A hyperfactorial divisibility
Darij Grinberg
brief version

Let us define a function $H : \mathbb{N} \to \mathbb{N}$ by

$$H(n) = \prod_{k=0}^{n-1} k! \qquad \text{for every } n \in \mathbb{N}.$$

Our goal is to prove the following theorem:

Theorem 0 (MacMahon). We have

$$H(b+c) H(c+a) H(a+b) \mid H(a) H(b) H(c) H(a+b+c)$$

for every $a \in \mathbb{N}$, every $b \in \mathbb{N}$ and every $c \in \mathbb{N}$.

Remark: Here, we denote by $\mathbb{N}$ the set $\{0, 1, 2, ...\}$ (and not the set $\{1, 2, 3, ...\}$, as some authors do).

Before we come to the proof, first some definitions:

Notations.

- Let R be a ring. Let $u \in \mathbb{N}$ and $v \in \mathbb{N}$, and let $a_{i,j}$ be an element of R for every $(i, j) \in \{1, 2, ..., u\} \times \{1, 2, ..., v\}$. Then, we denote by $(a_{i,j})_{1 \leq i \leq u}^{1 \leq j \leq v}$ the $u \times v$ matrix $A \in R^{u \times v}$ whose entry in row i and column j is $a_{i,j}$ for every $(i, j) \in \{1, 2, ..., u\} \times \{1, 2, ..., v\}$.

- Let R be a commutative ring with unity. Let $P \in R[X]$ be a polynomial. Let $j \in \mathbb{N}$. Then, we denote by $\operatorname{coeff}_j P$ the coefficient of the polynomial P before X^j. (In particular, this implies $\operatorname{coeff}_j P = 0$ for every $j > \deg P$.) Thus, for every $P \in R[X]$ and every $d \in \mathbb{N}$ satisfying $\deg P \leq d$, we have

$$P(X) = \sum_{k=0}^{d} \operatorname{coeff}_k (P) \cdot X^k.$$

- Let R be a ring. Let $n \in \mathbb{N}$. Let $a_1, a_2, ..., a_n$ be n elements of R. Then, $\operatorname{diag}(a_1, a_2, ..., a_n)$ will mean the diagonal $n \times n$ matrix whose diagonal entries are $a_1, a_2, ..., a_n$ (from top-left to bottom-right). In other words,
$$\operatorname{diag}(a_1, a_2, ..., a_n) = \left(\begin{cases} a_i, & \text{if } i = j; \\ 0, & \text{if } i \neq j \end{cases} \right)_{1 \leq i \leq n}^{1 \leq j \leq n}.$$

- If n and m are two integers, then the *binomial coefficient* $\binom{m}{n} \in \mathbb{Q}$ is defined by
$$\binom{m}{n} = \begin{cases} \dfrac{m(m-1)\cdots(m-n+1)}{n!}, & \text{if } n \geq 0; \\ 0, & \text{if } n < 0 \end{cases}.$$
It is well-known that $\binom{m}{n} \in \mathbb{Z}$ for all $n \in \mathbb{Z}$ and $m \in \mathbb{Z}$.

We are first going to prove a known fact from linear algebra:

Theorem 1 (Vandermonde determinant). Let R be a commutative ring with unity. Let $m \in \mathbb{N}$. Let a_1, a_2, ..., a_m be m elements of R. Then,

$$\det\left(\left(a_i^{j-1}\right)_{1\leq i\leq m}^{1\leq j\leq m}\right) = \prod_{\substack{(i,j)\in\{1,2,...,m\}^2;\\ i>j}} (a_i - a_j).$$

Actually we are more interested in a corollary - and generalization - of this fact:

Theorem 2 (generalized Vandermonde determinant). Let R be a commutative ring with unity. Let $m \in \mathbb{N}$. For every $j \in \{1, 2, ..., m\}$, let $P_j \in R[X]$ be a polynomial such that $\deg(P_j) \leq j-1$. Let a_1, a_2, ..., a_m be m elements of R. Then,

$$\det\left(\left(P_j(a_i)\right)_{1\leq i\leq m}^{1\leq j\leq m}\right) = \left(\prod_{j=1}^{m} \operatorname{coeff}_{j-1}(P_j)\right) \cdot \prod_{\substack{(i,j)\in\{1,2,...,m\}^2;\\ i>j}} (a_i - a_j).$$

