Functional equations and Lie algebras
Emanuela Petracci
2001/2002
Errata and questions by Darij Grinberg - I

This is a list of errors in Emanuela Petracci's thesis "Functional equations and Lie algebras" I found while reading parts of it. The word "you" always refers to the author of the thesis.

Despite the many errors, the thesis is a masterpiece of algebra. It provides (among other things) a proof of the Poincaré-Birkhoff-Witt for $\mathbb{Q}$-algebras which does not require the ground ring to be a field (so "for $\mathbb{Q}$-algebras" merely means that the ground ring is a commutative $\mathbb{Q}$-algebra). Among several such proofs (all of which are highly nontrivial), the one given in this thesis is probably the most conceptual one.

General errors

- There seems to be a bug in the style you are using: While there are no dots after "Convention" and "Remark" and the likes (for example, "Convention 1.1.1" and "Remark 1.1.1"), there *are* dots after "Definition" (for instance, "Definition. 1.1.1").
- The word "verify" is misused as a synonym for "satisfy" throughout the thesis (a mistake common of Francophone authors).

Chapter 1

- **Page 9, §1.1:** Replace "not-zero" by "non-zero".
- **Definition 1.1.1:** Replace "$m, n \in M$" by "$m \in M$".
- **Definition 1.1.2:** Replace "equipped of" by "equipped with'. (This mistake occurs in many places throughout the text.)
- **Between Definition 1.1.2 and Notation 1.1.1:** Remove the "and $\alpha \in \mathbb{K}$" part.

- **Example 1.1.1:** Replace "the set formal series in s" by "the set of formal series in z".
- **Example 1.1.2 b):** The last comma in "$\{v \otimes w; v \in M, w \in N,\}$" is misplaced – it should be outside the brackets.
- You do some kind of introduction to superalgebra in Chapter 1. If you want this to be self-contained, I think a definition of the notion of the tensor product of two superalgebras would be in place somewhere in §1.1: you use this notion later, and you never define it, although you define much more basic notions (like Example 1.1.2).
- **Example 1.1.2 c):** In "$T(M) := \mathbb{K} + (M \otimes M) + (M \otimes M \otimes M) + \cdots$", you forgot the M addend.
- **Between (1.8) and (1.9):** Replace "$X_1, .., X_n \in M$" by "$X_1, ..., X_n \in M$".
- **Second absatz of page 12:** Remove the "of" from "Because of $S(M)$ is a coalgebra". Also, the "coalgebra" here should probably be a "cocommutative coalgebra".
- **Remark 1.2.1:** It wouldn't hurt to explicitly remind the reader here that X denotes the "constant function" ($1 \mapsto X$, $S^n(M) \to \{0\}$ for $n \neq 0$).
- **Page 12, one line below remark 1.2.1:** "*formal vectors field*" should be "*formal vector fields*".
- **Page 12, one line below Definition 1.2.1:** A closing parenthesis was omitted in "$P(S(M)$".
- **Proof of Lemma 1.3.1:** This is correct, but I don't understand why you require $k \geq 1$ all the time. Wouldn't $k \geq 0$ be completely enough?
- **Remark 1.3.1:** Here and in the following, when you write "$p(X_1 + \cdots + X_n)$", you actually mean $p(X_1) + \cdots + p(X_n)$ (or, what is the same, $p(X_1 \cdots X_n)$). This appears so often in your paper that I am wondering whether it is some standard abuse of notation, or I am blind?
- **Remark 1.3.1:** Replace the ":=" by "=" in "$(\mathrm{ad}\, x)^0 (Y) := Y \in \mathfrak{g}_x$". This is not a definition of $(\mathrm{ad}\, x)^0 (Y)$; it is already defined.

- **First line of page 14:** You write: "As a consequence of the last remark, we can define". This is right, but there is no need to use the last remark here. A simpler way to check that $q(\operatorname{ad} x)(Y)$ is well-defined is the following: For every $m \in \mathbb{N}$, let $S_{\leq m}(\mathfrak{g})$ define the $\mathbb{K}$-subsupermodule $\bigoplus_{i=0}^{m} S^i(\mathfrak{g})$ of $S(\mathfrak{g})$. For every $m \in \mathbb{N}$, let $\mathfrak{g}_x^{\geq m}$ denote the $\mathbb{K}$-subsupermodule $\{f \in \mathfrak{g}_x \mid f(S_{\leq m}(\mathfrak{g})) = 0\}$ of $\mathfrak{g}_x$. Then, it is easy to show that $x \in \mathfrak{g}_x^{\geq 0}$, but every $m \in \mathbb{N}$ satisfies $(\operatorname{ad} x)(\mathfrak{g}_x^{\geq m}) \subseteq \mathfrak{g}_x^{\geq m+1}$. As a consequence, for every $Y \in \mathfrak{g}$, every sufficiently high $m \in \mathbb{N}$ satisfies $(\operatorname{ad} x)^m (Y) = 0$, and thus $q(\operatorname{ad} x)(Y)$ is well-defined.

