell; \\ 0, & \text{if } j\neq\ell \end{cases} = \sum_{\ell\in\{1,2,...,v\}} a_{i,\ell} \cdot \begin{cases} \beta_\ell, & \text{if } j=\ell; \\ 0, & \text{if } j\neq\ell \end{cases} \\
&= \sum_{\substack{\ell\in\{1,2,...,v\}; \\ \ell=j}} a_{i,\ell} \cdot \underbrace{\begin{cases} \beta_\ell, & \text{if } j=\ell; \\ 0, & \text{if } j\neq\ell \end{cases}}_{\substack{=\beta_\ell, \text{ since } \ell=j \\ \text{yields } j=\ell}} + \sum_{\substack{\ell\in\{1,2,...,v\}; \\ \ell\neq j}} a_{i,\ell} \cdot \underbrace{\begin{cases} \beta_\ell, & \text{if } j=\ell; \\ 0, & \text{if } j\neq\ell \end{cases}}_{\substack{=0, \text{ since } \ell\neq j \\ \text{yields } j\neq\ell}} \\
&= \sum_{\substack{\ell\in\{1,2,...,v\}; \\ \ell=j}} a_{i,\ell}\beta_\ell + \underbrace{\sum_{\substack{\ell\in\{1,2,...,v\}; \\ \ell\neq j}} a_{i,\ell} \cdot 0}_{=0} = \sum_{\substack{\ell\in\{1,2,...,v\}; \\ \ell=j}} a_{i,\ell}\beta_\ell = \sum_{\ell\in\{j\}} a_{i,\ell}\beta_\ell \\
&\qquad \left(\text{since } j\in\{1,2,...,v\} \text{ yields } \{\ell\in\{1,2,...,v\} \mid \ell=j\}=\{j\}\right) \\
&= a_{i,j}\beta_j.
\end{aligned}$$

Thus,

$$\left(a_{i,j}\right)_{1\leq i\leq u}^{1\leq j\leq v} \cdot \left(\begin{cases} \beta_i, & \text{if } j=i; \\ 0, & \text{if } j\neq i \end{cases}\right)_{1\leq i\leq v}^{1\leq j\leq v} = \left(\sum_{\ell=1}^{v} a_{i,\ell} \cdot \begin{cases} \beta_\ell, & \text{if } j=\ell; \\ 0, & \text{if } j\neq\ell \end{cases}\right)_{1\leq i\leq u}^{1\leq j\leq v} = \left(a_{i,j}\beta_j\right)_{1\leq i\leq u}^{1\leq j\leq v},$$

and thus, Lemma 7 **(b)** is proven.

(c) We have

$$\begin{aligned}
&\underbrace{\left(\begin{cases} \alpha_i, & \text{if } j=i; \\ 0, & \text{if } j\neq i \end{cases}\right)_{1\leq i\leq u}^{1\leq j\leq u} \cdot \left(a_{i,j}\right)_{1\leq i\leq u}^{1\leq j\leq v}}_{=\left(\alpha_i a_{i,j}\right)_{1\leq i\leq u}^{1\leq j\leq v} \text{ by Lemma 7 (a)}} \cdot \left(\begin{cases} \beta_i, & \text{if } j=i; \\ 0, & \text{if } j\neq i \end{cases}\right)_{1\leq i\leq v}^{1\leq j\leq v} \\
&= \left(\alpha_i a_{i,j}\right)_{1\leq i\leq u}^{1\leq j\leq v} \cdot \left(\begin{cases} \beta_i, & \text{if } j=i; \\ 0, & \text{if } j\neq i \end{cases}\right)_{1\leq i\leq v}^{1\leq j\leq v} = \left(\alpha_i a_{i,j}\beta_j\right)_{1\leq i\leq u}^{1\leq j\leq v}
\end{aligned}$$

by Lemma 7 **(b)** (applied to $\alpha_i a_{i,j}$ instead of $a_{i,j}$).

