An involutive introduction to symmetric functions

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`http://www.ma.rhul.ac.uk/~uvah099/Maths/Sym/SymFuncs2020.pdf`

version of 8 May 2020

Errata and addenda by Darij Grinberg

Errata and comments

- **pages 1–2, Preface:** Something similar to your solution to Question 21 appears in the proof of Theorem 6.3 of:

  Anthony Mendes, Jeffrey Remmel,

  Counting with Symmetric Functions,

  Springer 2015,

  and also in §9.2 of:

  Eric Egge,

  An Introduction to Symmetric Functions and Their Combinatorics,

  Student Mathematical Library **91**,

  AMS 2019.

  Might be worth a brief comparison.

- **page 4, §1.3:** "unital **C**-algebra isomorphisms" → "unital **C**-algebra automorphisms".

- **page 4, §1.3:** "sends the unit element 1 is sent to itself" → "sends the unit element 1 to itself".

- **page 21:** It is worth pointing out that Theorem 2.1 immediately yields a new proof of Proposition 1.16.

- **page 23, Definition 3.2:** After "*abacus* representing λ", add "(or, for short, *abacus for* λ)".

- **page 24, §3.2:** In the computation of $a_{(3,1)+(2,1,0)}$, the "$+x_3^2x_1^5 - x_1^2x_3^5$" part should be "$-x_3^2x_1^5 + x_1^2x_3^5$" (both signs need to be flipped).

- **page 25:** In the first case ("If there are no collisions") of the definition of $J(A,S)$, I briefly stumbled over the question of what to do if the first bead we want to move right is already in the rightmost position. Thinking about the purpose of the construction, I soon realized that in this case, the abacus is simply extended by one gap to the right before moving the bead. This is probably worth writing out.

- **page 27, proof of Corollary 3.9:** It might be worth explaining what a "Young's Rule addition" is (i.e., adding boxes in such a way that no two boxes are added in the same column).

- **page 28, §3.5, and many places below:** Let me note that "ribbon" and "border strip" are synonyms for "rim-hook" widely used in the literature.

- **page 28, §3.5:** I'd add the remark that (for any partition λ) we say that λ/λ is a 0-strip, and that its sign $\operatorname{sgn}(\lambda/\lambda)$ is defined to be 1 (contrary to what the definition of sign would suggest). This convention is important in making Corollary 3.13 work (keep in mind that α_i can be 0 in a composition α).

- **page 30, proof of Corollary 3.13:** I'd mention here that you are using Theorem 3.11 for all $r \in \mathbf{N}_0$, not just for $r \in \mathbf{N}$. (Of course, Theorem 3.11 for $r = 0$ is obvious.)

- **page 30, §3.6:** After "just observe that $P(1,2,2,1) = (2,2,1,1)$", add "$= P(1,1,2,2)$", in order to clarify what this has to do with 2-rim-hooks.

- **page 31, Definition 4.1:** I think an example illustrating the concepts of "excess" and "record" used in this definition would be helpful. For example, in order to find the 1-unpaired 1s in 121321132, we make the following table:
$$\begin{pmatrix} 1 & 2 & 1 & 3 & 2 & 1 & 1 & 3 & 2 \\ 1 & 0 & 1 & 1 & 0 & 1 & 2 & 2 & 1 \\ * & & & & & & * & & \end{pmatrix}.$$
The top row is the word $w = 121321132$. The middle row shows, for each entry of this word, the excess of 1s over 2s in the part of the word reaching up to this entry (when the word is read from left to right). The bottom row has an asterisk $*$ in each column where the excess achieves a new record; thus, the 1-unpaired 1s in w are exactly the entries which have a $*$ under them. A similar table can be made for finding 1-unpaired 2s.

- **page 32, proof of Lemma 4.2:** You write: "since every $k+1$ to the left of position i is paired, this new k is unpaired". I believe this isn't so simple. Couldn't this new k grab a $k+1$ to its left that was previously paired with some other k in w, and thus mess up the pairing of parentheses?

