The Hopf algebra of finite topologies and T-partitions

Loic Foissy and Claudia Malvenuto

version of 4 October 2014 (arXiv preprint arXiv:1407.0476v2)

Errata and addenda by Darij Grinberg

- **Various places:** You use the two different "spellings" "T-partition" (with a mathmode "T") and "T-partition" (with a textmode "T") synonymously. It would be better if you keep to one of them. I, personally, prefer the latter, since the former (falsely) suggests that it depends on some object called "T" (which does not exist and is a red herring; the actual argument is the "$\mathcal{T}$" in "T-partition of $\mathcal{T}$").

- **Page 2:** Replace "Retenauer" by "Reutenauer".

- **Page 2:** "in this the present text" $\to$ "in the present text".

- **Page 2:** "is totally ordered by the refinement" $\to$ "is partially ordered by refinement".

- **Page 2:** I think "the coproducts" should be "the coproduct", or not?

- **Page 3:** In the definition of a "strict T-partition", replace "and $k \leq_{\mathcal{T}} i$" by "and $j \leq_{\mathcal{T}} i$".

- **Page 3:** In the displayed equation

  $$\Gamma_{(1,1,1)}(\mathcal{T}) = \sum_{f \text{ generalized } T\text{-partition of } \mathbf{T}} f, \qquad \Gamma_{(1,0,0)}(\mathcal{T}) = \sum_{f \text{ strict } T\text{-partition of } \mathbf{T}} f,$$

  replace both appearances of "$\mathbf{T}$" by "$\mathcal{T}$".

- **Page 3:** "to defined" $\to$ "to define".

- **Page 4:** Somewhere here it would be good to point out that $\max \varnothing$ is to be understood as 0. (You use this convention when you write $\max f$ for a packed word f.)

- **Page 4, §1.1:** In the "$\Delta((511423))$" example, replace "$(1) \otimes (4312)$" by "$(11) \otimes (4312)$".

- **Page 4, §1.1:** It would be better if you use a different letter for the involution that you call j; you tend to use the letter "j" for integers too.

- **Page 5:** In the first displayed equation on page 5, replace "$\sigma \otimes \tau$" by "$\sigma \otimes \sigma'$".

- **Page 5:** Also, in the same equation, it would be good if you define $\sigma \otimes \sigma'$. (I assume that it means the permutation in $\mathfrak{S}_{n+n'}$ which sends every $k \in \{1, 2, \ldots, n+n'\}$ to $\begin{cases} \sigma(k), & \text{if } k \leq n; \\ \sigma'(k-n)+n, & \text{if } k > n \end{cases}$.)
- **Page 6:** "we define a special posets" should be "we define a special poset".
- **Page 6:** "(isoclasses) of special posets" should be "(isoclasses of) special posets".
- **Page 6:** In "Up to a unique isomorphism, we can assume that $P = [n]$ as a totally ordered set", replace "P" by "$(P, \leq_{tot})$".
- **Page 6:** Your definition of L is incompatible with the definition given in the reference [11]. (Also, the way how you identify linear extensions of special double posets with certain permutations is incompatible with how you do it in [11].) Maybe warn the reader about this, if this is your intention?
- **Page 7, §1.2:** "We represent a P-partition" should be "We represent a P-partition f".
- **Page 7, Remark:** In the summation subscript "f P-partition of w", replace "w" by "$\mathcal{P}$".
- **Page 7, §1.2:** Replace "$\{w \mid Pack(w) = \sigma\}$" by "$\{w \mid Std(w) = \sigma\}$".
- **Page 8, §2.1:** You write: "Moreover, the open sets of $\mathcal{T}$ are the ideals of $\leq_{\mathcal{T}}$". This was a neat exercise to prove, but maybe a reference to a proof in the literature would not hurt?
- **Page 8:** In "Moreover, $\leq_{T_\leq} = \leq$, and $\mathcal{T}_{\leq_{\mathcal{T}}} = \mathcal{T}$", replace "$T$" by "$\mathcal{T}$".
- **Page 8:** Replace "$\bar{i} \leq_{\mathcal{T}} \bar{j}$ if $i \leq j$" by "$\bar{i} \leq_{\mathcal{T}} \bar{j}$ if $i \leq_{\mathcal{T}} j$" (or generalize this definition of quotient posets to an arbitrary preorder).
- **Page 9, Definition 2:** I think "which" should be "whose" (both in (1) and in (2)).
- **Page 11, §2.3:** When you define $Std(\mathcal{T})$ and $\mathcal{T}_{|Y}$, it would be useful to point out how these definitions translate into the language of preorders. Namely:
  - If X is a finite totally ordered set of cardinality n, if $\mathcal{T}$ is a topology on X, and if ϕ is the unique increasing bijection from X to $[n]$, then two elements i and j of $[n]$ satisfy $i \leq_{Std(\mathcal{T})} j$ if and only if $\phi^{-1}(i) \leq_{\mathcal{T}} \phi^{-1}(j)$.
  - If X is a finite set, if $\mathcal{T}$ is a topology on X, and if $Y \subseteq X$, then two elements i and j of Y satisfy $i \leq_{\mathcal{T}_{|Y}} j$ if and only if $i \leq_{\mathcal{T}} j$.