- **Theorem 1.3.1:** The "$\left(\dfrac{q(t+u) - q(u)}{t} : [Y, Z]\right)$" should be $\left(\dfrac{q(t+u) - q(u)}{t} : [Y, Z]\right)_x$ (with an x index).

- **Proof of Theorem 1.3.1:** Three typos in the computation:
  - In the first line of the computation, "$\operatorname{ad} x)^k$" should be "$(\operatorname{ad} x)^k$".
  - In the second line of the computation, "$\left((\operatorname{ad} x)^k\right)(Z)$" should be "$\left((\operatorname{ad} x)^k (Z)\right)$".
  - In the third line of the computation, "$\left(u^k : [Y, Z]\right)$" should be "$\left(u^k : [Y, Z]\right)_x$".

Chapter 2

- **Page 15, one line below Remark 2.1.2:** You write: "$\Phi^a := id * \varphi^a \equiv \operatorname{Mult} \circ (1 \otimes \varphi^a) \circ \Delta$". The 1 here stands for *id*; maybe it would be better to just call it *id* (lest it be confused with the neutral element with respect to convolution).

- **Page 16, Lemma 2.1.1:** It might be helpful to explain how expressions like "$\varphi^a * Y$" are to be understood. (As far as I understand, in the expression "$\varphi^a * Y$", the terms φ^a and Y are understood to mean the maps $S(\mathfrak{g}) \xrightarrow{\varphi^a} \mathfrak{g} \xrightarrow{\text{inclusion}} S(\mathfrak{g})$ and $S(\mathfrak{g}) \xrightarrow{Y} \mathfrak{g} \xrightarrow{\text{inclusion}} S(\mathfrak{g})$, respectively.)

- **Proof of Lemma 2.1.1 ii):** I fear I don't understand this proof, although I suspect the problem is on my side and not on that of the proof's[1].

Anyway, here is a more down-to-earth proof of Lemma 2.1.1 *ii)*:

Proof of Lemma 2.1.1 ii): In the following, we are going to use the sumfree Sweedler notation for the comultiplication on $S(\mathfrak{g})$. Also we will assume that all vectors are even, since I don't want to struggle with the minus signs. I am pretty sure that the general case can be proven analogously.

We start with some straightforward observations:

Observation 1: Every $\alpha \in S(\mathfrak{g})$ and every $Y \in \mathfrak{g}$ satisfy $[x, Y](\alpha) = [x(\alpha), Y]$. (Here, on the left hand side, Y denotes the constant map $Y \in \mathfrak{g}_x$, as usual.)

Proof of Observation 1: The constant map $Y \in \mathfrak{g}_x = \operatorname{Hom}(S(\mathfrak{g}), \mathfrak{g})$ maps every $\beta \in S(\mathfrak{g})$ to $\varepsilon(\beta) Y \in \mathfrak{g}$. Thus,

$$[x, Y](\alpha) = \left[x\left(\alpha_{(1)}\right), \underbrace{Y\left(\alpha_{(2)}\right)}_{=\varepsilon\left(\alpha_{(2)}\right)Y}\right] = \left[x\left(\alpha_{(1)}\right), \varepsilon\left(\alpha_{(2)}\right) Y\right]$$
$$= \left[x\left(\underbrace{\alpha_{(1)}\varepsilon\left(\alpha_{(2)}\right)}_{=\alpha}\right), Y\right] = [x(\alpha), Y].$$

This proves Observation 1.

