A determinant identity for symmetric matrices

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0.1. The formula

A strange asymmetry is haunting linear algebra: Alternating matrices (i.e., square matrices A that satisfy $A^T = -A$ and have a diagonal full of zeroes) are known to have a rich and beautiful algebraic theory, while little has been said algebraically about symmetric matrices (i.e., square matrices A that satisfy $A^T = A$). It is as if symmetric matrices "belong to" the realm of analysis (where the spectral theorem and various decompositions help manage them over $\mathbb{R}$), whereas alternating matrices have their home in algebra and algebraic combinatorics (as seen most elegantly in the theory of Pfaffians [Loehr11, §12.12], [DreWen95], [Knuth95]).

In the 1970s, Procesi discovered a remarkable determinantal identity for symmetric matrices that disrupts this pattern. He used this identity to classify the invariants of the orthogonal group. In this note, we shall give it a new and simpler proof.

To state this identity, let us first introduce some notations.

If a and b are two integers, then $[a, b]$ shall denote the set $\{a, a+1, a+2, \ldots, b\}$ (this is empty if $a > b$). We let $[n]$ denote the set $[1, n] = \{1, 2, \ldots, n\}$, where n is any nonnegative integer. Given any set U, we let S_U denote the group of all permutations of U. In particular, $S_{[n]}$ is the well-known n-th symmetric group S_n. When the set U is finite, we will denote the sign of a permutation $\sigma \in S_U$ by $(-1)^\sigma$.

Let $\mathbb{K}$ be a commutative ring. Fix an integer $k \geq 0$. Let $A = (a_{i,j})_{1 \leq i,j \leq 2k} \in \mathbb{K}^{2k \times 2k}$ be a symmetric $2k \times 2k$-matrix. Given any two k-tuples $(i_1, i_2, \ldots, i_k)$ and $(j_1, j_2, \ldots, j_k)$ of elements of $[2k]$, we define $A_{i_1, i_2, \ldots, i_k}^{j_1, j_2, \ldots, j_k}$ to be the $k \times k$-matrix $\left(a_{i_x, j_y}\right)_{1 \leq x,y \leq k}$. If $i_1 < i_2 < \cdots < i_k$ and $j_1 < j_2 < \cdots < j_k$, then this matrix is a submatrix of A; otherwise, it is a "submatrix in a wider sense", allowing for repeated and permuted rows/columns.

The identity we shall be concerned with is the following:

Theorem 0.1.1. Let $p \in [k]$. Then,

$$\sum_{\sigma \in S_{[p,k+p]}} (-1)^\sigma \det \left(A_{1,2,\ldots,p-1,\sigma(p),\sigma(p+1),\ldots,\sigma(k)}^{\sigma(k+1),\sigma(k+2),\ldots,\sigma(k+p),k+p+1,k+p+2,\ldots,2k} \right) = 0.$$

Example 0.1.2. Let $k = 2$ and $p = 2$. Then, Theorem 0.1.1 says that

$$\sum_{\sigma \in S_{[2,4]}} (-1)^\sigma \det \left(A_{1,\sigma(2)}^{\sigma(3),\sigma(4)} \right) = 0.$$

Expanding this sum, we can rewrite this as

$$\det\left(A_{1,2}^{3,4}\right) - \det\left(A_{1,3}^{2,4}\right) - \det\left(A_{1,4}^{3,2}\right) - \det\left(A_{1,2}^{4,3}\right) + \det\left(A_{1,3}^{4,2}\right) + \det\left(A_{1,4}^{2,3}\right) = 0. \qquad (1)$$

The six addends on the left hand side actually come in three pairs of equal addends, since a determinant changes sign when two of its columns are swapped:

$$\det\left(A_{1,2}^{4,3}\right) = -\det\left(A_{1,2}^{3,4}\right);$$
$$\det\left(A_{1,3}^{4,2}\right) = -\det\left(A_{1,3}^{2,4}\right);$$
$$\det\left(A_{1,4}^{3,2}\right) = -\det\left(A_{1,4}^{2,3}\right).$$

Collecting these equal addends together, we can rewrite the equality (1) as

$$2\left(\det\left(A_{1,2}^{3,4}\right) - \det\left(A_{1,3}^{2,4}\right) + \det\left(A_{1,4}^{2,3}\right)\right) = 0.$$

Since this is a polynomial identity in the entries of A, we can divide it by 2 (working in a polynomial ring over $\mathbb{Z}$ instead of $\mathbb{K}$) and obtain

$$\det\left(A_{1,2}^{3,4}\right) - \det\left(A_{1,3}^{2,4}\right) + \det\left(A_{1,4}^{2,3}\right) = 0. \qquad (2)$$