- **Page 11, Proposition 6:** Replace "$n \geq 1$" by "$n \geq 0$".
- **Page 11, proof of Proposition 6:** At the beginning of this proof, I would suggest adding the observation that if X is a finite totally ordered set and if $\mathcal{T}$ is a topology on X, then

$$\Delta\left(Std\left(\mathcal{T}\right)\right) = \sum_{O\in\mathcal{T}} Std\left(\mathcal{T}_{|X\setminus O}\right) \otimes Std\left(\mathcal{T}_{|O}\right).$$

  This formula generalizes the formula that you use to define Δ, and it can be easily derived from the latter (since the open sets of $\mathcal{T}$ and the open sets of $Std\left(\mathcal{T}\right)$ are in an obvious 1-to-1 correspondence). You later tacitly use this formula when you compute $\left(\Delta \otimes Id\right) \circ \Delta\left(\mathcal{T}\right)$ and $\left(Id \otimes \Delta\right) \circ \Delta\left(\mathcal{T}\right)$.
- **Page 12:** It would be helpful to explain the notations $O.O'$ (meaning $O \sqcup O'\left(+n\right)$) and $O \downarrow [n']$ (meaning $O \sqcup [n']\left(+n\right)$), where $O \in \mathcal{T}$ and $O' \in \mathcal{T}'$ for two topologies $\mathcal{T} \in \mathbf{T}_n$ and $\mathcal{T}' \in \mathbf{T}_{n'}$.
- **Page 12:** On the second line of the computation that shows $\Delta\left(\mathcal{T}.\mathcal{T}'\right) = \Delta\left(\mathcal{T}\right).\Delta\left(\mathcal{T}'\right)$, replace "$\mathcal{T}'_{[n']\setminus O'}$" by "$\mathcal{T}'_{|[n']\setminus O'}$", and also replace "$\mathcal{T}\mid_{O'}$" by "$\mathcal{T}'\mid_{O'}$".
- **Page 12:** On the third line of the same computation, again replace "$\mathcal{T}\mid_{O'}$" by "$\mathcal{T}'\mid_{O'}$".
- **Page 12:** On the second-to-last line of the last computation of this page, replace "$\Delta\left(\mathcal{T}\right) - 1 \otimes \mathcal{T}'$" by "$\Delta\left(\mathcal{T}'\right) - 1 \otimes \mathcal{T}'$".
- **Page 12:** On the last line of the last computation of this page, replace "$\left(\mathcal{T} \otimes 1\right) \downarrow \Delta\left(\mathcal{T}\right)$" by "$\left(\mathcal{T} \otimes 1\right) \downarrow \Delta\left(\mathcal{T}'\right)$".
- **Page 13, §2.4:** Here is one observation that you tacitly use in some of your arguments: If X is a finite totally ordered set with total ordering $\leq_{tot}$, and if $\mathcal{T}$ is a topology on X such that $\mathcal{T}$ is T_0, then

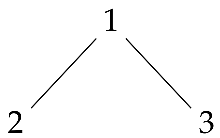
$$Std\left(\mathcal{T}\right) \cong \left(X, \leq_{\mathcal{T}}, \leq_{tot}\right) \tag{1}$$

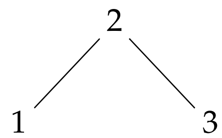
  as special posets.
- **Page 14, proof of Proposition 8:** Replace "$\sim'_{\mathcal{T}}$" by "$\sim_{\mathcal{T}'}$" (on the 4th line of page 14).
- **Page 14, proof of Proposition 8:** Replace "If $\mathcal{T} \in \mathbf{T}_n$, $n \geq 1$" by "If $\mathcal{T} \in \mathbf{T}_n$, $n \geq 0$".
- **Page 14, proof of Proposition 8:** Replace "$Std\left(O/\sim_{\mathcal{T}}\right)$" by "$O/\sim_{\mathcal{T}}$".
- **Page 14, proof of Proposition 8:** Remove the words "If $\mathcal{T}$ has k equivalence classes" (as you never get to use k).

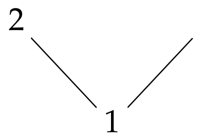
- **Page 14, proof of Proposition 8:** In the last displayed equation of this proof, you rewrite $([n] \setminus O) / \sim_{\mathcal{T}}$ as $\overline{Std\left(\mathcal{T}_{|[n]\setminus O}\right)}$, and rewrite $O / \sim_{\mathcal{T}}$ as $\overline{Std\left(\mathcal{T}_{|O}\right)}$. It would be good if you would add some explanation of why this rewriting is possible. The reason is that whenever Z is a subset of $[n]$ which is a union of equivalence classes of $\mathcal{T}$ [1], we have $Z / \sim_{\mathcal{T}} \cong \overline{Std\left(\mathcal{T}_{|Z}\right)}$ as special posets. This fact is straightforward to check (using (1)), but in my opinion is worth explicitly stating and explicitly referencing when you use it.