Observation 2: Every $\ell \in \mathbb{N}$, $b \in \mathfrak{g}$ and $\alpha \in S(\mathfrak{g})$ satisfy

$$\left((\operatorname{ad} x)\left((\operatorname{ad} x)^{\ell}(b)\left(\alpha_{(2)}\right)\right)\right)\left(\alpha_{(1)}\right) = (\operatorname{ad} x)^{\ell+1}(b)(\alpha).$$

[1] I have troubles understanding the equation "$\Phi^a_{\mathfrak{g}} \circ \Psi^b_{\mathfrak{g}} = \Phi^a_{(\mathfrak{g}_x)_y} \circ (id * \psi(\operatorname{ad} y)(b)) |_{S(\mathfrak{g})} = \Phi^a_{(\mathfrak{g}_x)_y} \circ \psi(\operatorname{ad} y)(b)^L |_{S(\mathfrak{g})}$". (It is not clear to me how to interpret the term $\psi(\operatorname{ad} y)(b)$ – as an element of $(\mathfrak{g}_x)_y$ regarded as a constant map $S\left((\mathfrak{g}_x)_y\right) \to (\mathfrak{g}_x)_y$, or as a (non-constant) map $S(\mathfrak{g}_x) \to \mathfrak{g}_x$ – in order for both equality signs to be valid.)

Proof of Observation 2: We have

$$\left(\underbrace{(\operatorname{ad} x)\left((\operatorname{ad} x)^{\ell}(b)\left(\alpha_{(2)}\right)\right)}_{=\left[x,(\operatorname{ad} x)^{\ell}(b)\left(\alpha_{(2)}\right)\right]} \right)\left(\alpha_{(1)}\right)$$

$$= \left[x, (\operatorname{ad} x)^{\ell}(b)\left(\alpha_{(2)}\right)\right]\left(\alpha_{(1)}\right) = \left[x\left(\alpha_{(1)}\right), (\operatorname{ad} x)^{\ell}(b)\left(\alpha_{(2)}\right)\right]$$

$$\left(\begin{array}{c} \text{by Observation 1, applied to } \alpha_{(1)} \text{ and } (\operatorname{ad} x)^{\ell}(b)\left(\alpha_{(2)}\right) \\ \text{instead of } \alpha \text{ and } Y \end{array} \right)$$

$$= \underbrace{\left[x, (\operatorname{ad} x)^{\ell}(b)\right]}_{=(\operatorname{ad} x)^{\ell+1}(b)}(\alpha) = (\operatorname{ad} x)^{\ell+1}(b)(\alpha),$$

thus proving Observation 2.

Observation 3: Every $q \in \mathbb{N}$, $k \in \mathbb{N}$, $b \in \mathfrak{g}$ and $\alpha \in S(\mathfrak{g})$ satisfy

$$\left((\operatorname{ad} x)^{q}\left((\operatorname{ad} x)^{k}(b)\left(\alpha_{(2)}\right)\right)\right)\left(\alpha_{(1)}\right) = (\operatorname{ad} x)^{q+k}(b)(\alpha).$$

Proof of Observation 3: We prove Observation 3 by induction over q. The induction base is the case when $q = 0$; this case is easy (it reduces to showing that $\left((\operatorname{ad} x)^{k}(b)\left(\alpha_{(2)}\right)\right)\left(\alpha_{(1)}\right) = (\operatorname{ad} x)^{k}(b)(\alpha)$, but this is clear since $(\operatorname{ad} x)^{k}(b)\left(\alpha_{(2)}\right)$ is a constant map and thus satisfies

$$\left((\operatorname{ad} x)^{k}(b)\left(\alpha_{(2)}\right)\right)\left(\alpha_{(1)}\right) = \varepsilon\left(\alpha_{(1)}\right) \cdot (\operatorname{ad} x)^{k}(b)\left(\alpha_{(2)}\right)$$

$$= (\operatorname{ad} x)^{k}(b)\left(\underbrace{\varepsilon\left(\alpha_{(1)}\right)\alpha_{(2)}}_{=\alpha}\right) = (\operatorname{ad} x)^{k}(b)(\alpha)$$

). For the induction step, we assume that some $q \in \mathbb{N}$ satisfies

$$\left((\operatorname{ad} x)^{q}\left((\operatorname{ad} x)^{k}(b)\left(\alpha_{(2)}\right)\right)\right)\left(\alpha_{(1)}\right) = (\operatorname{ad} x)^{q+k}(b)(\alpha) \tag{1}$$

for all $\alpha \in S(\mathfrak{g})$, and try to prove that

$$\left((\operatorname{ad} x)^{q+1}\left((\operatorname{ad} x)^{k}(b)\left(\alpha_{(2)}\right)\right)\right)\left(\alpha_{(1)}\right) = (\operatorname{ad} x)^{q+1+k}(b)(\alpha)$$

for all $\alpha \in S(\mathfrak{g})$. But this follows from

$$\left(\underbrace{(\operatorname{ad} x)^{q+1} \left((\operatorname{ad} x)^k (b) \left(\alpha_{(2)} \right) \right)}_{\substack{= (\operatorname{ad} x)\left((\operatorname{ad} x)^q \left((\operatorname{ad} x)^k (b) \left(\alpha_{(2)} \right) \right) \right) \\ = \left[x, (\operatorname{ad} x)^q \left((\operatorname{ad} x)^k (b) \left(\alpha_{(2)} \right) \right) \right]}} \right) \left(\alpha_{(1)} \right)$$