  Let me suggest two valid proofs of this claim (though I cannot say any of them is particularly readable).

  I shall refer to the third sentence of Lemma 4.2 ("Changing the letters [...] entries of w") as Lemma 4.2 (b).

  First proof of Lemma 4.2 (b): Let w' be the word obtained from w by the change indicated in Lemma 4.2 (b).

Regard the ks and $(k+1)$s in w as closing and opening parentheses, respectively. The paired ks and the paired $(k+1)$s then correspond to parentheses that are paired according to the usual rules of bracketing. This pairing has the following property: Between any paired parenthesis and its partner[1], there are no unpaired parentheses[2]. Therefore, any change to the unpaired parentheses in w does not interfere with the paired parentheses; in particular, it does not render their pairing invalid[3]. In general, such a change might introduce some new paired parentheses; however, the change indicated in Lemma 4.2 (b) cannot do this, because it replaces the unpaired subword $k^c (k+1)^d$ by a subword of the form $k^{c'} (k+1)^{d'}$, which clearly creates no opportunity for further pairing. Therefore, the paired parentheses in w' are exactly the paired parentheses in w (in particular, they occupy the same positions in w' as in w); consequently, the k-unpaired entries of w' are in the same positions as the k-unpaired entries of w. This proves Lemma 4.2.

Second proof of Lemma 4.2 (b): We proceed by strong induction on the length of the word. Thus, we fix our w, k, c, d, c' and d', but we assume that Lemma 4.2 (b) is already proven for all words shorter than w in the place of w.

A word is said to be *simple* if it has the form $(k+1)\, vk$, where v is a word (possibly empty) containing neither of the letters k and $k+1$. (Of course, the letter k is fixed here.) Let w' be the word obtained from w by the change indicated in Lemma 4.2 (b).

If the word w contains no simple factor, then Lemma 4.2 (b) is obvious (indeed, in this case, all ks and all $(k+1)$s are unpaired in w, and the same holds for w'). We thus assume that the word w contains a simple factor. In this case, we choose some simple factor of w; we denote this factor by u, and we let p and q be the positions (in w) of its first and last letter. For any word z having at least q letters, we let $\overline{z}$ be the word obtained from z by removing the letters at positions $p, p+1, \ldots, q$.

Now, the pairing of the ks and $(k+1)$s in w (regarded as closing and opening parentheses) has the following property: The $k+1$ in position p is paired with the k in position q (since there are no ks and no $(k+1)$s between them), and the pairing of the remaining ks and $(k+1)$s in w is precisely the same as if the simple factor u (starting at position p and ending at position q) was absent (i.e., it is the same as for the word $\overline{w}$). Exactly

[1]The *partner* of a paired parenthesis is the other parenthesis that it is paired with.

[2]In fact, any unpaired parenthesis between them would have prevented them from getting paired with each other.

[3]"Invalid" would mean that two parentheses that were paired to each other before the change could end up not paired to each other after the change. This cannot happen, because there were no unpaired entries between them (as we have just seen), and so none of the letters between them have changed.

the same holds for the word w', because the simple factor u is unaffected by the change that transforms w into w' (indeed, the change only modifies unpaired letters, but there are no unpaired letters in u). Hence, in order to prove Lemma 4.2 (b) for our word w, it suffices to prove Lemma 4.2 (b) for the word $\overline{w}$ (since the word $\overline{w'}$ is obtained from $\overline{w}$ by the same change that transforms w into w'). But this follows from the induction hypothesis, since the word $\overline{w}$ is shorter than w. This concludes the proof of Lemma 4.2 (b).