Thus, Lemma 7 **(c)** is proven.

(d) The matrix $\left(\begin{cases} \alpha_i, & \text{if } j=i; \\ 0, & \text{if } j\neq i \end{cases}\right)_{1\leq i\leq u}^{1\leq j\leq u}$ is diagonal (since $\begin{cases} \alpha_i, & \text{if } j=i; \\ 0, & \text{if } j\neq i \end{cases} = 0$ for every $i\in\{1,2,...,u\}$ and $j\in\{1,2,...,u\}$ satisfying $j\neq i$). Since the determinant of a diagonal matrix equals the product of its diagonal entries, this yields

$$\det\left(\left(\begin{cases} \alpha_i, & \text{if } j=i; \\ 0, & \text{if } j\neq i \end{cases}\right)_{1\leq i\leq u}^{1\leq j\leq u}\right) = \prod_{i=1}^{u} \underbrace{\begin{cases} \alpha_i, & \text{if } i=i; \\ 0, & \text{if } i\neq i \end{cases}}_{=\alpha_i, \text{ since } i=i} = \prod_{i=1}^{u} \alpha_i.$$

Similarly,

$$\det\left(\left(\begin{cases} \beta_i, & \text{if } j=i; \\ 0, & \text{if } j\neq i \end{cases}\right)_{1\leq i\leq v}^{1\leq j\leq v}\right) = \prod_{i=1}^{v} \beta_i.$$

Lemma 7 **(c)** yields

$$\left(\alpha_i a_{i,j}\beta_j\right)_{1\leq i\leq u}^{1\leq j\leq v} = \left(\begin{cases} \alpha_i, & \text{if } j=i; \\ 0, & \text{if } j\neq i \end{cases}\right)_{1\leq i\leq u}^{1\leq j\leq u} \cdot \left(a_{i,j}\right)_{1\leq i\leq u}^{1\leq j\leq v} \cdot \left(\begin{cases} \beta_i, & \text{if } j=i; \\ 0, & \text{if } j\neq i \end{cases}\right)_{1\leq i\leq v}^{1\leq j\leq v}.$$

Thus, if $u=v$, then

$$\begin{aligned}
&\det\left(\left(\alpha_i a_{i,j}\beta_j\right)_{1\leq i\leq u}^{1\leq j\leq v}\right) \\
&= \det\left(\left(\begin{cases} \alpha_i, & \text{if } j=i; \\ 0, & \text{if } j\neq i \end{cases}\right)_{1\leq i\leq u}^{1\leq j\leq u} \cdot \left(a_{i,j}\right)_{1\leq i\leq u}^{1\leq j\leq v} \cdot \left(\begin{cases} \beta_i, & \text{if } j=i; \\ 0, & \text{if } j\neq i \end{cases}\right)_{1\leq i\leq v}^{1\leq j\leq v}\right) \\
&= \underbrace{\det\left(\left(\begin{cases} \alpha_i, & \text{if } j=i; \\ 0, & \text{if } j\neq i \end{cases}\right)_{1\leq i\leq u}^{1\leq j\leq u}\right)}_{=\prod_{i=1}^{u}\alpha_i} \cdot \det\left(\left(a_{i,j}\right)_{1\leq i\leq u}^{1\leq j\leq v}\right) \cdot \underbrace{\det\left(\left(\begin{cases} \beta_i, & \text{if } j=i; \\ 0, & \text{if } j\neq i \end{cases}\right)_{1\leq i\leq v}^{1\leq j\leq v}\right)}_{=\prod_{i=1}^{v}\beta_i} \\
&= \left(\prod_{i=1}^{u}\alpha_i\right)\cdot\left(\prod_{i=1}^{v}\beta_i\right)\cdot\det\left(\left(a_{i,j}\right)_{1\leq i\leq u}^{1\leq j\leq v}\right).
\end{aligned}$$

Thus, Lemma 7 **(d)** is proven.