Both Theorems 1 and 2 can be deduced from the following lemma:

Lemma 3. Let R be a commutative ring with unity. Let $m \in \mathbb{N}$. For every $j \in \{1, 2, ..., m\}$, let $P_j \in R[X]$ be a polynomial such that $\deg(P_j) \leq j-1$. Let a_1, a_2, ..., a_m be m elements of R. Then,

$$\det\left(\left(P_j(a_i)\right)_{1\leq i\leq m}^{1\leq j\leq m}\right) = \left(\prod_{j=1}^{m} \operatorname{coeff}_{j-1}(P_j)\right) \cdot \det\left(\left(a_i^{j-1}\right)_{1\leq i\leq m}^{1\leq j\leq m}\right).$$

Proof of Lemma 3. For every $j \in \{1, 2, ..., m\}$, we have $P_j(X) = \sum_{k=0}^{m-1} \operatorname{coeff}_k(P_j) \cdot X^k$ (since $\deg(P_j) \leq j-1 \leq m-1$). Thus, for every $i \in \{1, 2, ..., m\}$ and $j \in \{1, 2, ..., m\}$, we have

$$P_j(a_i) = \sum_{k=0}^{m-1} \operatorname{coeff}_k(P_j) \cdot a_i^k = \sum_{k=0}^{m-1} a_i^k \cdot \operatorname{coeff}_k(P_j) = \sum_{k=1}^{m} a_i^{k-1} \cdot \operatorname{coeff}_{k-1}(P_j)$$

(here we substituted $k-1$ for k in the sum). Hence,

$$\left(P_j(a_i)\right)_{1\leq i\leq m}^{1\leq j\leq m} = \left(a_i^{j-1}\right)_{1\leq i\leq m}^{1\leq j\leq m} \cdot \left(\operatorname{coeff}_{i-1}(P_j)\right)_{1\leq i\leq m}^{1\leq j\leq m}.$$

But the matrix $\left(\operatorname{coeff}_{i-1}(P_j)\right)_{1\leq i\leq m}^{1\leq j\leq m}$ is upper triangular (since $\operatorname{coeff}_{i-1}(P_j) = 0$ for every $i \in \{1, 2, ..., m\}$ and $j \in \{1, 2, ..., m\}$ satisfying $i > j$ [1]); hence,

[1] because $i > j$ yields $i-1 > j-1$, thus $i-1 > \deg(P_j)$ (since $\deg(P_j) \leq j-1$) and therefore $\operatorname{coeff}_{i-1}(P_j) = 0$

$\det\left(\left(\operatorname{coeff}_{i-1}(P_j)\right)_{1\le i\le m}^{1\le j\le m}\right) = \prod_{j=1}^{m} \operatorname{coeff}_{j-1}(P_j)$ (since the determinant of an upper triangular matrix equals the product of its diagonal entries).

Now,

$$\begin{aligned}\det\left(\left(P_j(a_i)\right)_{1\le i\le m}^{1\le j\le m}\right) &= \det\left(\left(a_i^{j-1}\right)_{1\le i\le m}^{1\le j\le m} \cdot \left(\operatorname{coeff}_{i-1}(P_j)\right)_{1\le i\le m}^{1\le j\le m}\right) \\ &= \det\left(\left(a_i^{j-1}\right)_{1\le i\le m}^{1\le j\le m}\right) \cdot \underbrace{\det\left(\left(\operatorname{coeff}_{i-1}(P_j)\right)_{1\le i\le m}^{1\le j\le m}\right)}_{=\prod_{j=1}^{m}\operatorname{coeff}_{j-1}(P_j)} \\ &= \left(\prod_{j=1}^{m} \operatorname{coeff}_{j-1}(P_j)\right) \cdot \det\left(\left(a_i^{j-1}\right)_{1\le i\le m}^{1\le j\le m}\right),\end{aligned}$$

and thus, Lemma 3 is proven.