$$= \left[x, (\operatorname{ad} x)^q \left((\operatorname{ad} x)^k (b) \left(\alpha_{(2)} \right) \right) \right] \left(\alpha_{(1)} \right)$$

$$= \left[x \left(\left(\alpha_{(1)} \right)_{(1)} \right), \left((\operatorname{ad} x)^q \left((\operatorname{ad} x)^k (b) \left(\alpha_{(2)} \right) \right) \right) \left(\left(\alpha_{(1)} \right)_{(2)} \right) \right]$$

$$= \left[x \left(\alpha_{(1)} \right), \underbrace{\left((\operatorname{ad} x)^q \left((\operatorname{ad} x)^k (b) \left(\left(\alpha_{(2)} \right)_{(2)} \right) \right) \right) \left(\left(\alpha_{(2)} \right)_{(1)} \right)}_{\substack{= (\operatorname{ad} x)^{q+k} (b) \left(\alpha_{(2)} \right) \\ \text{(by (1), applied to } \alpha_{(2)} \text{ instead of } \alpha\text{)}}} \right]$$

(by coassociativity)

$$= \left[x \left(\alpha_{(1)} \right), (\operatorname{ad} x)^{q+k} (b) \left(\alpha_{(2)} \right) \right]$$

$$= \underbrace{\left[x, (\operatorname{ad} x)^{q+k} (b) \right]}_{= (\operatorname{ad} x)^{(q+k)+1} (b) = (\operatorname{ad} x)^{q+1+k} (b)} (\alpha) = (\operatorname{ad} x)^{q+1+k} (b) (\alpha).$$

Thus, Observation 3 is proven.

Observation 4: Every $f \in \operatorname{Hom}(S(\mathfrak{g}), S(\mathfrak{g}))$, every $Y \in \mathfrak{g}$ and every $\alpha \in S(\mathfrak{g})$ satisfy

$$(f * Y)(\alpha) = f(\alpha) Y$$

(where the expression Y in "$f * Y$" is regarded as a map $S(\mathfrak{g}) \to S(\mathfrak{g})$ by first considering it as a constant map $S(\mathfrak{g}) \to \mathfrak{g}$ and then composing it with the inclusion map $\mathfrak{g} \to S(\mathfrak{g})$).

Proof of Observation 4: We have

$$(f * Y)(\alpha) = f\left(\alpha_{(1)}\right) \underbrace{Y\left(\alpha_{(2)}\right)}_{\substack{=Y\varepsilon\left(\alpha_{(2)}\right) \\ \text{(by the definition} \\ \text{of the constant map } Y)}} = f\left(\alpha_{(1)}\right) Y\varepsilon\left(\alpha_{(2)}\right)$$

$$= f\left(\underbrace{\alpha_{(1)}\varepsilon\left(\alpha_{(2)}\right)}_{=\alpha}\right) Y = f(\alpha) Y.$$

This proves Observation 4.
Now let us prove Lemma 2.1.1 *ii)*: We want to show that

$$\Phi^a \circ \Psi^b = id * \left(\varphi^a * \psi^b - \left(\frac{\varphi(t+u) - \varphi(t)}{u}\psi(u) : [a,b]\right)_x\right).$$

In order to do that, it is enough to show that every $\alpha \in S(\mathfrak{g})$ satisfies

$$\left(\Phi^a \circ \Psi^b\right)(\alpha) = \left(id * \left(\varphi^a * \psi^b - \left(\frac{\varphi(t+u) - \varphi(t)}{u}\psi(u) : [a,b]\right)_x\right)\right)(\alpha).$$