More generally, proceeding as in Example 0.1.2, we can reduce Theorem 0.1.1 as follows:

Theorem 0.1.3. Let $p \in [k]$. Let $T_{[p,k+p]}$ be the subgroup of $S_{[p,k+p]}$ consisting of those permutations σ that fix the subset $[p,k]$ (that is, satisfy $\sigma([p,k]) = [p,k]$). Let L be a left transversal (i.e., a system of distinct representatives for the left cosets) of $T_{[p,k+p]}$ in $S_{[p,k+p]}$. Then,

$$\sum_{\sigma \in L} (-1)^{\sigma} \det\left(A_{1,2,\ldots,p-1,\sigma(p),\sigma(p+1),\ldots,\sigma(k)}^{\sigma(k+1),\sigma(k+2),\ldots,\sigma(k+p),k+p+1,k+p+2,\ldots,2k}\right) = 0.$$

Theorem 0.1.3 appears in [Proces06, §13.8.3, Lemma] (and, in a more inchoate form, in [DeCPro76, Lemma 5.2]), where it is used to identify the invariants of the orthogonal group.

As in Example 0.1.2, we can easily derive Theorem 0.1.3 from Theorem 0.1.1: Just observe that the determinant $\det\left(A_{1,2,\ldots,p-1,\sigma(p),\sigma(p+1),\ldots,\sigma(k)}^{\sigma(k+1),\sigma(k+2),\ldots,\sigma(k+p),k+p+1,k+p+2,\ldots,2k}\right)$ gets multiplied by the sign $(-1)^{\tau}$ when we multiply the permutation $\sigma \in$

$S_{[p,k+p]}$ by a permutation $\tau \in T_{[p,k+p]}$ from the right (since this merely permutes the rows and the columns of the matrix, and the signs of the relevant permutations multiply to $(-1)^{\tau}$). This observation lets us bunch equal addends in Theorem 0.1.3 together into blocks, thus yielding

$$\left|T_{[p,k+p]}\right| \cdot \sum_{\sigma \in L} (-1)^{\sigma} \det\left(A^{\sigma(k+1),\sigma(k+2),\ldots,\sigma(k+p),k+p+1,k+p+2,\ldots,2k}_{1,2,\ldots,p-1,\sigma(p),\sigma(p+1),\ldots,\sigma(k)}\right) = 0.$$

Finally, we can divide by the positive integer $\left|T_{[p,k+p]}\right|$ after making the WLOG assumption that the entries of A are indeterminates in a polynomial ring over $\mathbb{Z}$.

Hence, it remains to prove Theorem 0.1.1. This was done in [Proces06, §13.8.3, Lemma] and [DeCPro76, Lemma 5.2] by a complicated induction. We shall instead give a simple proof that uses nothing but the definition of a determinant.

0.2. The proof

Proof of Theorem 0.1.1. We extend each permutation $\sigma \in S_{[p,k+p]}$ to a permutation $\widehat{\sigma} \in S_{[2k]}$ by setting

$$\widehat{\sigma}(i) := \begin{cases} \sigma(i), & \text{if } i \in [p, k+p]; \\ i, & \text{if } i \notin [p, k+p] \end{cases} \qquad \text{for all } i \in [2k].$$

This extended permutation $\widehat{\sigma}$ agrees with the original σ on the inputs $p, p+1, \ldots, k+p$. Thus, it shall lead to no confusion if we simply write $\sigma(i)$ for $\widehat{\sigma}(i)$ whenever $i \in [2k]$. We will therefore do so, for brevity's sake. Thus, we can rewrite the determinant

$$\det\left(A^{\sigma(k+1),\sigma(k+2),\ldots,\sigma(k+p),k+p+1,k+p+2,\ldots,2k}_{1,2,\ldots,p-1,\sigma(p),\sigma(p+1),\ldots,\sigma(k)}\right)$$

in Theorem 0.1.1 in the nicer form

$$\det\left(A^{\sigma(k+1),\sigma(k+2),\ldots,\sigma(2k)}_{\sigma(1),\sigma(2),\ldots,\sigma(k)}\right).$$