Proof of Theorem 1. For every $j \in \{1, 2, ..., m\}$, define a polynomial $P_j \in R[X]$ by $P_j(X) = \prod_{k=1}^{j-1}(X - a_k)$. Then, P_j is a monic polynomial of degree $j-1$. In other words, $\deg(P_j) = j-1$ and $\operatorname{coeff}_{j-1}(P_j) = 1$ for every $j \in \{1, 2, ..., m\}$. Thus, Lemma 3 yields

$$\det\left(\left(P_j(a_i)\right)_{1\le i\le m}^{1\le j\le m}\right) = \left(\prod_{j=1}^{m} \operatorname{coeff}_{j-1}(P_j)\right) \cdot \det\left(\left(a_i^{j-1}\right)_{1\le i\le m}^{1\le j\le m}\right). \qquad (1)$$

But the matrix $\left(P_j(a_i)\right)_{1\le i\le m}^{1\le j\le m}$ is lower triangular (since $P_j(a_i) = 0$ for every $i \in \{1, 2, ..., m\}$ and $j \in \{1, 2, ..., m\}$ satisfying $i < j$, as follows quickly from the definition of P_j); hence, $\det\left(\left(P_j(a_i)\right)_{1\le i\le m}^{1\le j\le m}\right) = \prod_{j=1}^{m} P_j(a_j)$ (since the determinant of a lower triangular matrix equals the product of its diagonal entries). Thus, (1) becomes

$$\prod_{j=1}^{m} P_j(a_j) = \left(\prod_{j=1}^{m} \underbrace{\operatorname{coeff}_{j-1}(P_j)}_{=1}\right) \cdot \det\left(\left(a_i^{j-1}\right)_{1\le i\le m}^{1\le j\le m}\right) = \det\left(\left(a_i^{j-1}\right)_{1\le i\le m}^{1\le j\le m}\right).$$

But $P_j(X) = \prod_{k=1}^{j-1}(X - a_k)$ yields $P_j(a_j) = \prod_{k=1}^{j-1}(a_j - a_k)$, so that

$$\begin{aligned}\det\left(\left(a_i^{j-1}\right)_{1\le i\le m}^{1\le j\le m}\right) &= \prod_{j=1}^{m} P_j(a_j) = \prod_{j=1}^{m}\prod_{k=1}^{j-1}(a_j - a_k) = \prod_{\substack{(j,k)\in\{1,2,...,m\}^2;\\ k<j}} (a_j - a_k) \\ &= \prod_{\substack{(i,j)\in\{1,2,...,m\}^2;\\ j<i}} (a_i - a_j) = \prod_{\substack{(i,j)\in\{1,2,...,m\}^2;\\ i>j}} (a_i - a_j).\end{aligned}$$

Hence, Theorem 1 is proven.

Now, Theorem 2 immediately follows from Lemma 3 and Theorem 1.

A consequence of Theorem 2:

Corollary 4. Let R be a commutative ring with unity. Let $m \in \mathbb{N}$. Let a_1, a_2, ..., a_m be m elements of R. Then,

$$\det\left(\left(\prod_{k=1}^{j-1}(a_i - k)\right)_{1 \leq i \leq m}^{1 \leq j \leq m}\right) = \prod_{\substack{(i,j) \in \{1,2,...,m\}^2; \\ i > j}} (a_i - a_j).$$

Proof of Corollary 4. For every $j \in \{1, 2, ..., m\}$, define a polynomial $P_j \in R[X]$ by $P_j(X) = \prod_{k=1}^{j-1}(X - k)$. Then, P_j is a monic polynomial of degree $j - 1$. In other words, $\deg(P_j) = j - 1$ and $\operatorname{coeff}_{j-1}(P_j) = 1$ for every $j \in \{1, 2, ..., m\}$. Thus, applying Theorem 2 to these polynomials P_j yields the assertion of Corollary 4.