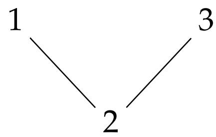
- **Page 14, commutative diagram:** I would rather not use the label "ι" for the inclusion map $\mathbf{H_{SP}} \to \mathbf{H_T}$, given that ι already means something different.

- **Page 15, Definition 9:** Replace "$\mathcal{T} \in \mathbf{T}_l$" by "$\mathcal{T}' \in \mathbf{T}_l$".

- **Page 15, Proposition 10:** I think the claim that "ι is an isometry for this pairing" is wrong, unless I incorrectly understood the definition of ι. For a counterexample, let $\mathcal{T} =$ 

and $\mathcal{T}' =$ 

(where both $\mathcal{T}$ and $\mathcal{T}'$ are T_0, and I draw them as the Hasse diagrams of their posets). Then, $Pic(\mathcal{T}, \mathcal{T}')$ has one element (namely, the map sending $1, 2, 3$ to $3, 1, 2$, respectively), but $Pic(\iota(\mathcal{T}), \iota(\mathcal{T}'))$ is empty (since $i(\mathcal{T}) =$ 

and $\iota(\mathcal{T}') =$ 

).
On the other hand, if you change the definition of ι so that ι also reverses the labelling of the ground set, then this new ι is an isometry for the pairing (in fact, then you have $Pic(\mathcal{T}, \mathcal{T}') = Pic(\iota(\mathcal{T}), \iota(\mathcal{T}'))$ as sets), but then Proposition 6 (3) no longer holds.

- **Page 15, proof of Proposition 10:** It would be helpful to observe that $n = n_1 + n_2$ (directly after "Let $f \in Pic(\mathcal{T}_1.\mathcal{T}_2, \mathcal{T})$").

- **Page 15, proof of Proposition 10:** You write: "Moreover, by restriction, $Std\left(f_{|[n_1]}\right)$ is a picture between $\mathcal{T}_1$ and $Std\left(\mathcal{T}_{|[n]\setminus O_f}\right)$ and $Std\left(f_{|[n_1+n_2]\setminus[n_2]}\right)$ is a picture between $\mathcal{T}_2$ and $Std\left(\mathcal{T}_{|O}\right)$", replace "$\mathcal{T}_{|O}$" by "$\mathcal{T}_{|O_f}$". Also, I think you should define what "$Std\left(f_{|[n_1]}\right)$" and "$Std\left(f_{|[n_1+n_2]\setminus[n_2]}\right)$" mean. The definition that you are using and omitting seems to be the following one:

[1] In particular, Z can be any open set of $\mathcal{T}$ or any closed set of $\mathcal{T}$.

Let X_1 be a finite totally ordered set of cardinality m_1. Let ϕ_1 be the unique increasing bijection from X_1 to $[m_1]$. Let $\mathcal{S}_1$ be a topology on X_1.

Let X_2 be a finite totally ordered set of cardinality m_2. Let ϕ_2 be the unique increasing bijection from X_2 to $[m_2]$. Let $\mathcal{S}_2$ be a topology on X_2.

Let f be a map $X_1 \to X_2$. Then, $Std(f)$ is defined to be the map $\phi_2 \circ f \circ \phi_1^{-1} : [m_1] \to [m_2]$.

- **Page 16, proof of Proposition 10:** In "$\sharp Pic\left(\mathcal{T}_1, Std\left(\mathcal{T}_{|[n]\setminus O}\right)\right) \times \sharp Pic\left(\mathcal{T}_2, Std\left(\mathcal{T}_{|O}\right)\right)$", replace the "$\times$" sign by a "$\cdot$".
- **Page 16, proof of Proposition 10:** Replace "$\left\langle Pic\left(\mathcal{T}_2, Std(\mathcal{T}_{|O}\right)\right\rangle$" by "$\left\langle \mathcal{T}_2, Std\left(\mathcal{T}_{|O}\right)\right\rangle$".
- **Page 16, §3.1:** It would make more sense if you define your partial order not just on $\mathbf{T}_n$, but more generally on the set of all topologies on X whenever X is a finite set. This more general notion is used in the first step of your proof of Theorem 12 (3).
- **Page 18, proof of Theorem 12 (1):** Replace "$I_2 = I \cap ([k+l] \setminus [l])\,(-k)$" by "$I_2 = (I \cap ([k+l] \setminus [l]))\,(-k)$".
- **Page 18, proof of Theorem 12 (1):** You have not defined what $(I \cap ([k+l] \setminus [l]))\,(-k)$ means. (To fix this, add "The set $O\,(-n)$ is the set $\{k - n \mid k \in O\}$" immediately after the sentence "The set $O\,(+n)$ is the set $\{k + n \mid k \in O\}$" on page 9.)
- **Page 18, proof of Theorem 12 (1):** Replace "$I_1 \sqcup I_2\,[k]$" by "$I_1 \sqcup I_2\,(+k)$".
- **Page 18, proof of Theorem 12 (2):** Replace "any open sets" by "any open set".
- **Page 19, proof of Theorem 12 (3):** I personally find the equality sign between