Thus, for each $\sigma \in S_{[p,k+p]}$, we have

$$\begin{aligned}
&\det\left(A^{\sigma(k+1),\sigma(k+2),\ldots,\sigma(k+p),k+p+1,k+p+2,\ldots,2k}_{1,2,\ldots,p-1,\sigma(p),\sigma(p+1),\ldots,\sigma(k)}\right) \\
&= \det\left(A^{\sigma(k+1),\sigma(k+2),\ldots,\sigma(2k)}_{\sigma(1),\sigma(2),\ldots,\sigma(k)}\right) \\
&= \det\left(a_{\sigma(i),\ \sigma(k+j)}\right)_{1 \le i,j \le k} \qquad \left(\text{by the definition of } A^{\sigma(k+1),\sigma(k+2),\ldots,\sigma(2k)}_{\sigma(1),\sigma(2),\ldots,\sigma(k)}\right) \\
&= \sum_{\zeta \in S_k} (-1)^{\zeta} \prod_{i=1}^{k} a_{\sigma(i),\ \sigma(k+\zeta(i))} \qquad (\text{by the definition of a determinant}).
\end{aligned}$$

Multiplying this with $(-1)^\sigma$ and summing the result over all $\sigma \in S_{[p,k+p]}$, we obtain

$$\begin{aligned}
&\sum_{\sigma\in S_{[p,k+p]}} (-1)^\sigma \det\left(A^{\sigma(k+1),\sigma(k+2),\ldots,\sigma(k+p),k+p+1,k+p+2,\ldots,2k}_{1,2,\ldots,p-1,\sigma(p),\sigma(p+1),\ldots,\sigma(k)}\right)\\
&= \sum_{\sigma\in S_{[p,k+p]}} (-1)^\sigma \sum_{\zeta\in S_k} (-1)^\zeta \prod_{i=1}^{k} a_{\sigma(i),\ \sigma(k+\zeta(i))}\\
&= \sum_{\zeta\in S_k} (-1)^\zeta \sum_{\sigma\in S_{[p,k+p]}} (-1)^\sigma \prod_{i=1}^{k} a_{\sigma(i),\ \sigma(k+\zeta(i))}. \qquad (3)
\end{aligned}$$

Now, let $\zeta \in S_k$ be arbitrary. We shall show that

$$\sum_{\sigma\in S_{[p,k+p]}} (-1)^\sigma \prod_{i=1}^{k} a_{\sigma(i),\ \sigma(k+\zeta(i))} = 0. \qquad (4)$$

Indeed, the set $[p, k+p]$ has size $|[p, k+p]| = k+1 > k$. Hence, by the pigeonhole principle, there exists some $j \in [k]$ such that both numbers j and $k+\zeta(j)$ belong to the set $[p, k+p]$ [1]. Consider this j. The two numbers j and $k+\zeta(j)$ are furthermore distinct (since the former is $\leq k$ while the latter is $> k$); thus, the transposition $\tau := t_{j,k+\zeta(j)} \in S_{[p,k+p]}$ exists (since j and $k+\zeta(j)$ belong to $[p, k+p]$) and has sign $(-1)^\tau = -1$ (since any transposition has sign -1). Since $S_{[p,k+p]}$ is a group, we have $\sigma\tau \in S_{[p,k+p]}$ for each $\sigma \in S_{[p,k+p]}$ (because $\tau \in S_{[p,k+p]}$).

When we replace a permutation $\sigma \in S_{[p,k+p]}$ by the composition $\sigma\tau$, the sign $(-1)^\sigma$ flips (since $(-1)^{\sigma\tau} = (-1)^\sigma \underbrace{(-1)^\tau}_{=-1} = -(-1)^\sigma$), but the product

[1] *Proof.* Assume the contrary. Thus, $|\{j, k+\zeta(j)\} \cap [p, k+p]| \leq 1$ for each $j \in [k]$. Hence,

$$\left|\bigcup_{j=1}^{k} (\{j, k+\zeta(j)\} \cap [p, k+p])\right| \leq \sum_{j=1}^{k} \underbrace{|\{j, k+\zeta(j)\} \cap [p, k+p]|}_{\leq 1} \leq \sum_{j=1}^{k} 1 = k.$$

In view of

$$\bigcup_{j=1}^{k} (\{j, k+\zeta(j)\} \cap [p, k+p]) = \underbrace{\left(\bigcup_{j=1}^{k} \{j, k+\zeta(j)\}\right)}_{\substack{=\{1,2,\ldots,k,k+\zeta(1),k+\zeta(2),\ldots,k+\zeta(k)\}\\ =[2k] \\ \text{(since } \zeta\in S_k \text{ ensures that the numbers } k+\zeta(1),k+\zeta(2),\ldots,k+\zeta(k)\\ \text{are precisely } k+1,k+2,\ldots,2k)}} \cap [p, k+p]$$

$$= [2k] \cap [p, k+p] = [p, k+p],$$

this rewrites as $|[p, k+p]| \leq k$. But this contradicts $|[p, k+p]| > k$.