Also notice that:

Lemma 5. Let $m \in \mathbb{N}$. Then,

$$\prod_{\substack{(i,j) \in \{1,2,...,m\}^2; \\ i > j}} (i - j) = H(m).$$

Proof of Lemma 5. We have

$$\prod_{\substack{(i,j) \in \{1,2,...,m\}^2; \\ i > j}} (i - j) = \prod_{\substack{(i,j) \in \{0,1,...,m-1\}^2; \\ i > j}} (i - j)$$

(here we shifted i and j by 1, which doesn't change anything since $i - j$ remains constant)

$$= \prod_{i \in \{0,1,...,m-1\}} \prod_{\substack{j \in \{0,1,...,m-1\}; \\ i > j}} (i - j) = \prod_{i \in \{0,1,...,m-1\}} \prod_{j=0}^{i-1} (i - j) = \prod_{i \in \{0,1,...,m-1\}} \underbrace{\prod_{j=1}^{i} j}_{=i!}$$

(here, we substituted j for $i - j$ in the second product)

$$= \prod_{i \in \{0,1,...,m-1\}} i! = \prod_{k=0}^{m-1} k! = H(m).$$

Hence, Lemma 5 is proven.

Now, we notice that every $a \in \mathbb{N}$, every $b \in \mathbb{N}$ and every $c \in \mathbb{N}$ satisfy

$$H(a + b + c) = \prod_{k=0}^{a+b+c-1} k! = \left(\underbrace{\prod_{k=0}^{a+b-1} k!}_{=H(a+b)}\right) \cdot \prod_{k=a+b}^{a+b+c-1} k! = H(a + b) \cdot \prod_{k=a+b}^{a+b+c-1} k!$$

$$= H(a + b) \cdot \prod_{i=1}^{c} (a + b + i - 1)!$$

(here we substituted $a + b + i - 1$ for k in the product), (2)

$$H(b+c) = \prod_{k=0}^{b+c-1} k! = \left(\underbrace{\prod_{k=0}^{b-1} k!}_{=H(b)} \right) \cdot \prod_{k=b}^{b+c-1} k! = H(b) \cdot \prod_{k=b}^{b+c-1} k! = H(b) \cdot \prod_{i=1}^{c} (b+i-1)!$$

$$(\text{here we substituted } b+i-1 \text{ for } k \text{ in the product}), \qquad (3)$$

$$H(c+a) = \prod_{k=0}^{c+a-1} k! = \left(\underbrace{\prod_{k=0}^{a-1} k!}_{=H(a)} \right) \cdot \prod_{k=a}^{c+a-1} k! = H(a) \cdot \prod_{k=a}^{c+a-1} k! = H(a) \cdot \prod_{i=1}^{c} (a+i-1)!$$

$$(\text{here we substituted } a+i-1 \text{ for } k \text{ in the product}). \qquad (4)$$

Next, a technical lemma.

Lemma 6. For every $i \in \mathbb{N}$ and $j \in \mathbb{N}$ satisfying $i \geq 1$ and $j \geq 1$, we have

$$\binom{a+b+i-1}{a+i-j} = \frac{(a+b+i-1)!}{(a+i-1)! \cdot (b+j-1)!} \cdot \prod_{k=1}^{j-1} (a+i-k).$$

The *proof* of this lemma is completely straightforward: Either we have $a+i-j \geq 0$ and Lemma 6 follows from standard manipulations with binomial coefficients, or we have $a+i-j < 0$ and Lemma 6 follows from $\binom{a+b+i-1}{a+i-j} = 0$ and $\prod_{k=1}^{j-1} (a+i-k) = 0$.

Another trivial lemma:

Lemma 7. Let R be a commutative ring with unity. Let $u \in \mathbb{N}$, and let $a_{i,j}$ be an element of R for every $(i,j) \in \{1,2,...,u\}^2$.

Let α_1, α_2, ..., α_u be u elements of R. Let β_1, β_2, ..., β_u be u elements of R. Then,

$$\det\left((\alpha_i a_{i,j} \beta_j)_{1 \leq i \leq u}^{1 \leq j \leq u} \right) = \prod_{i=1}^{u} \alpha_i \cdot \prod_{i=1}^{u} \beta_i \cdot \det\left((a_{i,j})_{1 \leq i \leq u}^{1 \leq j \leq u} \right).$$