  $$\sum_{O \in \mathcal{T}} \sum_{\substack{\mathcal{T}' \leq \mathcal{T},\ O \in \mathcal{T}', \\ ([n] \setminus O) <_{\mathcal{T}'} O}} R_{Std\left(\mathcal{T}'_{|[n]\setminus O}\right)} \otimes R_{Std\left(\mathcal{T}'_{|O}\right)}$$

  and

  $$\sum_{O \in \mathcal{T}} \sum_{\substack{\mathcal{S} \leq Std\left(\mathcal{T}_{|[n]\setminus O}\right), \\ \mathcal{S}' \leq Std\left(\mathcal{T}_{|O}\right)}} R_{\mathcal{S}} \otimes R_{\mathcal{S}'}$$

  a bit underexplained. Here is how I would explain why these terms are equal:

  First, we need a lemma:

  Lemma A. Let $O \in \mathcal{T}$, $\mathcal{S}' \leq \mathcal{T}_{|O}$ and $\mathcal{S} \leq \mathcal{T}_{|[n]\setminus O}$.

(a) There exists a unique topology $\mathcal{T}' \in \mathbf{T}_n$ such that $\mathcal{S}' = \mathcal{T}'_{|O}$, $\mathcal{S} = \mathcal{T}'_{|[n]\setminus O}$ and $([n] \setminus O) <_{\mathcal{T}'} O$.

(b) This topology $\mathcal{T}'$ is given by $\mathcal{T}' = \{\Omega \cup O \mid \Omega \in \mathcal{S}\} \cup \mathcal{S}'$.

(c) This topology $\mathcal{T}'$ further satisfies $\mathcal{T}' \leq \mathcal{T}$ and $O \in \mathcal{T}'$.

Proof of Lemma A. Parts **(a)** and **(b)** of Lemma A are merely the result of your First step, but with $\mathcal{S}$ and $\mathcal{S}'$ renamed as $\mathcal{S}'$ and $\mathcal{S}$.

(c) From **(b)**, we have $\mathcal{T}' = \{\Omega \cup O \mid \Omega \in \mathcal{S}\} \cup \mathcal{S}'$. Thus, $O \in \mathcal{T}'$. It remains to prove that $\mathcal{T}' \leq \mathcal{T}$. In other words, we need to prove that $I \in \mathcal{T}$ for every $I \in \mathcal{T}'$. Indeed, let $I \in \mathcal{T}'$. Then, $I \in \mathcal{T}' = \{\Omega \cup O \mid \Omega \in \mathcal{S}\} \cup \mathcal{S}'$, so that either $I \in \{\Omega \cup O \mid \Omega \in \mathcal{S}\}$ or $I \in \mathcal{S}'$. In the first case, we have $I = \Omega \cup O$ for some $\Omega \in \mathcal{S}$; now, this Ω satisfies $\Omega \in \mathcal{S} \subseteq \mathcal{T}_{|[n]\setminus O}$ and thus $\Omega \cup O \in \mathcal{T}$, so that $I = \Omega \cup O \in \mathcal{T}$. In the second case, $I \in \mathcal{S}' \subseteq \mathcal{T}_{|O} \subseteq \mathcal{T}$ (since O is open in $\mathcal{T}$). Hence, we have proven $I \in \mathcal{T}$ in both cases. Thus, $\mathcal{T}' \leq \mathcal{T}$ is proven, and thus the proof of Lemma A **(c)** is complete.

As a consequence of Lemma A, we have:

Lemma B. Let $O \in \mathcal{T}$, $\mathcal{S}' \leq \mathcal{T}_{|O}$ and $\mathcal{S} \leq \mathcal{T}_{|[n]\setminus O}$. Then, there exists a unique topology $\mathcal{T}' \in \mathbf{T}_n$ such that $\mathcal{T}' \leq \mathcal{T}$, $O \in \mathcal{T}'$, $([n] \setminus O) <_{\mathcal{T}'} O$, $\mathcal{S} = \mathcal{T}'_{|[n]\setminus O}$ and $\mathcal{S}' = \mathcal{T}'_{|O}$.

Proof of Lemma B. The uniqueness follows from Lemma A **(a)**, while the existence follows from parts **(a)** and **(c)** of Lemma A.