Since

$$\left(\Phi^a \circ \Psi^b\right)(\alpha)$$

$$= \Phi^a \left(\underbrace{\Psi^b(\alpha)}_{=\left(id*\psi^b\right)(\alpha)=\alpha_{(1)}\psi^b\left(\alpha_{(2)}\right)=\left(\psi^b\left(\alpha_{(2)}\right)\right)^L\left(\alpha_{(1)}\right)} \right) = \Phi^a\left(\left(\psi^b\left(\alpha_{(2)}\right)\right)^L\left(\alpha_{(1)}\right)\right)$$

$$= \left(\Phi^a \circ \left(\psi^b\left(\alpha_{(2)}\right)\right)^L\right)\left(\alpha_{(1)}\right)$$

$$= \left(id * \left(\varphi^a * \left(\psi^b\left(\alpha_{(2)}\right)\right) - \left(\frac{\varphi(t+u)-\varphi(t)}{u} : \left[a, \psi^b\left(\alpha_{(2)}\right)\right]\right)_x\right)\right)\left(\alpha_{(1)}\right)$$

$$\left(\begin{array}{c} \text{since Lemma 2.1.1 } i) \text{ (applied to } Y = \psi^b\left(\alpha_{(2)}\right) \text{) yields} \\ \Phi^a \circ \left(\psi^b\left(\alpha_{(2)}\right)\right)^L \\ = id * \left(\varphi^a * \left(\psi^b\left(\alpha_{(2)}\right)\right) - \left(\frac{\varphi(t+u)-\varphi(t)}{u} : \left[a, \psi^b\left(\alpha_{(2)}\right)\right]\right)_x\right) \end{array}\right)$$

$$= \underbrace{\left(id * \varphi^a * \left(\psi^b\left(\alpha_{(2)}\right)\right)\right)\left(\alpha_{(1)}\right)}_{\substack{=(id*\varphi^a)\left(\alpha_{(1)}\right)\cdot\psi^b\left(\alpha_{(2)}\right) \\ \text{(by Observation 4, applied to} \\ id*\varphi^a,\ \psi^b\left(\alpha_{(2)}\right) \text{ and } \alpha_{(1)} \\ \text{instead of } f,\ Y \text{ and } \alpha)}}$$

$$- \underbrace{\left(id * \left(\frac{\varphi(t+u)-\varphi(t)}{u} : \left[a, \psi^b\left(\alpha_{(2)}\right)\right]\right)_x\right)\left(\alpha_{(1)}\right)}_{=\left(\alpha_{(1)}\right)_{(1)}\left(\frac{\varphi(t+u)-\varphi(t)}{u}:\left[a,\psi^b\left(\alpha_{(2)}\right)\right]\right)_x\left(\left(\alpha_{(1)}\right)_{(2)}\right)}$$

$$= \underbrace{(id * \varphi^a)\left(\alpha_{(1)}\right) \cdot \psi^b\left(\alpha_{(2)}\right)}_{=\left(id*\varphi^a*\psi^b\right)(\alpha)}$$

$$- \underbrace{\left(\alpha_{(1)}\right)_{(1)}\left(\frac{\varphi(t+u)-\varphi(t)}{u} : \left[a, \psi^b\left(\alpha_{(2)}\right)\right]\right)_x\left(\left(\alpha_{(1)}\right)_{(2)}\right)}_{\substack{=\alpha_{(1)}\left(\frac{\varphi(t+u)-\varphi(t)}{u}:\left[a,\psi^b\left(\alpha_{(3)}\right)\right]\right)_x\left(\alpha_{(2)}\right) \\ =\alpha_{(1)}\left(\frac{\varphi(t+u)-\varphi(t)}{u}:\left[a,\psi^b\left(\left(\alpha_{(2)}\right)_{(2)}\right)\right]\right)_x\left(\left(\alpha_{(2)}\right)_{(1)}\right)}}$$

$$= \left(id * \varphi^a * \psi^b\right)(\alpha) - \alpha_{(1)}\left(\frac{\varphi(t+u)-\varphi(t)}{u} : \left[a, \psi^b\left(\left(\alpha_{(2)}\right)_{(2)}\right)\right]\right)_x\left(\left(\alpha_{(2)}\right)_{(1)}\right)$$

and

$$\left(id * \left(\varphi^a * \psi^b - \left(\frac{\varphi(t+u)-\varphi(t)}{u}\psi(u):[a,b]\right)_x\right)\right)(\alpha)$$
$$= \left(id * \varphi^a * \psi^b\right)(\alpha) - \underbrace{\left(id * \left(\frac{\varphi(t+u)-\varphi(t)}{u}\psi(u):[a,b]\right)_x\right)(\alpha)}_{=\alpha_{(1)}\left(\frac{\varphi(t+u)-\varphi(t)}{u}\psi(u):[a,b]\right)_x\left(\alpha_{(2)}\right)}$$
$$= \left(id * \varphi^a * \psi^b\right)(\alpha) - \alpha_{(1)}\left(\frac{\varphi(t+u)-\varphi(t)}{u}\psi(u):[a,b]\right)_x\left(\alpha_{(2)}\right),$$

this rewrites as

$$\left(id * \varphi^a * \psi^b\right)(\alpha) - \alpha_{(1)}\left(\frac{\varphi(t+u)-\varphi(t)}{u}:\left[a,\psi^b\left(\left(\alpha_{(2)}\right)_{(2)}\right)\right]\right)_x\left(\left(\alpha_{(2)}\right)_{(1)}\right)$$
$$= \left(id * \varphi^a * \psi^b\right)(\alpha) - \alpha_{(1)}\left(\frac{\varphi(t+u)-\varphi(t)}{u}\psi(u):[a,b]\right)_x\left(\alpha_{(2)}\right).$$