This is clear because the matrix $(\alpha_i a_{i,j} \beta_j)_{1 \leq i \leq u}^{1 \leq j \leq u}$ can be written as the product

$$\operatorname{diag}(\alpha_1, \alpha_2, ..., \alpha_u) \cdot (a_{i,j})_{1 \leq i \leq u}^{1 \leq j \leq u} \cdot \operatorname{diag}(\beta_1, \beta_2, ..., \beta_u),$$

and thus

$$\begin{aligned} &\det\left((\alpha_i a_{i,j} \beta_j)_{1 \leq i \leq u}^{1 \leq j \leq u} \right) \\ &= \det\left(\operatorname{diag}(\alpha_1, \alpha_2, ..., \alpha_u) \cdot (a_{i,j})_{1 \leq i \leq u}^{1 \leq j \leq u} \cdot \operatorname{diag}(\beta_1, \beta_2, ..., \beta_u) \right) \\ &= \underbrace{\det(\operatorname{diag}(\alpha_1, \alpha_2, ..., \alpha_u))}_{=\prod_{i=1}^{u} \alpha_i} \cdot \det\left((a_{i,j})_{1 \leq i \leq u}^{1 \leq j \leq u} \right) \cdot \underbrace{\det(\operatorname{diag}(\beta_1, \beta_2, ..., \beta_u))}_{=\prod_{i=1}^{u} \beta_i}. \end{aligned}$$

Now, back to proving Theorem 0:

We have

$$\det\left(\left(\binom{a+b+i-1}{a+i-j}\right)_{1\leq i\leq c}^{1\leq j\leq c}\right)$$
$$=\det\left(\left(\frac{(a+b+i-1)!}{(a+i-1)!\cdot(b+j-1)!}\cdot\prod_{k=1}^{j-1}(a+i-k)\right)_{1\leq i\leq c}^{1\leq j\leq c}\right) \qquad \text{(by Lemma 6)}$$
$$=\det\left(\left(\frac{(a+b+i-1)!}{(a+i-1)!}\cdot\prod_{k=1}^{j-1}(a+i-k)\cdot\frac{1}{(b+j-1)!}\right)_{1\leq i\leq c}^{1\leq j\leq c}\right)$$
$$=\prod_{i=1}^{c}\frac{(a+b+i-1)!}{(a+i-1)!}\cdot\prod_{i=1}^{c}\frac{1}{(b+i-1)!}\cdot\det\left(\left(\prod_{k=1}^{j-1}(a+i-k)\right)_{1\leq i\leq c}^{1\leq j\leq c}\right)$$

(by Lemma 7, applied to $R=\mathbb{Q}$, $u=c$, $a_{i,j}=\prod\limits_{k=1}^{j-1}(a+i-k)$, $\alpha_i=\dfrac{(a+b+i-1)!}{(a+i-1)!}$ and $\beta_i=\dfrac{1}{(b+i-1)!}$). Since

$$\det\left(\left(\prod_{k=1}^{j-1}(a+i-k)\right)_{1\leq i\leq c}^{1\leq j\leq c}\right)=\prod_{\substack{(i,j)\in\{1,2,\ldots,c\}^2;\\ i>j}}\left(\underbrace{(a+i)-(a+j)}_{=i-j}\right)$$

(by Corollary 4, applied to $R=\mathbb{Z}$, $m=c$ and $a_i=a+i$ for every $i\in\{1,2,...,c\}$)

$$=\prod_{\substack{(i,j)\in\{1,2,\ldots,c\}^2;\\ i>j}}(i-j)=H(c) \qquad \text{(by Lemma 5, applied to } m=c\text{)},$$

this becomes

$$\det\left(\left(\binom{a+b+i-1}{a+i-j}\right)_{1\leq i\leq c}^{1\leq j\leq c}\right)=\prod_{i=1}^{c}\frac{(a+b+i-1)!}{(a+i-1)!}\cdot\prod_{i=1}^{c}\frac{1}{(b+i-1)!}\cdot H(c). \tag{5}$$

Now,

$$\frac{H(a)\,H(b)\,H(c)\,H(a+b+c)}{H(b+c)\,H(c+a)\,H(a+b)}$$

$$= \frac{H(a)\,H(b)\,H(c)\,H(a+b)\cdot\prod_{i=1}^{c}(a+b+i-1)!}{\left(H(b)\cdot\prod_{i=1}^{c}(b+i-1)!\right)\cdot\left(H(a)\cdot\prod_{i=1}^{c}(a+i-1)!\right)\cdot H(a+b)}$$