Now that Lemma B is proven, we have

$$\begin{aligned}
&\sum_{O \in \mathcal{T}} \sum_{\substack{\mathcal{T}' \leq \mathcal{T},\ O \in \mathcal{T}', \\ ([n]\setminus O) <_{\mathcal{T}'} O}} R_{Std\left(\mathcal{T}'_{|[n]\setminus O}\right)} \otimes R_{Std\left(\mathcal{T}'_{|O}\right)} \\
&= \sum_{O \in \mathcal{T}} \sum_{\substack{\mathcal{S} \leq \mathcal{T}_{|[n]\setminus O}, \\ \mathcal{S}' \leq \mathcal{T}_{|O}}} \underbrace{\sum_{\substack{\mathcal{T}' \leq \mathcal{T},\ O \in \mathcal{T}', \\ ([n]\setminus O) <_{\mathcal{T}'} O, \\ \mathcal{S} = \mathcal{T}'_{|[n]\setminus O},\ \mathcal{S}' = \mathcal{T}'_{|O}}} R_{Std(\mathcal{S})} \otimes R_{Std(\mathcal{S}')}}_{\substack{\text{this sum has precisely one term} \\ \text{(due to Lemma B), and thus equals} \\ R_{Std(\mathcal{S})} \otimes R_{Std(\mathcal{S}')}}} \\
&\qquad \left(\text{since } \mathcal{T}'_{|[n]\setminus O} \leq \mathcal{T}_{|[n]\setminus O} \text{ and } \mathcal{T}'_{|O} \leq \mathcal{T}_{|O} \text{ for every } \mathcal{T}' \leq \mathcal{T}\right) \\
&= \sum_{O \in \mathcal{T}} \sum_{\substack{\mathcal{S} \leq \mathcal{T}_{|[n]\setminus O}, \\ \mathcal{S}' \leq \mathcal{T}_{|O}}} R_{Std(\mathcal{S})} \otimes R_{Std(\mathcal{S}')} \\
&= \sum_{O \in \mathcal{T}} \sum_{\substack{\mathcal{S} \leq Std\left(\mathcal{T}_{|[n]\setminus O}\right), \\ \mathcal{S}' \leq Std\left(\mathcal{T}_{|O}\right)}} R_{\mathcal{S}} \otimes R_{\mathcal{S}'}
\end{aligned}$$

(here, we substituted $Std(\mathcal{S})$ and $Std(\mathcal{S}')$ for $\mathcal{S}$ and $\mathcal{S}'$ in the sum).

- **Page 19, proof of Theorem 12 (3):** In "We used the first step for the third equality", replace "third" by "fourth" (or by a larger number if you add in intermediate steps).
- **Page 19, Definition 13:** In (2), replace "Let f a generalized" by "Let f be a generalized".
- **Page 19, Definition 13:** In (2), replace "$i, j \in [n]$" by "$i, j, k \in [n]$".
- **Page 19, Definition 13:** Again, I'd recommend generalizing this definition from the case $\mathcal{T} \in \mathbf{T}_n$ to the case when $\mathcal{T}$ is a topology on a finite set X equipped with a total order.[2] I think the generalized definition will look like this:

  Definition 13'. Let X be a finite totally ordered set. Let $\mathcal{T}$ be a topology on X.

  (1) A *generalized T-partition* of $\mathcal{T}$ is a surjective map $f : X \to [p]$ such that if $i \leq_{\mathcal{T}} j$ in X, then $f(i) \leq f(j)$ in $[p]$. If f is a generalized T-partition of $\mathcal{T}$, we shall represent it by the packed word $f(x_1) f(x_2) \ldots f(x_n)$, where $(x_1, x_2, \ldots, x_n)$ is the list of the elements of X in increasing order (with respect to the total order on X).

  (2) Let f be a generalized T-partition of $\mathcal{T}$. We shall say that f is a *(strict) T-partition* if for all $i, j \in X$:

  – $i <_{\mathcal{T}} j$ and $i > j$ implies that $f(i) < f(j)$ in $[p]$.

  – If $i < j < k$, $i \sim_{\mathcal{T}} k$ and $f(i) = f(j) = f(k)$, then $i \sim_{\mathcal{T}} j$ and $j \sim_{\mathcal{T}} k$.

  (3) The set of generalized T-partitions of $\mathcal{T}$ is denoted by $\mathcal{P}(\mathcal{T})$; the set of (strict) T-partitions of $\mathcal{T}$ is denoted by $\mathcal{P}_s(\mathcal{T})$.

  (4) If $f \in \mathcal{P}(\mathcal{T})$, we put:

$$\ell_1(f) = \sharp \left\{ (i,j) \in X^2 \mid i <_{\mathcal{T}} j,\ i < j, \text{ and } f(i) = f(j) \right\},$$
$$\ell_2(f) = \sharp \left\{ (i,j) \in X^2 \mid i <_{\mathcal{T}} j,\ i > j, \text{ and } f(i) = f(j) \right\},$$
$$\ell_3(f) = \sharp \{ (i,j,k) \in X^3 \mid i < j < k,\ i \sim_{\mathcal{T}} k, \\ i \not\sim_{\mathcal{T}} j,\ j \not\sim_{\mathcal{T}} k \text{ and } f(i) = f(j) = f(k) \}.$$

  It is useful to notice that if $\mathcal{T}$ and $\mathcal{S}$ are two topologies on finite totally ordered sets X and Y, and if $\phi : X \to Y$ is an isomorphism of totally ordered sets which is, at the same time, a homeomorphism between $\mathcal{T}$ and $\mathcal{S}$, then the generalized T-partitions of X are in 1-to-1 correspondence with