- **page 33, proof of Lemma 4.4:** You write: "If t' is not semistandard then $t\left(a-1,b\right) = k$". This requires proof. A priori, it is clear that if t' is not semistandard, then either $t\left(a-1,b\right) = k$ or $t\left(a,b-1\right) = k+1$ (or both). To obtain your claim, we need to rule out that $t\left(a,b-1\right) = k+1$. Fortunately, this is easy: If we had $t\left(a,b-1\right) = k+1$, then the letter $k+1$ of $\mathrm{w}\left(t\right)$ corresponding to the entry $k+1$ in position $\left(a,b-1\right)$ of t would be a k-unpaired $k+1$ (indeed, the letter immediately following it is a k-unpaired $k+1$, but there is a fact (easily proven using Definition 4.1) that if a letter p in a word w is a $k+1$, and if the letter immediately following it is a k-unpaired $k+1$, then the letter p must also be a k-unpaired $k+1$), but this would contradict the fact that the leftmost unpaired $k+1$ in $\mathrm{w}\left(t\right)$ is the letter corresponding to the entry $t\left(a,b\right)$ (which is further right than the letter we are talking about).

  For some reason, every argument I make about coplactic maps degenerates into a run-on sentence like this...

- **page 33, proof of Lemma 4.4:** You say "S_k and S_kE_k are involutions". Well, almost... In order to be able to say that S_k is an involution, you need to extend S_k to a map $\mathrm{SSYT}\left(\mu,\alpha\right) \to \mathrm{SSYT}\left(\mu,\alpha\right)$ (rather than merely $\mathrm{SSYT}_k\left(\mu,\alpha\right) \to \mathrm{SSYT}_{k+1}\left(\mu,\alpha\right)$). An involution must be a bijection from a set to itself, not to another set. Likewise, the map S_kE_k is not in itself an involution, but if you combine the maps $S_kE_k : \mathrm{SSYT}_{k+1}\left(\mu,\alpha\right) \to \mathrm{SSYT}_{k+1}\left(\mu,\alpha'-\epsilon\left(k\right)\right)$ for all α into one large map $S_kE_k : \mathrm{SSYT}_{k+1}\left(\mu\right) \to \mathrm{SSYT}_{k+1}\left(\mu\right)$, where $\mathrm{SSYT}_{k+1}\left(\mu\right) = \bigsqcup_\alpha \mathrm{SSYT}_{k+1}\left(\mu,\alpha\right)$, then this large map S_kE_k is an involution.

- **page 35, proof of Lemma 4.7:** "$\sigma = \mathrm{id}_{\mathrm{Sym}_N}$ by Question 22" $\to$ "$\sigma = \mathrm{id}_{\mathrm{Sym}_N}$ by Question 22 (b)".

- **page 35, proof of Lemma 4.7:** At the very end of this proof, it wouldn't hurt to explicitly mention that t is the unique element of $\mathrm{SSYT}\left(\lambda,\lambda\right)$ because $\left|\mathrm{SSYT}\left(\lambda,\lambda\right)\right| = K_{\lambda\lambda} = 1$ by Question 11 (c).

- **page 35:** You write: "J has a unique fixed point in $\mathcal{T}$ if $\mu = \lambda$, and otherwise none".

This is not quite obvious. In order to prove that an unlatticed tableau $t \in \mathcal{T}$ cannot be a fixed point of J, you need to observe that the content of $J(t)$ is different from the content of t (because Question 22 (c) shows that $\lambda \cdot (\sigma(k, k+1)) \neq \lambda \cdot \sigma$).

- **page 37, proof of Theorem 4.10:** You write: "it follows that applying J therefore cancels all contributions to c_μ except those coming from tableaux $t \in \mathcal{T}$ such that $J(t) = t$". Let me explain this in a bit more detail:

  Define the *sign* $\operatorname{sgn}(t)$ of a tableau $t \in \mathcal{T}$ to be $\operatorname{sgn} \sigma$, where σ is the unique permutation in Sym_N satisfying $t \in \operatorname{SSYT}(\mu, \lambda \cdot \sigma)$. (The uniqueness of this σ follows from Question 22 (c).) Then, we can rewrite the definition of c_μ as $c_\mu = \sum_{t \in \mathcal{T}} \operatorname{sgn}(t)$ (using the fact that the σ in the preceding sentence is unique). Thus, a sign-reversing involution on $\mathcal{T}$ should help simplify c_μ. And Lemma 4.7 (i) shows precisely that the involution J is sign-reversing on the unlatticed tableaux $t \in \mathcal{T}$.