(by (2), (3) and (4))

$$= \frac{\prod_{i=1}^{c}(a+b+i-1)!}{\prod_{i=1}^{c}(b+i-1)!\cdot\prod_{i=1}^{c}(a+i-1)!}\cdot H(c) = \underbrace{\frac{\prod_{i=1}^{c}(a+b+i-1)!}{\prod_{i=1}^{c}(a+i-1)!}}_{=\prod_{i=1}^{c}\frac{(a+b+i-1)!}{(a+i-1)!}}\cdot\underbrace{\frac{1}{\prod_{i=1}^{c}(b+i-1)!}}_{=\prod_{i=1}^{c}\frac{1}{(b+i-1)!}}\cdot H(c)$$

$$= \left(\prod_{i=1}^{c}\frac{(a+b+i-1)!}{(a+i-1)!}\right)\cdot\left(\prod_{i=1}^{c}\frac{1}{(b+i-1)!}\right)\cdot H(c)$$

$$= \det\left(\left(\binom{a+b+i-1}{a+i-j}\right)_{1\leq i\leq c}^{1\leq j\leq c}\right) \qquad \text{(by (5))} \qquad (6)$$

$$\in \mathbb{Z}$$

(since $\left(\binom{a+b+i-1}{a+i-j}\right)_{1\leq i\leq c}^{1\leq j\leq c} \in \mathbb{Z}^{c\times c}$). In other words,

$$H(b+c)\,H(c+a)\,H(a+b) \mid H(a)\,H(b)\,H(c)\,H(a+b+c).$$

Thus, Theorem 0 is finally proven.

Remarks.

1. Theorem 0 was briefly mentioned (with a combinatorial interpretation, but without proof) on the first page of [1]. It also follows from the formula (2.1) in [3] (since $\frac{H(a)\,H(b)\,H(c)\,H(a+b+c)}{H(b+c)\,H(c+a)\,H(a+b)} = \prod_{i=1}^{c}\frac{(a+b+i-1)!\,(i-1)!}{(a+i-1)!\,(b+i-1)!}$), or, equivalently, the formula (2.17) in [4]. It is also generalized in [2], Section 429 (where one has to consider the limit $x\to 1$).

2. We can prove more:

Theorem 8. For every $a\in\mathbb{N}$, every $b\in\mathbb{N}$ and every $c\in\mathbb{N}$, we have

$$\frac{H(a)\,H(b)\,H(c)\,H(a+b+c)}{H(b+c)\,H(c+a)\,H(a+b)}$$
$$= \det\left(\left(\binom{a+b+i-1}{a+i-j}\right)_{1\leq i\leq c}^{1\leq j\leq c}\right) = \det\left(\left(\binom{a+b}{a+i-j}\right)_{1\leq i\leq c}^{1\leq j\leq c}\right).$$

We recall a useful fact to help us in the proof:

Theorem 9, the Vandermonde convolution identity. Let $x \in \mathbb{Z}$ and $y \in \mathbb{Z}$. Let $q \in \mathbb{Z}$. Then,

$$\binom{x+y}{q} = \sum_{k\in\mathbb{Z}} \binom{x}{k} \binom{y}{q-k}.$$

(The sum on the right hand side is an infinite sum, but only finitely many of its addends are nonzero.)

Proof of Theorem 8. For every $i \in \{1, 2, ..., c\}$ and every $j \in \{1, 2, ..., c\}$, we have

$$\begin{aligned}
\binom{a+b+i-1}{a+i-j} &= \sum_{k\in\mathbb{Z}} \binom{a+b}{k} \binom{i-1}{a+i-j-k} \\
&\qquad \text{(by Theorem 9, applied to } x = a+b,\ y = i-1 \text{ and } q = a+i-j\text{)} \\
&= \sum_{\ell\in\mathbb{Z}} \binom{a+b}{a-j+\ell} \binom{i-1}{i-\ell} \qquad \text{(here we substituted } a-j+\ell \text{ for } k \text{ in the sum)} \\
&= \sum_{\ell=1}^{c} \binom{a+b}{a-j+\ell} \binom{i-1}{i-\ell} \\
&\qquad \left(\begin{array}{c} \text{here, we restricted the summation from } \ell \in \mathbb{Z} \text{ to } \ell \in \{1, 2, ..., c\}\,, \\ \text{which doesn't change the sum because} \\ \dbinom{a+b}{a-j+\ell} \dbinom{i-1}{i-\ell} = 0 \text{ for all } \ell \in \mathbb{Z} \diagdown \{1, 2, ..., c\} \end{array}\right) \\
&= \sum_{\ell=1}^{c} \binom{i-1}{i-\ell} \binom{a+b}{a-j+\ell}.
\end{aligned}$$