[2] This generality is used on page 21 (in notation like "$\mathcal{P}\left(\mathcal{T}_{|O}\right)$").

the generalized T-partitions of Y, and the same holds for strict T-partitions. If we represent these (generalized and strict) T-partitions by packed words, then we actually have $\mathcal{P}(\mathcal{T}) = \mathcal{P}(\mathcal{S})$ and $\mathcal{P}_s(\mathcal{T}) = \mathcal{P}_s(\mathcal{S})$. As a consequence, if X is a finite totally ordered set of cardinality n, and if $\mathcal{T}$ is a topology on X, then the unique increasing bijection $\phi : X \to [n]$ is an isomorphism of totally ordered sets which is, at the same time, a homeomorphism between $\mathcal{T}$ and $Std(\mathcal{T})$. Thus, if we represent (generalized and strict) T-partitions by packed words, then we have $\mathcal{P}(\mathcal{T}) = \mathcal{P}(Std(\mathcal{T}))$ and $\mathcal{P}_s(\mathcal{T}) = \mathcal{P}_s(Std(\mathcal{T}))$. You are tacitly using this rather often.

- **Page 20, Proposition 14:** Remove the spurious closing parenthesis in "$j \circ \Gamma_{(q_1,q_2,q_3)})$".
- **Page 20, Remarks (2):** This is not very precise. It would be more correct to say that if $\mathcal{T} \in \mathbf{T}$ is T_0, then a strict T-partition of $\mathcal{T}$ is the same as a packed word which is, at the same time, a (P, id)-partition in Stanley's sense [16], where P is the poset associated to $\mathcal{T}$ and $\mathrm{id} : [n] \to [n]$ is the identity map (which we use as a labelling). Unlike you, Stanley does not require his (P, ω)-partitions to be packed, and he uses the word "P-partition" (without the ω) for the particular case when the labelling ω is natural (which the identity map not always is!).
- **Page 21:** Replace "Then $f = h_{|O} = h_{|O'} = f'$ and $g = Pack\left(h_{|[n]\setminus O}\right) = Pack\left(h_{|[n]\setminus O}\right) = g'$" by "Then $f = h_{|[n]\setminus O} = h_{|[n]\setminus O'} = f'$ and $g = h_{|O}[-\max(f)] = h_{|O'}[-\max(f')] = g'$ (where, for any set S, any function $\varphi : S \to \mathbb{Z}$ and any integer z, we let $\varphi[-z]$ be the function $S \to \mathbb{Z}$, $i \mapsto \varphi(i) - z$)". (I would rather not use the "*Pack*" operator here, as it can mean different things; we do **not** want to "pack" the domain.)
- **Page 21:** Replace "$O = h^{-1}(\{k+1, \ldots \max(h)\})$" by "$O = h^{-1}(\{k+1, \ldots, \max(h)\})$".
- **Page 21:** Replace "$g = Pack\left(h_{|O}\right)$" by "$g = h_{|O}[-k]$".
- **Pages 22-23, §4.2:** Again, it would help to generalize the definition of $\mathcal{L}(\mathcal{T})$ to the case when $\mathcal{T}$ is a topology on an arbitrary finite totally ordered set rather than on $[n]$. This generalization is similar to the generalization I suggested for Definition 13, and you are already using it on page 24 when you write things like "$\mathcal{L}\left(\mathcal{T}_{|O}\right)$".
- **Page 22, proof of Proposition 14:** Replace "$\mathcal{T}_w$" by "$\mathcal{T}_f$".
- **Page 22, proof of Proposition 14:** On the same line, replace "$\Gamma_q(T_f)$" by "$\Gamma_q(\mathcal{T}_f)$".