- **page 39, proof of Theorem 5.3:** In Claim 1, it might be better to replace "$\dfrac{a_j!}{C_{1j}! \cdots C_{kj}!}$" by "$\dbinom{a_j}{C_{1,j}, \ldots, C_{k,j}}$" (after perhaps reminding the reader of the definition of multinomial coefficients: namely, if $u_1, u_2, \ldots, u_k$ are k nonnegative integers, and if $v = u_1 + u_2 + \cdots + u_k$, then the multinomial coefficient $\dbinom{v}{u_1, u_2, \ldots, u_k}$ is defined to be the positive integer $\dfrac{v!}{u_1! u_2! \cdots u_k!}$). After all, you always write it as a multinomial coefficient later on.

- **page 41, proof of Lemma 5.4:** In the last computation of this proof, you are tacitly using the identity $\langle s_\lambda, h_\mu \rangle = K_{\lambda\mu}$ (for any partitions λ and μ). This is probably worth stating earlier on.

- **page 41, proof of Lemma 5.6:** I'd replace "Comparing (5.2) and (5.3) we get" by the somewhat more detailed "The definition of ω yields $\omega(h_\mu) = e_\mu$. In view of (5.2) and (5.3), this rewrites as".

- **page 42, §5.3, Alternative proof:** I don't know how detailed this all is supposed to be, but I feel like there are a lot of silent steps here. In particular, it would help pointing out (probably somewhere in §3) that an abacus of a partition λ can be obtained by vertically reflecting an abacus of its conjugate λ' and turning beads into gaps and vice versa. This is a beautiful (yet simple) fact, and explains why the border-strip tableaux for λ are in bijection with those of λ'.

- **page 42, §5.4:** Have you ever defined what a skew-partition is, and how its Young diagram is defined?

  (A *skew-partition* is a pair (λ, ν) of two partitions λ and ν satisfying $[\nu] \subseteq [\lambda]$. It is written as λ/ν, and its Young diagram $[\lambda/\nu]$ is defined to be the set difference $[\lambda] - [\nu]$. A tableau of shape λ/ν is defined just as a tableau of usual shape.)

- **page 43, §5.5:** It can't possibly hurt to say somewhere that the "ω-involution" means the involution ω.

- **page 43, proof of Proposition 5.7:** I am not sure how you conclude that this irreducible constituent in the last sentence is actually the image of s_μ. (This is not hard to check – e.g., there is a standard trick that uses $\langle \chi^\mu, \chi^\mu \rangle = \langle s_\mu, s_\mu \rangle = 1$ to show that χ^μ is $\pm$ an irreducible character, and then we can use $\langle \chi^\mu, \pi^\mu \rangle = 1 > 0$ to conclude that the $\pm$ is in fact a $+$.)

- **page 43, proof of Corollary 5.10:** Strictly speaking, you have not shown that all irreducible characters of Sym_n are of the form χ^λ, so the proof is incomplete. (I am not saying that this is difficult, but it needs a couple more lines.)

- **page 46, (5.7):** Replace "g" by "σ" or vice versa.

- **page 46, proof of Lemma 5.12:** Replace "of S_n" by "of Sym_n".

- **page 47, proof of Theorem 5.14:** It is worth stating explicitly the fact that you are using for the last equality sign in the long computation. This fact says that if A is a finite group, and if B and C are two subgroups of A, then

  $$\left\langle 1 \uparrow_B^A \downarrow_C^A, 1 \right\rangle_C = (\text{the number of double cosets } BaC \text{ with } a \in A).$$

  This can indeed be derived from Mackey's formula or from the interpretation of $1 \uparrow_B^A$ as a coset space character.