Thus,

$$\begin{aligned}
\left(\binom{a+b+i-1}{a+i-j}\right)_{1\leq i\leq c}^{1\leq j\leq c} &= \left(\sum_{\ell=1}^{c} \binom{i-1}{i-\ell} \binom{a+b}{a-j+\ell}\right)_{1\leq i\leq c}^{1\leq j\leq c} \\
&= \left(\binom{i-1}{i-j}\right)_{1\leq i\leq c}^{1\leq j\leq c} \cdot \left(\binom{a+b}{a-j+i}\right)_{1\leq i\leq c}^{1\leq j\leq c} \\
&= \left(\binom{i-1}{i-j}\right)_{1\leq i\leq c}^{1\leq j\leq c} \cdot \left(\binom{a+b}{a+i-j}\right)_{1\leq i\leq c}^{1\leq j\leq c}. \qquad (7)
\end{aligned}$$

Now, the matrix $\left(\binom{i-1}{i-j}\right)_{1\leq i\leq c}^{1\leq j\leq c}$ is lower triangular (since $\binom{i-1}{i-j} = 0$ for every $i \in \{1, 2, ..., m\}$ and $j \in \{1, 2, ..., m\}$ satisfying $i < j$). Since the determinant of an lower triangular matrix equals the product of its diagonal entries, this yields

$$\det\left(\left(\binom{i-1}{i-j}\right)_{1\leq i\leq c}^{1\leq j\leq c}\right) = \prod_{j=1}^{m} \underbrace{\binom{j-1}{j-j}}_{=\binom{j-1}{0}=1} = 1. \qquad (8)$$

Now,

$$\det\left(\left(\binom{a+b+i-1}{a+i-j}\right)_{1\leq i\leq c}^{1\leq j\leq c}\right)=\det\left(\left(\binom{i-1}{i-j}\right)_{1\leq i\leq c}^{1\leq j\leq c}\cdot\left(\binom{a+b}{a+i-j}\right)_{1\leq i\leq c}^{1\leq j\leq c}\right)$$

$$\text{(by (7))}$$

$$=\underbrace{\det\left(\left(\binom{i-1}{i-j}\right)_{1\leq i\leq c}^{1\leq j\leq c}\right)}_{=1\text{ by (8)}}\cdot\det\left(\left(\binom{a+b}{a+i-j}\right)_{1\leq i\leq c}^{1\leq j\leq c}\right)$$

$$=\det\left(\left(\binom{a+b}{a+i-j}\right)_{1\leq i\leq c}^{1\leq j\leq c}\right).$$

Combined with (6), this yields Theorem 8.

3. We notice a particularly known consequence of Corollary 4:

Corollary 10. Let $m\in\mathbb{N}$. Let a_1, a_2, ..., a_m be m integers. Then,

$$\det\left(\left(\binom{a_i-1}{j-1}\right)_{1\leq i\leq m}^{1\leq j\leq m}\right)\cdot H(m)=\prod_{\substack{(i,j)\in\{1,2,...,m\}^2;\\ i>j}}(a_i-a_j).$$

In particular,

$$H(m)\mid\prod_{\substack{(i,j)\in\{1,2,...,m\}^2;\\ i>j}}(a_i-a_j).$$

Proof of Corollary 10. For every $i\in\{1,2,...,m\}$ and $j\in\{1,2,...,m\}$, we have

$$\binom{a_i-1}{j-1}=\frac{\prod\limits_{k=1}^{j-1}(a_i-k)}{(j-1)!}=1\cdot\prod_{k=1}^{j-1}(a_i-k)\cdot\frac{1}{(j-1)!}. \tag{9}$$