- **Page 22, proof of Proposition 14:** In the last paragraph of this proof, replace "(q_2, q_2, q_3)" by "(q_2, q_1, q_3)" (twice).
- **Page 22, proof of Proposition 14:** In the last paragraph of this proof, replace "$q_2^{\ell_2(g)} q_1^{\ell_1(g)} q_2^{\ell_2(g)}$" by "$q_2^{\ell_2(g)} q_1^{\ell_1(g)} q_3^{\ell_3(g)}$".
- **Page 23, Remark:** Replace "$\mathcal{L}(T) \subseteq P(T)$" by "$\mathcal{L}(T) \subseteq \mathcal{P}(T)$".
- **Page 23, Proposition 16:** Replace "$f(1), \ldots f(n)$" by "$f(1), \ldots, f(n)$".
- **Page 23, proof of Proposition 16:** Replace "$f(1), \ldots f(n)$" by "$f(1), \ldots, f(n)$" (twice).
- **Page 23, proof of Proposition 16:** On the second line of the displayed computation, replace "$f \sqcup\!\sqcup f'$" by "$f' \sqcup\!\sqcup f''$".
- **Page 24, proof of Proposition 16:** "are union of" $\rightarrow$ "are unions of".
- **Page 24, proof of Proposition 16:** Replace "of both $\mathbf{H}_{\mathcal{T}}$ and" by "of both $\mathbf{H_T}$ and".
- **Page 24:** Here you have used the surjectivity of L to prove that $(\mathbf{WQSym}, \sqcup\!\sqcup, \Delta)$ is a Hopf algebra. In your place, I would add a remark that the same argument (but with L replaced by $\Gamma_{(1,0,0)}$) can be showed that **WQSym** with its standard structure (that is, $(\mathbf{WQSym}, ., \Delta)$) is a Hopf algebra. This fact is, of course, commonplace, but there is virtue in having a readable proof that does not use the alphabet doubling trick. (My problem with the alphabet doubling trick is the scarcity of authors bothering to explain it in a way that is formally correct.)
- **Page 25, proof of Lemma 18:** Replace "As $i \sim_{\mathcal{T}} k$" by "As $i \sim_{\mathcal{T}_f} k$".
- **Page 26, proof of Proposition 19, First step:** You write: "This is the generalization of lemma 18". I think you are only generalizing the $\Longrightarrow$ part of Lemma 18 here.
- **Page 26, proof of Proposition 19, First step:** Replace "If $i \not\sim_{\mathcal{T}} j$," by "Assume that $i \not\sim_{\mathcal{T}} j$." (because you are carrying this assumption through the next few sentences).
- **Page 26, proof of Proposition 19, Second step:** When you say "necessarily the $C_{p,r}$ are intervals", I think you mean "no element of $C_{p,s}$ is between two elements of $C_{p,r}$ for any $s \neq r$" (so, the $C_{p,r}$ are intersections of intervals with $g^{-1}(\{p\})$, although I don't think this perspective is helpful).
- **Page 26, proof of Proposition 19, Second step:** In the Unicity proof, replace "are all distincts" by "are all distinct".

- **Page 26, proof of Proposition 19, Second step:** In the Unicity proof, it would be better to replace "If $p < q$ and $c_{q,s} < c_{p,r}$" by "If $p < q$ and $c_{q,s} \leq c_{p,r}$". (While this is equivalent, it simplifies the argument.)

- **Page 26, proof of Proposition 19, Second step:** In the Existence proof, replace "if $x \in C_{p,r}$" by "if $i \in C_{p,r}$".

- **Page 26, proof of Proposition 19, Second step:** In the Existence proof, replace "$p = g(x) \leq g(y) = q$" by "$p = g(i) \leq g(j) = q$".

- **Page 26, proof of Proposition 19, Second step:** In the Existence proof, replace "$f(x) < f(y)$" by "$f(i) < f(j)$".

- **Page 27, proof of Proposition 19, Second step:** In the Existence proof, replace "$p = g(x) \leq g(y) = q$" by "$p = g(i) \leq g(j) = q$".

- **Page 27, proof of Proposition 19, Third step:** Replace "$x', y' \in \mathbf{H}_{\mathcal{T}}$" by "$x', y' \in \mathbf{H}_{\mathbf{T}}$".

- **Page 28, proof of Corollary 28:** Replace "This is the first step of the proof of proposition 19" by "The first equality sign follows from the first and the second step of Proposition 19; the second equality sign follows from Lemma 18".

- **Page 28, Remark:** "for all packed word" $\to$ "for any packed word".

- **Page 29, §4.4:** You write: "a T-partition of $\mathcal{T}$ is a P-partition of the poset associated to $\mathcal{T}$, in Stanley's sense [16]". This is not precise; see my comment on page 20, Remarks (2) for how to correct it.

- **Page 29, §4.4:** In the second commutative diagram, I think the "$\theta_{(1,0,0)}$" arrow should be "$\varphi_{(1,0,0)}$".

- **Page 30, proof of Lemma 24, part 1:** Replace "and $j > i$" by "and $i > j$".

- **Page 30, proof of Lemma 24, part 2:** After "Hence, if $c_p = c_q$", add "and $p < q$".

- **Page 30, proof of Lemma 24, part 2:** Before "By definition of the standardization", add "Then" (to make clear that the $p < q$ and $c_p \neq c_q$ hypotheses still apply).

- **Page 30, proof of Lemma 25:** Replace "σ^{-1} is increasing on $i_p, \ldots, j_p$" by "σ^{-1} is increasing on $\{i_p, \ldots, j_p\}$".

- **Page 30, proof of Lemma 25:** Replace "$\sigma^{-1}(i) = k$ and $\sigma^{-i}(j) = l$" by "$\sigma(i) = k$ and $\sigma(j) = l$".

- **Page 30, proof of Lemma 25:** On the last line of page 30, you write "$k < l$". I would add some more details on why this is true: We have $i = \sigma^{-1}(k)$ and $j = \sigma^{-1}(l)$. Now, $f\left(\underbrace{\sigma^{-1}(k)}_{=i}\right) = f(i) = p < q = f\left(\underbrace{j}_{=\sigma^{-1}(l)}\right) = f\left(\sigma^{-1}(l)\right)$. Recalling our construction of f, we see that this yields $k < l$ (since $f\left(\sigma^{-1}(g)\right)$ weakly increases with g).