- **pages 50–51, Question 7:** Here is an easier way to solve part (g) (which also shows that you can replace "$\ell \in \mathbf{N}$" by "$\ell \in \mathbf{N}_0$"):

  Step 1: We observe that every $N \geq 0$ satisfies

  $$\sum_{i=0}^{N} \binom{N}{i} d_{(1^i)} = N!. \tag{1}$$

  (This follows by noticing that $\binom{N}{i} d_{(1^i)}$ is the number of permutations $\sigma \in \operatorname{Sym}_N$ that have exactly i fixed points.)

  Step 2: Now, fix $n \in \mathbf{N}_0$. For each $\ell \in \{0, 1, \ldots, n\}$, we set

  $$w_\ell = (-1)^{\ell+1} \sum_{m=\ell+1}^{n} \binom{m-1}{\ell} \binom{n}{m} d_{(1^{n-m})}. \tag{2}$$

Thus, our goal is to prove that

$$d_{(1^n)} = \frac{n!}{\ell!} d_{(1^\ell)} + w_\ell \qquad \text{for each } \ell \in \{0, 1, \ldots, n\}.$$

This we shall prove by induction over $n - \ell$. The base case ($n - \ell = 0$) is obvious (since $w_n = 0$). For the induction step, it suffices to prove that

$$\frac{n!}{\ell!} d_{(1^\ell)} + w_\ell = \frac{n!}{(\ell+1)!} d_{(1^{\ell+1})} + w_{\ell+1} \tag{3}$$

for each $\ell \in \{0, 1, \ldots, n-1\}$. Thus we shall focus on proving (3).

Step 3: Fix $\ell \in \{0, 1, \ldots, n-1\}$. The definition of $w_{\ell+1}$ yields

$$\begin{aligned} w_{\ell+1} &= (-1)^{\ell+2} \sum_{m=\ell+2}^{n} \binom{m-1}{\ell+1} \binom{n}{m} d_{(1^{n-m})} \\ &= (-1)^{\ell+2} \sum_{m=\ell+1}^{n} \binom{m-1}{\ell+1} \binom{n}{m} d_{(1^{n-m})} \\ &\qquad \left(\begin{array}{c} \text{here, we have extended the range of the sum by one} \\ \text{extra addend, which is zero} \\ \text{(since } \dbinom{m-1}{\ell+1} = 0 \text{ when } m = \ell+1\text{)} \end{array} \right) \\ &= -(-1)^{\ell+1} \sum_{m=\ell+1}^{n} \binom{m-1}{\ell+1} \binom{n}{m} d_{(1^{n-m})}. \end{aligned}$$