Hence, it will be enough to prove that

$$\alpha_{(1)}\left(\frac{\varphi(t+u)-\varphi(t)}{u}:\left[a,\psi^b\left(\left(\alpha_{(2)}\right)_{(2)}\right)\right]\right)_x\left(\left(\alpha_{(2)}\right)_{(1)}\right)$$
$$= \alpha_{(1)}\left(\frac{\varphi(t+u)-\varphi(t)}{u}\psi(u):[a,b]\right)_x\left(\alpha_{(2)}\right). \quad (2)$$

This will clearly be proven if we succeed to show that every $\beta \in S(\mathfrak{g})$ satisfies

$$\left(\frac{\varphi(t+u)-\varphi(t)}{u}:\left[a,\psi^b\left(\beta_{(2)}\right)\right]\right)_x\left(\beta_{(1)}\right)$$
$$= \left(\frac{\varphi(t+u)-\varphi(t)}{u}\psi(u):[a,b]\right)_x(\beta) \quad (3)$$

(because applying (3) to $\beta = \alpha_{(2)}$ and multiplying with $\alpha_{(1)}$, we will obtain (2)). So let us prove (3).

Let us show a somewhat stronger assertion: let us show that every polynomial $P \in \mathbb{K}[t,u]$ satisfies

$$\left(P:\left[a,\psi^b\left(\beta_{(2)}\right)\right]\right)_x\left(\beta_{(1)}\right) = \left(P\cdot\psi(u):[a,b]\right)_x(\beta). \quad (4)$$

Once this equality (4) is proven, (3) will immediately follow (by setting $P = \dfrac{\varphi(t+u) - \varphi(t)}{u}$). So let us prove (4):
Since the equality (4) is linear in ψ and P and continuous in ψ, we can WLOG assume that $\psi = z^k$ for some $k \in \mathbb{N}$, and that $P = t^r u^q$ for some $r \in \mathbb{N}$ and $q \in \mathbb{N}$. Then,

$$\begin{aligned}
&\left(P : \left[a, \psi^b\left(\beta_{(2)}\right)\right]\right)_x \left(\beta_{(1)}\right) \\
&= \left(t^r u^q : \left[a, \psi^b\left(\beta_{(2)}\right)\right]\right)_x \left(\beta_{(1)}\right) = \left[(\operatorname{ad} x)^r (a), (\operatorname{ad} x)^q \left(\psi^b\left(\beta_{(2)}\right)\right)\right] \left(\beta_{(1)}\right) \\
&= \left[((\operatorname{ad} x)^r (a)) \left(\left(\beta_{(1)}\right)_{(1)}\right), ((\operatorname{ad} x)^q \left(\psi^b\left(\beta_{(2)}\right)\right)) \left(\left(\beta_{(1)}\right)_{(2)}\right)\right] \\
&= \left[((\operatorname{ad} x)^r (a)) \left(\beta_{(1)}\right), \left((\operatorname{ad} x)^q \left(\psi^b\left(\left(\beta_{(2)}\right)_{(2)}\right)\right)\right) \left(\left(\beta_{(2)}\right)_{(1)}\right)\right]
\end{aligned}$$

and

$$\begin{aligned}
&\Bigg(\underbrace{P \cdot \psi(u)}_{= t^r u^q \cdot u^k = t^r u^{q+k}} : [a, b]\Bigg)_x (\beta) \\
&= \left(t^r u^{q+k} : [a, b]\right)_x (\beta) = \left[(\operatorname{ad} x)^r (a), (\operatorname{ad} x)^{q+k} (b)\right] (\beta) \\
&= \left[((\operatorname{ad} x)^r (a)) \left(\beta_{(1)}\right), \left((\operatorname{ad} x)^{q+k} (b)\right) \left(\beta_{(2)}\right)\right].
\end{aligned}$$