Now, let us prove Theorem 0:

Proof of Theorem 0. Let $a\in\mathbb{N}$, $b\in\mathbb{N}$ and $c\in\mathbb{N}$. We have

$$\begin{aligned}
&\det\left(\left(\binom{a+b+i-1}{a+i-j}\right)_{1\leq i\leq c}^{1\leq j\leq c}\right) \\
&= \det\left(\left(\frac{(a+b+i-1)!}{(a+i-1)!\cdot(b+j-1)!}\cdot\prod_{k=1}^{j-1}(a+i-k)\right)_{1\leq i\leq c}^{1\leq j\leq c}\right) \qquad \text{(by Lemma 6)} \\
&= \det\left(\left(\frac{(a+b+i-1)!}{(a+i-1)!}\cdot\prod_{k=1}^{j-1}(a+i-k)\cdot\frac{1}{(b+j-1)!}\right)_{1\leq i\leq c}^{1\leq j\leq c}\right) \\
&= \left(\prod_{i=1}^{c}\frac{(a+b+i-1)!}{(a+i-1)!}\right)\cdot\left(\prod_{i=1}^{c}\frac{1}{(b+i-1)!}\right)\cdot\det\left(\left(\prod_{k=1}^{j-1}(a+i-k)\right)_{1\leq i\leq c}^{1\leq j\leq c}\right)
\end{aligned}$$

(by Lemma 7 **(d)**, applied to $R=\mathbb{Q}$, $u=c$, $v=c$, $a_{i,j}=\prod_{k=1}^{j-1}(a+i-k)$, $\alpha_i=\frac{(a+b+i-1)!}{(a+i-1)!}$ and $\beta_i=\frac{1}{(b+i-1)!}$). Since

$$\begin{aligned}
\det\left(\left(\prod_{k=1}^{j-1}(a+i-k)\right)_{1\leq i\leq c}^{1\leq j\leq c}\right) &= \prod_{\substack{(i,j)\in\{1,2,...,c\}^2;\\ i>j}}\left(\underbrace{(a+i)-(a+j)}_{=i-j}\right) \\
&\qquad \text{(by Corollary 4, applied to } R=\mathbb{Z},\ m=c \text{ and } a_i=a+i \text{ for every } i\in\{1,2,...,c\}) \\
&= \prod_{\substack{(i,j)\in\{1,2,...,c\}^2;\\ i>j}}(i-j) = H(c) \qquad \text{(by Lemma 5, applied to } m=c),
\end{aligned}$$

this becomes

$$\det\left(\left(\binom{a+b+i-1}{a+i-j}\right)_{1\leq i\leq c}^{1\leq j\leq c}\right) = \left(\prod_{i=1}^{c}\frac{(a+b+i-1)!}{(a+i-1)!}\right)\cdot\left(\prod_{i=1}^{c}\frac{1}{(b+i-1)!}\right)\cdot H(c). \tag{8}$$

Now,

$$\begin{aligned}
&\frac{H(a)H(b)H(c)H(a+b+c)}{H(b+c)H(c+a)H(a+b)} \\
&= \frac{H(a)H(b)H(c)H(a+b)\cdot\prod_{i=1}^{c}(a+b+i-1)!}{\left(H(b)\cdot\prod_{i=1}^{c}(b+i-1)!\right)\cdot\left(H(a)\cdot\prod_{i=1}^{c}(a+i-1)!\right)\cdot H(a+b)} \\
&\qquad \text{(by (4), (5) and (6))} \\
&= \frac{\prod_{i=1}^{c}(a+b+i-1)!}{\left(\prod_{i=1}^{c}(b+i-1)!\right)\cdot\left(\prod_{i=1}^{c}(a+i-1)!\right)}\cdot H(c) \\
&= \underbrace{\frac{\prod_{i=1}^{c}(a+b+i-1)!}{\prod_{i=1}^{c}(a+i-1)!}}_{=\prod_{i=1}^{c}\frac{(a+b+i-1)!}{(a+i-1)!}}\cdot\underbrace{\frac{1}{\prod_{i=1}^{c}(b+i-1)!}}_{=\prod_{i=1}^{c}\frac{1}{(b+i-1)!}}\cdot H(c) \\
&= \left(\prod_{i=1}^{c}\frac{(a+b+i-1)!}{(a+i-1)!}\right)\cdot\left(\prod_{i=1}^{c}\frac{1}{(b+i-1)!}\right)\cdot H(c) \\
&= \det\left(\left(\binom{a+b+i-1}{a+i-j}\right)_{1\leq i\leq c}^{1\leq j\leq c}\right) \qquad \text{(by (8))} \qquad (9) \\
&\in \mathbb{Z}
\end{aligned}$$