$\prod_{i=1}^{k} a_{\sigma(i),\ \sigma(k+\zeta(i))}$ remains unchanged (since the only factor that sees any change is the factor for $i = j$, which changes from $a_{\sigma(j),\ \sigma(k+\zeta(j))}$ to

$$\begin{aligned} a_{(\sigma\tau)(j),\ (\sigma\tau)(k+\zeta(j))} &= a_{\sigma(k+\zeta(j)),\ \sigma(j)} && \text{(since } \tau \text{ swaps } j \text{ with } k+\zeta(j)) \\ &= a_{\sigma(j),\ \sigma(k+\zeta(j))} && \text{(since the matrix } A \text{ is symmetric)}, \end{aligned}$$

which of course just means that it stays the same). Hence, by pairing up each even permutation $\sigma \in S_{[p,k+p]}$ with the odd permutation $\sigma\tau \in S_{[p,k+p]}$ in the sum (4), we cause all addends to cancel with their respective partners. Thus, the whole sum simplifies to 0. This proves (4).

Now, forget that we fixed ζ. We thus have proved (4) for each $\zeta \in S_k$. Now, (3) becomes

$$\begin{aligned} &\sum_{\sigma \in S_{[p,k+p]}} (-1)^{\sigma} \det\left(A^{\sigma(k+1),\sigma(k+2),\ldots,\sigma(k+p),k+p+1,k+p+2,\ldots,2k}_{1,2,\ldots,p-1,\sigma(p),\sigma(p+1),\ldots,\sigma(k)} \right) \\ &= \sum_{\zeta \in S_k} (-1)^{\zeta} \underbrace{\sum_{\sigma \in S_{[p,k+p]}} (-1)^{\sigma} \prod_{i=1}^{k} a_{\sigma(i),\ \sigma(k+\zeta(i))}}_{\substack{=0 \\ \text{(by (4))}}} = 0. \end{aligned}$$

This proves Theorem 0.1.1. □

Remark 0.2.1. In our above proof, we did not use the symmetry of A in full; we only used the requirement

$$a_{u,v} = a_{v,u} \qquad \text{for all } u, v \in [p, k+p]. \tag{5}$$

Indeed, when we argued that $a_{\sigma(k+\zeta(j)),\ \sigma(j)} = a_{\sigma(j),\ \sigma(k+\zeta(j))}$, we could easily observe that $\sigma(k+\zeta(j)) \in [p, k+p]$ (since the choice of j ensures that $k + \zeta(j) \in [p, k+p]$, and since σ preserves the set $[p, k+p]$) and $\sigma(j) \in [p, k+p]$ (since the choice of j ensures that $j \in [p, k+p]$, and since σ preserves the set $[p, k+p]$). Thus, we obtain slightly more general versions of Theorem 0.1.1 and Theorem 0.1.3.

0.3. Generalizations and variants

Our above proof of Theorem 0.1.1 offers several opportunities for variation. Among other things:

- The signs $(-1)^{\zeta}$ in the definition of a determinant can be removed or replaced by other conjugation-invariant functions of ζ; this causes the determinants to be replaced by permanents or immanants [GouJac92].

- The signs $(-1)^\sigma$ in Theorem 0.1.1 can be removed, but then the matrix A must be assumed to be skew-symmetric rather than symmetric. Or, to be more precise, we only need to assume that $a_{u,v} = -a_{v,u}$ for any two distinct(!) elements u and v of $[p, k+p]$.

Some of these variants, however, are trivial. Indeed, if $k > 1$, then the variant of Theorem 0.1.1 with determinants replaced by permanents falls prey to a trivial cancellation argument (since the permanent of a matrix does not change if we swap some columns or some rows). The same happens with the version of Theorem 0.1.1 for skew-symmetric matrices and with no $(-1)^\sigma$ signs. However, if we make both changes at once (i.e., remove the $(-1)^\sigma$ signs, require A to be skew-symmetric, and replace determinants by permanents), then we obtain the following nontrivial result:

Theorem 0.3.1. Let $p \in [k]$. Instead of assuming that A be symmetric, let us assume that $a_{u,v} = -a_{v,u}$ for any two distinct elements u and v of $[p, k+p]$. Then,

$$\sum_{\sigma \in S_{[p,k+p]}} \operatorname{per}\left(A^{\sigma(k+1),\sigma(k+2),\ldots,\sigma(k+p),k+p+1,k+p+2,\ldots,2k}_{1,2,\ldots,p-1,\sigma(p),\sigma(p+1),\ldots,\sigma(k)}\right) = 0,$$

where per B denotes the permanent of a matrix B.

Question 0.3.2. Can we prove Theorem 0.1.3 directly by sign-reversing involution, without having to divide by a positive integer?

References

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