The equality (4) thus transforms into

$$\begin{aligned}
&\left[((\operatorname{ad} x)^r (a)) \left(\beta_{(1)}\right), \left((\operatorname{ad} x)^q \left(\psi^b\left(\left(\beta_{(2)}\right)_{(2)}\right)\right)\right) \left(\left(\beta_{(2)}\right)_{(1)}\right)\right] \\
&= \left[((\operatorname{ad} x)^r (a)) \left(\beta_{(1)}\right), \left((\operatorname{ad} x)^{q+k} (b)\right) \left(\beta_{(2)}\right)\right]. \qquad (5)
\end{aligned}$$

It thus remains to prove (5).
Since $\psi^b = \underbrace{\psi}_{= z^k} (\operatorname{ad} x)(b) = (\operatorname{ad} x)^k (b)$, every $\gamma \in S(\mathfrak{g})$ satisfies

$$\begin{aligned}
\left((\operatorname{ad} x)^q \left(\psi^b\left(\gamma_{(2)}\right)\right)\right) \left(\gamma_{(1)}\right) &= \left((\operatorname{ad} x)^q \left((\operatorname{ad} x)^k (b) \left(\gamma_{(2)}\right)\right)\right) \left(\gamma_{(1)}\right) \\
&= (\operatorname{ad} x)^{q+k} (b) (\gamma)
\end{aligned}$$

(by Observation 3, applied to γ instead of α). Applying this to $\gamma = \beta_{(2)}$ and taking the Lie bracket with $((\operatorname{ad} x)^r (a)) \left(\beta_{(1)}\right)$, we obtain (5). As explained above, this completes the proof of Lemma 2.1.1 *ii)*.

- **Proof of Theorem 2.1.1:** The first three lines of this proof don't seem to belong into this proof. Neither does the last line of the computation. Also, there are some typos:
  - On the third line of the computation, "$\frac{\psi(t+u)-\psi(t)}{u}\varphi(u)$" should be $\frac{\psi(t+u)-\psi(u)}{t}\varphi(t)$.
  - On the fourth line of the computation, *I think* there should be a $(-1)^{\text{something}}$ term in front of the second fraction. I am not exactly sure here since I have never been following the (-1) signs carefully.

- **Lemma 2.1.2:** In this lemma (and its proof), "$\mathbb{N}$" should be replaced by $\mathbb{N}\setminus\{0\}$. (Here I am assuming that $\mathbb{N}$ contains 0 in your terminology. This assumption is reinforced by the statement of Remark 1.3.1.)

- **Proof of Lemma 2.1.3:** Replace "$(w(t,u),[\alpha,\beta])_x$" by "$(w(t,u):[\alpha,\beta])_x$" (three times).

- **Proof of Lemma 2.1.3:** In the formula, there are two commata instead of one on the right hand side.

- **Proof of Theorem 2.1.2:** You write: "By theorem 1.2.1, this identity is equivalent to". In my opinion, what you are using here is not Theorem 1.2.1, but simply the $*$-invertibility of *id*.

- **Remark 2.2.2:** Replace "$\frac{1}{e^z-1}$" by $\frac{z}{e^z-1}$. Also, replace "$\sum\limits_{k\geq 0}^{\infty}$" by $\sum\limits_{k\geq 0}$.

- **Proof of Theorem 2.2.4:** In the first line of this proof, "$\omega(,u)$" should be $\omega(t,u)$.

- **Lemma 2.2.3:** You might want to change "$\mathbb{K}[t]/t^N$" into $\mathbb{K}_0[t]/t^N$. (In fact, you only consider $\varphi\in\mathbb{K}_0[t]/t^N$ in the proof. I am not sure whether this is because the other case is not interesting enough to you, or you can easily rule it out.)

- **Theorem 2.2.5:** Replace "t^n" by t^N (I think).

- **First line of §2.3:** Replace "commutating" by "commuting".