(since $\left(\binom{a+b+i-1}{a+i-j}\right)_{1\leq i\leq c}^{1\leq j\leq c} \in \mathbb{Z}^{c\times c}$). In other words,

$$H(b+c)H(c+a)H(a+b) \mid H(a)H(b)H(c)H(a+b+c).$$

Thus, Theorem 0 is proven.

Remarks.

1. Theorem 0 was briefly mentioned (with a combinatorial interpretation, but without proof) on the first page of [1]. It also follows from the formula (2.1) in [3] (since $\frac{H(a)H(b)H(c)H(a+b+c)}{H(b+c)H(c+a)H(a+b)} = \prod_{i=1}^{c}\frac{(a+b+i-1)!(i-1)!}{(a+i-1)!(b+i-1)!}$), or, equivalently, the formula (2.17) in [4]. It is also generalized in [2], Section 429 (where one has to consider the limit $x \to 1$).

2. We can prove more:

Theorem 8. For every $a \in \mathbb{N}$, every $b \in \mathbb{N}$ and every $c \in \mathbb{N}$, we have

$$\frac{H(a)\,H(b)\,H(c)\,H(a+b+c)}{H(b+c)\,H(c+a)\,H(a+b)} = \det\left(\left(\binom{a+b+i-1}{a+i-j}\right)_{1\leq i\leq c}^{1\leq j\leq c}\right) = \det\left(\left(\binom{a+b}{a+i-j}\right)_{1\leq i\leq c}^{1\leq j\leq c}\right).$$

We recall a useful fact to help us in the proof:

Theorem 9, the Vandermonde convolution identity. Let $x \in \mathbb{Z}$ and $y \in \mathbb{Z}$. Let $q \in \mathbb{Z}$. Then,

$$\binom{x+y}{q} = \sum_{k\in\mathbb{Z}} \binom{x}{k}\binom{y}{q-k}.$$

(The sum on the right hand side is an infinite sum, but only finitely many of its addends are nonzero.)

Proof of Theorem 8. For every $i \in \{1, 2, ..., c\}$ and every $j \in \{1, 2, ..., c\}$, we have

$$\binom{a+b+i-1}{a+i-j} = \sum_{k\in\mathbb{Z}} \binom{a+b}{k}\binom{i-1}{a+i-j-k}$$

(by Theorem 9, applied to $x = a+b$, $y = i-1$ and $q = a+i-j$)

$$= \sum_{\ell\in\mathbb{Z}} \binom{a+b}{a-j+\ell}\binom{i-1}{a+i-j-(a-j+\ell)}$$