Subtracting this equality from (2), we find

$$
\begin{aligned}
& w_\ell - w_{\ell+1} \\
&= (-1)^{\ell+1} \sum_{m=\ell+1}^{n} \binom{m-1}{\ell} \binom{n}{m} d_{(1^{n-m})} \\
&\qquad - \left(- (-1)^{\ell+1} \sum_{m=\ell+1}^{n} \binom{m-1}{\ell+1} \binom{n}{m} d_{(1^{n-m})} \right) \\
&= (-1)^{\ell+1} \sum_{m=\ell+1}^{n} \underbrace{\left(\binom{m-1}{\ell} + \binom{m-1}{\ell+1} \right)}_{\substack{= \binom{m}{\ell+1} \\ \text{(by the recursion of the} \\ \text{binomial coefficients)}}} \binom{n}{m} d_{(1^{n-m})} \\
&= (-1)^{\ell+1} \sum_{m=\ell+1}^{n} \underbrace{\binom{m}{\ell+1} \binom{n}{m}}_{\substack{= \binom{n}{\ell+1} \binom{n-(\ell+1)}{n-m} \\ \text{(by straightforward manipulations)}}} d_{(1^{n-m})} \\
&= (-1)^{\ell+1} \binom{n}{\ell+1} \underbrace{\sum_{m=\ell+1}^{n} \binom{n-(\ell+1)}{n-m} d_{(1^{n-m})}}_{\substack{= \sum_{i=0}^{n-(\ell+1)} \binom{n-(\ell+1)}{i} d_{(1^i)} \\ \text{(here, we have substituted } i \\ \text{for } n-m \text{ in the sum)}}} \\
&= (-1)^{\ell+1} \binom{n}{\ell+1} \underbrace{\sum_{i=0}^{n-(\ell+1)} \binom{n-(\ell+1)}{i} d_{(1^i)}}_{\substack{= (n-(\ell+1))! \\ \text{(by (1) (applied to } N = n-(\ell+1)))}} \\
&= (-1)^{\ell+1} \underbrace{\binom{n}{\ell+1} (n-(\ell+1))!}_{= \frac{n!}{(\ell+1)!}} = (-1)^{\ell+1} \frac{n!}{(\ell+1)!}.
\end{aligned}
$$

Comparing this with

$$\frac{n!}{(\ell+1)!} \underbrace{d_{(1^{\ell+1})}}_{\substack{=(\ell+1)d_{(1^\ell)}+(-1)^{\ell+1} \\ \text{(by the well-known recursion} \\ \text{for derangement numbers)}}} - \frac{n!}{\ell!} d_{(1^\ell)}$$
$$= \frac{n!}{(\ell+1)!} \left((\ell+1)\, d_{(1^\ell)} + (-1)^{\ell+1} \right) - \frac{n!}{\ell!} d_{(1^\ell)}$$
$$= \underbrace{\frac{n!}{(\ell+1)!} (\ell+1)}_{=\frac{n!}{\ell!}} d_{(1^\ell)} + \frac{n!}{(\ell+1)!} (-1)^{\ell+1} - \frac{n!}{\ell!} d_{(1^\ell)}$$
$$= \frac{n!}{\ell!} d_{(1^\ell)} + \frac{n!}{(\ell+1)!} (-1)^{\ell+1} - \frac{n!}{\ell!} d_{(1^\ell)} = \frac{n!}{(\ell+1)!} (-1)^{\ell+1} = (-1)^{\ell+1} \frac{n!}{(\ell+1)!},$$

we obtain

$$w_\ell - w_{\ell+1} = \frac{n!}{(\ell+1)!} d_{(1^{\ell+1})} - \frac{n!}{\ell!} d_{(1^\ell)}.$$

This is clearly equivalent to (3). Thus, (3) is proven. This completes the induction step.

- **page 54, Question 24:** I know it's a stupid remark, but you have never actually defined the notion of a "coplactic map". (It just means one of the maps E_k, F_k and S_k defined in §4.2.)
- **page 55, Question 25:** Question 25 (b) is precisely the claim of Lemma 5.6.
- **page 58, solution to Question 2:** Let me add that part (b) of the Question can also be easily solved without using part (a). One such solution appears in the solution to Exercise 2.2.9 in

  Darij Grinberg and Victor Reiner,

  Hopf Algebras in Combinatorics,

  version of 20 April 2020,

  `http://www.cip.ifi.lmu.de/~grinberg/algebra/HopfComb-sols.pdf`
  (also available at arXiv:1409.8356v6)

  (beware that the numbering on my website might have changed by the time you're reading this, but the numbering on arXiv:1409.8356v6 will never change).

  Incidentally, a generalization of your Question 2 appears in Propositions 1.1 and 1.2 of

  C. DeConcini, David Eisenbud, and C. Procesi,

Young Diagrams and Determinantal Varieties,
Inventiones math. 56 (1980), pp. 129–165.

- **page 64, solution to Question 21:** Maybe explain what “disjoint union” means (in “disjoint union of rim-hooks”).