Therefore,

$$\det\left(\left(\binom{a_i-1}{j-1}\right)_{1\leq i\leq m}^{1\leq j\leq m}\right)=\det\left(\left(1\cdot\prod_{k=1}^{j-1}(a_i-k)\cdot\frac{1}{(j-1)!}\right)_{1\leq i\leq m}^{1\leq j\leq m}\right)$$

$$=\underbrace{\prod_{i=1}^{m}1}_{=1}\cdot\underbrace{\prod_{i=1}^{m}\frac{1}{(i-1)!}}_{=\prod\limits_{k=0}^{m-1}\frac{1}{k!}=\frac{1}{\prod\limits_{i=1}^{m-1}k!}=\frac{1}{H(m)}}\cdot\underbrace{\det\left(\left(\prod_{k=1}^{j-1}(a_i-k)\right)_{1\leq i\leq m}^{1\leq j\leq m}\right)}_{=\prod\limits_{\substack{(i,j)\in\{1,2,...,m\}^2;\\ i>j}}(a_i-a_j)\text{ by Corollary 4}}$$

$$\left(\text{by Lemma 7, applied to } R=\mathbb{Q},\ u=m,\ a_{i,j}=\prod_{k=1}^{j-1}(a_i-k),\ \alpha_i=1\text{ and }\beta_i=\frac{1}{(i-1)!}\right)$$

$$=\frac{1}{H(m)}\cdot\prod_{\substack{(i,j)\in\{1,2,...,m\}^2;\\ i>j}}(a_i-a_j),$$

so that

$$\prod_{\substack{(i,j)\in\{1,2,...,m\}^2;\\ i>j}} (a_i - a_j) = \det\left(\left(\binom{a_i - 1}{j-1}\right)_{1\leq i\leq m}^{1\leq j\leq m}\right) \cdot H(m).$$

Thus,

$$H(m) \mid \prod_{\substack{(i,j)\in\{1,2,...,m\}^2;\\ i>j}} (a_i - a_j)$$

(since $\det\left(\left(\underbrace{\binom{a_i - 1}{j-1}}_{\in\mathbb{Z}}\right)_{1\leq i\leq m}^{1\leq j\leq m}\right) \in \mathbb{Z}$). Thus, Corollary 10 is proven.

Corollary 11. Let $m \in \mathbb{N}$. Let a_1, a_2, ..., a_m be m integers. Then,

$$\det\left(\left(\binom{a_i}{j-1}\right)_{1\leq i\leq m}^{1\leq j\leq m}\right) \cdot H(m) = \prod_{\substack{(i,j)\in\{1,2,...,m\}^2;\\ i>j}} (a_i - a_j).$$

Proof of Corollary 11. This follows from Corollary 10, applied to a_i+1 instead of a_i.

References

[1] Theresia Eisenkölbl, *Rhombus tilings of a hexagon with three fixed border tiles*, J. Combin. Theory Ser. A 88 (1999), 368-378; arXiv:math/9712261v2 [math.CO].
http://arxiv.org/abs/math/9712261v2

[2] Percy Alexander MacMahon, *Combinatory Analysis, vol. 2*, Cambridge University Press, 1916; reprinted by Chelsea, New York, 1960.
http://www.archive.org/details/combinatoryanaly02macmuoft [2]

[3] Christian Krattenthaler, *Advanced Determinant Calculus: A Complement*, Linear Algebra Appl. 411 (2005), 68-166; arXiv:math/0503507v2 [math.CO].
http://arxiv.org/abs/math.CO/0503507v2

[4] Christian Krattenthaler, *Advanced Determinant Calculus*, S\'eminaire Lotharingien Combin. 42 (1999) (The Andrews Festschrift), paper B42q, 67 pp; arXiv:math/9902004v3 [math.CO].
http://arxiv.org/abs/math/9902004v3

[2] See also:

- Percy Alexander MacMahon, *Combinatory Analysis, vol. 1*, Cambridge University Press, 1915, http://www.archive.org/details/combinatoryanal01macmuoft;
- Percy Alexander MacMahon, *An introduction to Combinatory analysis*, Cambridge University Press, 1920, http://www.archive.org/details/introductiontoco00macmrich.