- **(2.16):** The lower arrow of this commutative diagram should be $F_{\mathfrak{h}}$.
- **Proof of Theorem 2.4.1:** A comma is missing in "$x_1, ...x_{n+1}$".
- **Proof of Theorem 2.4.1:** Replace the "$\mathfrak{g}$" by an "$\mathfrak{h}$" in "Let $Y := F_{\mathfrak{g}}(x_1 \cdots x_n \otimes x_{n+1})$".
- **Proof of Theorem 2.4.1:** Replace every letter "X" in "$f_{t,i} : X_j \mapsto \begin{cases} X_j, & j \neq i \\ X_i t, & j = i \end{cases}$" by a lowercase "$x$".
- **Proof of Theorem 2.4.1:** Replace "$f_{t,i}(Y)$" by "$\widetilde{f_{t,i}}(Y)$".
- **Proof of Theorem 2.4.1:** I don't understand how you obtain "$Y = \sum_{i=1}^{n+1} Y_{1,i}$". However, it is completely enough to know that "$Y = \sum_{i\geq 0} Y_{1,i}$", and this is obvious.
- **Proof of Theorem 2.4.1:** Replace "$i \in \{i, ..., n+1\}$" by "$i \in \{1, ..., n+1\}$".
- **Proof of Theorem 2.4.1:** Replace "brackets of n elements" by "brackets of $n+1$ elements".
- **Proof of Lemma 2.5.1:** "using (2.2)" should be "using (2.1)" in my opinion.
- **Proof of Theorem 2.5.1 ii):** In the formula, you write "$\left(\Phi^{a_1} \circ \Phi^{g(a_j)} \circ \Phi^{a_n}\right)(1)$". This should be

  $$\left(\Phi^{a_1} \circ \cdots \circ \Phi^{a_{j-1}} \circ \Phi^{g(a_j)} \circ \Phi^{a_{j+1}} \circ \cdots \circ \Phi^{a_n}\right)(1).$$

  (You can leave out the $\Phi^{a_{j-1}}$ and $\Phi^{a_{j+1}}$ terms if you wish, but at least the $\cdots$ should be there.)
- **Proof of Theorem 2.5.1 ii):** You write "$[g_2, \Phi_1^a] = 1 \otimes \varphi_1^a \circ (g_2 \otimes 1 + 1 \otimes g_2 - \Delta \circ g_2) + \Phi^{g(a)}$". This should be

  $$[g_2, \Phi_1^a] = m \circ (1 \otimes \varphi_1^a) \circ ((g_2 \otimes 1 + 1 \otimes g_2) \circ \Delta - \Delta \circ g_2) + \Phi^{g(a)}.$$

  (Besides I don't understand why you are renaming *id* as 1 again, but it's fine for me.)
- **Proof of Lemma 2.5.2:** "From identity (1.2)" should be "From identity (2.2)".

- **Remark 2.5.5:** Replace "Dedeking" by "Dedekind".
- **Remark 2.5.5:** There is a useless parenthesis before "it is shown that β is one-to-one".

Chapter 3

- **Page 31:** Replace "if A is a $\mathbb{K}$-algebra equipped with a comultiplication Δ" by "if A is a $\mathbb{K}$-coalgebra with a comultiplication Δ".
- **(3.1):** The right hand side should be $\partial(X_1) \circ \cdots \circ \partial(X_n)(f) \mid_0$ rather than $\partial(X_1) \circ \cdots \circ \partial(X_n) \mid_0 (f)$.
- **Theorem 3.1.1:** I believe "$-\varphi_c(\operatorname{ad} x)(a)$" should be "$-(\varphi_c(\operatorname{ad} x)(a))^T$".
- **Proof of Theorem 3.1.1:** The minus sign in "$-(-1)^{p(X_1 \cdots X_n)p(a)} \langle X_1 \cdots X_n, \xi_c^a(f) \rangle$" should be removed. (The minus sign should only appear later due to the definition of the dual of a representation of a Lie algebra.)
- **Remark 3.1.1:** I don't understand what is meant by "the evaluation of ξ_c^a in $X \in \mathfrak{g}_0$".

Chapter 4

- **Page 36:** It would be good to clarify if the notions of "$\mathbb{K}$-supersymmetric space" (or "$\mathbb{K}$-super symmetric space") and "$\mathbb{K}$-symmetric space" are used interchangeably. (I think they are, but I am not sure.)
- **Page 36, Example 4.0.1:** Doesn't the example i) only work when $\mathbb{K} = \mathbb{K}_0$?
- **Page 36, proof of Lemma 4.0.2:** Replace "Φ_c et Φ_d" by "Φ_c and Φ_d".
- **Page 37, Theorem 4.0.2:** Replace "A representations" by "A representation".
- **Page 40, §4.2:** Replace "of finite rang" by "of finite rank".
- **Page 40, §4.3:** Replace "we give a an example" by "we give an example".

Chapter 5

- **Page 47, §5.1:** Replace "for any $X, Y \in \mathfrak{g}$" by "for any $X, Y \in M$".
- **Page 47, §5.1:** Replace "We say that α is *not-degenerate*" by "If $M = \mathfrak{g}$ and $N = \mathbb{K}$, then we say that α is *non-degenerate*".
- Generally, replace every appearance of "not-degenerate" in the text by "non-degenerate".

Chapter 6

- .

Appendix

- .