(here we substituted $a-j+\ell$ for k in the sum)

$$= \sum_{\ell\in\mathbb{Z}} \binom{a+b}{a-j+\ell}\binom{i-1}{i-\ell}$$

$$= \underbrace{\sum_{\substack{\ell\in\mathbb{Z};\\ (0\leq i-\ell\leq i-1 \text{ is true})}}}_{=\sum_{\substack{\ell\in\mathbb{Z};\\ 0\leq i-\ell\leq i-1}} = \sum_{\substack{\ell\in\mathbb{Z};\\ 1\leq \ell\leq i}} \text{ (since } 0\leq i-\ell\leq i-1 \text{ is equivalent to } 1\leq\ell\leq i)} \binom{a+b}{a-j+\ell}\binom{i-1}{i-\ell} + \sum_{\substack{\ell\in\mathbb{Z};\\ (0\leq i-\ell\leq i-1 \text{ is false})}} \binom{a+b}{a-j+\ell} \underbrace{\binom{i-1}{i-\ell}}_{\substack{=0, \text{ since } i-1\geq 0 \text{ and}\\ (0\leq i-\ell\leq i-1 \text{ is false})}}$$

$$= \sum_{\substack{\ell\in\mathbb{Z};\\ 1\leq\ell\leq i}} \binom{a+b}{a-j+\ell}\binom{i-1}{i-\ell} + \underbrace{\sum_{\substack{\ell\in\mathbb{Z};\\ (0\leq i-\ell\leq i-1 \text{ is false})}} \binom{a+b}{a-j+\ell}\cdot 0}_{=0} = \underbrace{\sum_{\substack{\ell\in\mathbb{Z};\\ 1\leq\ell\leq i}}}_{=\sum_{\ell=1}^{i}} \underbrace{\binom{a+b}{a-j+\ell}\binom{i-1}{i-\ell}}_{=\binom{i-1}{i-\ell}\binom{a+b}{a-j+\ell}}$$

$$= \sum_{\ell=1}^{i} \binom{i-1}{i-\ell}\binom{a+b}{a-j+\ell} = \sum_{\ell=1}^{c} \binom{i-1}{i-\ell}\binom{a+b}{a-j+\ell}$$

$$\left(\begin{array}{c}\text{here we replaced the } \sum_{\ell=1}^{i} \text{ sign by an } \sum_{\ell=1}^{c} \text{ sign, since all addends for } \ell > i \\ \text{are zero (as } \binom{i-1}{i-\ell} = 0 \text{ for } \ell > i\text{, since } i-\ell < 0 \text{ for } \ell > i\text{) and since } c \geq i\end{array}\right).$$

Thus,

$$\begin{aligned}
\left(\binom{a+b+i-1}{a+i-j}\right)_{1\leq i\leq c}^{1\leq j\leq c} &= \left(\sum_{\ell=1}^{c}\binom{i-1}{i-\ell}\binom{a+b}{a-j+\ell}\right)_{1\leq i\leq c}^{1\leq j\leq c} \\
&= \left(\binom{i-1}{i-j}\right)_{1\leq i\leq c}^{1\leq j\leq c} \cdot \left(\underbrace{\binom{a+b}{a-j+i}}_{=\binom{a+b}{a+i-j}}\right)_{1\leq i\leq c}^{1\leq j\leq c} \\
&= \left(\binom{i-1}{i-j}\right)_{1\leq i\leq c}^{1\leq j\leq c} \cdot \left(\binom{a+b}{a+i-j}\right)_{1\leq i\leq c}^{1\leq j\leq c}. \qquad (10)
\end{aligned}$$

Now, the matrix $\left(\binom{i-1}{i-j}\right)_{1\leq i\leq c}^{1\leq j\leq c}$ is lower triangular (since $\binom{i-1}{i-j} = 0$ for every $i \in \{1, 2, ..., m\}$ and $j \in \{1, 2, ..., m\}$ satisfying $i < j$ [6]). Since the determinant of an lower triangular matrix equals the product of its diagonal entries, this yields

$$\det\left(\left(\binom{i-1}{i-j}\right)_{1\leq i\leq c}^{1\leq j\leq c}\right) = \prod_{j=1}^{m} \underbrace{\binom{j-1}{j-j}}_{=\binom{j-1}{0}=1} = \prod_{j=1}^{m} 1 = 1. \qquad (11)$$