- **Page 31, proof of Lemma 25:** I would expect some more detail on why "By definition of f, i is the greatest element of C_p, with $p = f(i)$". The argument that I have (maybe not the shortest possible) is as follows: Let $p = f(i)$. Recall that $\sigma \in Std(f)$; thus, σ is increasing on C_p. By definition of f, we have $l \leq k$ for every l satisfying $f\left(\sigma^{-1}(l)\right) = f\left(\sigma^{-1}(k)\right)$ (since $f\left(\sigma^{-1}(k+1)\right) = f\left(\sigma^{-1}(k)\right) + 1$). Substituting $\sigma(u)$ for l here, and recalling that $f\left(\underbrace{\sigma^{-1}(k)}_{=i}\right) = f(i) = p$ and $k = \sigma(i)$, we can rewrite this as follows: We have $\sigma(u) \leq \sigma(i)$ for every u satisfying $f(u) = p$. In other words, $\sigma(u) \leq \sigma(i)$ for every $u \in C_p$. Thus, $u \leq i$ for every $u \in C_p$ (since σ is increasing on C_p). Hence, i is the greatest element of C_p.

  The argument why "$j = \sigma^{-1}(k+1)$ is the smallest element of C_{p+1}" is similar.

- **Page 31, proof of Lemma 25:** In Case (1) of the proof of $f\left(\sigma^{-1}(i)\right) = g\left(\sigma^{-1}(i)\right)$, I would add "$= g\left(\sigma^{-1}(i)\right) + 1$" before "$= g\left(\sigma^{-1}(i+1)\right)$". The same change could be made in Case (2), and a similar change in Case (3).

- **Page 31, proof of Lemma 25:** In the proof of $f\left(\sigma^{-1}(i)\right) = g\left(\sigma^{-1}(i)\right)$, I would define $p = f\left(\sigma^{-1}(i)\right)$ in all three Cases (1), (2) and (3), not just in Case (1) as you currently do.

- **Page 31, proof of Theorem 23, part 1, $\Longrightarrow$ direction:** You write: "By construction of $\psi_\sigma(I)$". But I don't see how what comes after this follows from the construction of $\psi_\sigma(I)$. Instead, I would say it follows from $I = M(f)$.

- **Page 31, proof of Theorem 23, part 1, $\Longrightarrow$ direction:** Replace "we should have $f(k') > f(l')$" by "we should have $f(k') \geq f(l')$".

- **Page 31, proof of Theorem 23, part 1, $\Longrightarrow$ direction:** Do you actually use the "$g(k') < g(l')$" observation? I don't see how.

- **Page 31, proof of Theorem 23, part 1, $\Longrightarrow$ direction:** Replace "So for all $k' \in C_q$, $l' \in C_{q+1}$, $k' < l'$" by "So for all $k' \in C'_q$ and $l' \in C'_{q+1}$, we have $k' \leq k < l'$" (note that I added inverted commas and a k).

- **Page 31, proof of Theorem 23, part 1, $\Longleftarrow$ direction:** Replace "If $g(k) \leq g(l)$, we put $\sigma(k) = i$ and $\sigma(l) = j$. By construction of $\psi_\sigma(J)$, $i < j$. By construction of $\psi_\sigma(I)$, $f(k) = f\left(\sigma^{-1}(i)\right) =\leq f\left(\sigma^{-1}(j)\right) = f(l)$." (which is not completely true[3]) by the following argument: "If $g(k) < g(l)$, then we have $\sigma(k) < \sigma(l)$ (since $\sigma = Std(g)$) and thus $f(k) \leq f(l)$ (as $\sigma = Std(f)$). The same conclusion holds if $g(k) = g(l)$ (this follows directly from the previous bullet point). Thus, if $g(k) \leq g(l)$, then $f(k) \leq f(l)$."

- **Page 31, proof of Theorem 23, part 1, $\Longleftarrow$ direction:** I don't understand how you prove that if $g(k) < g(l)$ and $k > l$, then $f(k) < f(l)$. [4] I would show this differently:

  Let k and l be such that $g(k) < g(l)$ and $k > l$. As shown in the previous bullet point, we have $f(k) \leq f(l)$ (since $g(k) \leq g(l)$). If $f(k) \neq f(l)$, we are thus done. Hence, assume that $f(k) = f(l)$. Then, $\sigma(k) > \sigma(l)$ (since $\sigma = Std(f)$ and $k > l$). But $g(k) < g(l)$ and thus $\sigma(k) < \sigma(l)$ (since $\sigma = Std(g)$), which contradicts $\sigma(k) > \sigma(l)$. This contradiction finishes the proof.

- **Page 31, Remark:** Replace "$Pack(f) = \sigma$" by "$Std(f) = \sigma$".

[3] I replaced the $\leq$ sign in $g(k) \leq g(l)$ by a $<$ sign, since otherwise your $i < j$ claim does not hold (unless we also assume $k < l$, which is not the optimal way).

[4] More precisely, I don't understand how you get $k \leq l$. Maybe you are using the Remark from page 29, but I do not see how.