Now,

$$\begin{aligned}
\det\left(\left(\binom{a+b+i-1}{a+i-j}\right)_{1\leq i\leq c}^{1\leq j\leq c}\right) &= \det\left(\left(\binom{i-1}{i-j}\right)_{1\leq i\leq c}^{1\leq j\leq c} \cdot \left(\binom{a+b}{a+i-j}\right)_{1\leq i\leq c}^{1\leq j\leq c}\right) \\
&\qquad (\text{by } (10)) \\
&= \underbrace{\det\left(\left(\binom{i-1}{i-j}\right)_{1\leq i\leq c}^{1\leq j\leq c}\right)}_{=1 \text{ by } (11)} \cdot \det\left(\left(\binom{a+b}{a+i-j}\right)_{1\leq i\leq c}^{1\leq j\leq c}\right) \\
&= \det\left(\left(\binom{a+b}{a+i-j}\right)_{1\leq i\leq c}^{1\leq j\leq c}\right).
\end{aligned}$$

Combined with (9), this yields

$$\begin{aligned}
&\frac{H(a)\,H(b)\,H(c)\,H(a+b+c)}{H(b+c)\,H(c+a)\,H(a+b)} \\
&= \det\left(\left(\binom{a+b+i-1}{a+i-j}\right)_{1\leq i\leq c}^{1\leq j\leq c}\right) = \det\left(\left(\binom{a+b}{a+i-j}\right)_{1\leq i\leq c}^{1\leq j\leq c}\right).
\end{aligned}$$

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[6] because $i < j$ yields $i - j < 0$ and thus $\binom{i-1}{i-j} = 0$

Thus, Theorem 8 is proven.

3. We notice a particularly known consequence of Corollary 4:

Corollary 10. Let $m \in \mathbb{N}$. Let a_1, a_2, ..., a_m be m integers. Then,

$$\det \left(\left(\binom{a_i - 1}{j - 1} \right)_{1 \leq i \leq m}^{1 \leq j \leq m} \right) \cdot H(m) = \prod_{\substack{(i,j) \in \{1,2,...,m\}^2; \\ i > j}} (a_i - a_j). \quad (12)$$

In particular,

$$H(m) \mid \prod_{\substack{(i,j) \in \{1,2,...,m\}^2; \\ i > j}} (a_i - a_j).$$

Proof of Corollary 10. Corollary 4 (applied to $R = \mathbb{Z}$) yields

$$\det \left(\left(\prod_{k=1}^{j-1} (a_i - k) \right)_{1 \leq i \leq m}^{1 \leq j \leq m} \right) = \prod_{\substack{(i,j) \in \{1,2,...,m\}^2; \\ i > j}} (a_i - a_j). \quad (13)$$

Now, for every $i \in \{1, 2, ..., m\}$ and $j \in \{1, 2, ..., m\}$, we have

$$\binom{a_i - 1}{j - 1} = \frac{\prod_{k=0}^{(j-1)-1} ((a_i - 1) - k)}{(j-1)!} = \frac{1}{(j-1)!} \prod_{k=0}^{(j-1)-1} ((a_i - 1) - k)$$

$$= \frac{1}{(j-1)!} \prod_{k=1}^{j-1} \left(\underbrace{(a_i - 1) - (k - 1)}_{= a_i - k} \right)$$

(here we substituted $k - 1$ for k in the product)

$$= \frac{1}{(j-1)!} \prod_{k=1}^{j-1} (a_i - k) = 1 \cdot \left(\prod_{k=1}^{j-1} (a_i - k) \right) \cdot \frac{1}{(j-1)!}. \quad (14)$$

Therefore,

$$\det\left(\left(\binom{a_i-1}{j-1}\right)_{1\leq i\leq m}^{1\leq j\leq m}\right)=\det\left(\left(1\cdot\left(\prod_{k=1}^{j-1}(a_i-k)\right)\cdot\frac{1}{(j-1)!}\right)_{1\leq i\leq m}^{1\leq j\leq m}\right)$$

$$=\left(\underbrace{\prod_{i=1}^{m}1}_{=1}\right)\cdot\underbrace{\left(\prod_{i=1}^{m}\frac{1}{(i-1)!}\right)}_{=\frac{1}{\prod_{i=1}^{m}(i-1)!}}\cdot\underbrace{\det\left(\left(\prod_{k=1}^{j-1}(a_i-k)\right)_{1\leq i\leq m}^{1\leq j\leq m}\right)}_{\substack{=\prod_{(i,j)\in\{1,2,\ldots,m\}^2;\ i>j}(a_i-a_j)\\ \text{(by (13))}}}$$

$$\left(\begin{array}{c}\text{by Lemma 7 }\textbf{(d)}\text{, applied to }R=\mathbb{Q},\ u=m,\ v=m,\\ a_{i,j}=\prod_{k=1}^{j-1}(a_i-k),\ \alpha_i=1\text{ and }\beta_i=\frac{1}{(i-1)!}\end{array}\right)$$

$$=\frac{1}{\prod_{i=1}^{m}(i-1)!}\cdot\prod_{\substack{(i,j)\in\{1,2,\ldots,m\}^2;\\ i>j}}(a_i-a_j),$$

so that

$$\prod_{\substack{(i,j)\in\{1,2,\ldots,m\}^2;\\ i>j}}(a_i-a_j)=\det\left(\left(\binom{a_i-1}{j-1}\right)_{1\leq i\leq m}^{1\leq j\leq m}\right)\cdot\prod_{i=1}^{m}(i-1)!$$

$$=\det\left(\left(\binom{a_i-1}{j-1}\right)_{1\leq i\leq m}^{1\leq j\leq m}\right)\cdot\underbrace{\prod_{k=0}^{m-1}k!}_{=H(m)}$$

(here we substituted k for $i-1$ in the product)

$$=\det\left(\left(\binom{a_i-1}{j-1}\right)_{1\leq i\leq m}^{1\leq j\leq m}\right)\cdot H(m).$$

Thus,

$$H(m)\mid\prod_{\substack{(i,j)\in\{1,2,\ldots,m\}^2;\\ i>j}}(a_i-a_j)$$

(since $\det\left(\left(\underbrace{\binom{a_i-1}{j-1}}_{\in\mathbb{Z}}\right)_{1\leq i\leq m}^{1\leq j\leq m}\right)\in\mathbb{Z}$). Thus, Corollary 10 is proven.

Corollary 11. Let $m\in\mathbb{N}$. Let a_1, a_2, ..., a_m be m integers. Then,

$$\det\left(\left(\binom{a_i}{j-1}\right)_{1\leq i\leq m}^{1\leq j\leq m}\right)\cdot H(m)=\prod_{\substack{(i,j)\in\{1,2,\ldots,m\}^2;\\ i>j}}(a_i-a_j).$$

Proof of Corollary 11. The equality (12) (applied to a_i+1 instead of a_i) yields

$$\det\left(\left(\binom{(a_i+1)-1}{j-1}\right)_{1\leq i\leq m}^{1\leq j\leq m}\right)\cdot H(m) = \prod_{\substack{(i,j)\in\{1,2,\ldots,m\}^2;\\ i>j}} \underbrace{((a_i+1)-(a_j+1))}_{=a_i-a_j} = \prod_{\substack{(i,j)\in\{1,2,\ldots,m\}^2;\\ i>j}} (a_i-a_j).$$

Since $(a_i+1)-1=a_i$ for every $i\in\{1,2,\ldots,m\}$, this rewrites as

$$\det\left(\left(\binom{a_i}{j-1}\right)_{1\leq i\leq m}^{1\leq j\leq m}\right)\cdot H(m) = \prod_{\substack{(i,j)\in\{1,2,\ldots,m\}^2;\\ i>j}} (a_i-a_j).$$

This proves Corollary 11.

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[7] See also:

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