Promotion and Rowmotion
Jessica Striker, Nathan Williams
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Errata and comments, 2 July 2013

- **Page 3, Theorem 2.8:** Replace "C_n" by "C_{nm}".
- **Page 6, Lemma 4.1:** I personally would write "$x \mapsto gx$" instead of your "$x \to gx$", as I belong to the set of mathematicians who make a distinction between the $\to$ and $\mapsto$ arrows.
- **Page 8, after the proof of Corollary 4.9:** In "$\varnothing = I_0 \subset I_1 \cdots \subset I_n = \mathcal{P}$", it would be better to add a "$\subset$" sign after the "I_1". (This appears at least twice in the text.)
- **Page 8, after the proof of Corollary 4.9:** Replace "in the singleton set $I_{i+1} - I_i$" by "in the singleton set $I_i - I_{i-1}$" (unless you have suddenly decided that linear extensions are bijections to $\{0, 1, \ldots, n-1\}$ rather than $\{1, 2, \ldots, n\}$).
- **Page 9, Definition 4.13:** Replace "R" by "$\mathcal{R}$".
- **Page 10, proof of Lemma 5.1:** Replace "$\prod\limits_{j=1}^{n} g_i$" by "$\prod\limits_{i=1}^{n} g_i$" on the first line and also on the last line of the proof.
- I'm unhappy with your notation $\prod\limits_{i=\alpha}^{\beta} f(i)$, which (I believe) you define to be something like $\begin{cases} f(\alpha) f(\alpha+1) \cdots f(\beta), & \text{if } \alpha \leq \beta; \\ f(\alpha) f(\alpha-1) \cdots f(\beta), & \text{if } \alpha \geq \beta \end{cases}$. I understand that this notation is an answer to the noncommutativity of the groups in which the products live, but I'd prefer $\overrightarrow{\prod\limits_{i=\alpha}^{\beta}}$ and $\overleftarrow{\prod\limits_{i=\alpha}^{\beta}}$ instead. The problem with your notation is that it conflicts with the classical convention that an empty product like $\prod\limits_{i=\alpha}^{\alpha-1} f(i)$ has to mean 1 (this is, for example, the meaning of the products in Definition 4.8 when $k = n = 0$, or the meaning of the product in Theorem 5.4 when $m = 1$).
- **Page 14:** On the first line of this page, I think both "J"'s in "$e_i + e_n \in J$ if and only if $e_i - e_n \in J$" should be "I"'s.
- **Page 16:** You write: "A word containing parentheses is called *balanced* if the number of left parentheses is always greater than or equal to the number of right parentheses." I'd wish the word "always" be more concretized (you probably mean "in every initial subword", not "in every subword").

- **Page 20, Definition 8.7:** To be honest I don't find this very readable. What exactly are the "outward-pointing edges"; what does "every second" mean (starting where?); are the paths directed (you say "begin" and "end" yet you don't draw arrows); and don't you want to say that the paths should not cross?
- **Page 21, §8.3:** In the first line of §8.3, the "2n" should be in mathmode (i. e., inside dollar signs).

Here is something more substantial, but probably not an error on your side: I don't understand your proof of Lemma 5.1. But I have an alternative one, which seems easier to formalize. Let me sketch it:

Alternative proof of Lemma 5.1: First, let me notice that the conditions on the g_i can be replaced by the following one:

$$g_i g_j = g_j g_i \text{ for every two } i \in \{1, 2, \ldots, n\} \text{ and } j \in \{1, 2, \ldots, n\} \text{ satisfying } |i - j| > 1. \tag{1}$$

Of course, (1) is a consequence of the conditions you give, but it is much more general and I wouldn't be surprised if Lemma 5.1 is also useful in cases where your conditions are not satisfied (Hecke algebras?). (That said, it is also not necessary to require the g_i to be generators...)

We will show that

$$\left(\begin{array}{c} \text{for any } k \in \{1, 2, \ldots, n\} \text{ and any } \omega \in \mathfrak{S}_k \text{ and } \nu \in \mathfrak{S}_k, \text{ there exists} \\ \text{an } \alpha \in \langle g_1, g_2, \ldots, g_{k-1} \rangle \text{ such that } \overrightarrow{\prod_{i=1}^{k}} g_{\omega(i)} = \alpha \cdot \left(\overrightarrow{\prod_{i=1}^{k}} g_{\nu(i)} \right) \cdot \alpha^{-1} \end{array} \right) \tag{2}$$

(where my $\overrightarrow{\prod\limits_{i=1}^{k}}$ means what you would call $\prod\limits_{i=1}^{k}$). Once this is shown, we will immediately obtain the claim of Lemma 5.1 just by applying this to $k = n$. Notice how (2) says $\alpha \in \langle g_1, g_2, \ldots, g_{k-1} \rangle$, not $\alpha \in \langle g_1, g_2, \ldots, g_k \rangle$; this is what makes (2) prone to induction over k.

Proof of (2): Of course, (2) is clear for $k = 1$, so we only need to do the induction step.

Let $\omega \in \mathfrak{S}_k$ and $\nu \in \mathfrak{S}_k$. It is easy to find an $\omega' \in \mathfrak{S}_{k-1}$ and $\gamma \in \langle g_1, g_2, \ldots, g_{k-1} \rangle$ such that

$$\overrightarrow{\prod_{i=1}^{k}} g_{\omega(i)} = \gamma g_k \cdot \left(\overrightarrow{\prod_{i=1}^{k-1}} g_{\omega'(i)} \right) \cdot \gamma^{-1}. \tag{3}$$

[1] Similarly, we can find a $\nu' \in \mathfrak{S}_{k-1}$ and $\delta \in \langle g_1, g_2, \ldots, g_{k-1} \rangle$ such that

$$\overrightarrow{\prod_{i=1}^{k}} g_{\nu(i)} = \delta g_k \cdot \left(\overrightarrow{\prod_{i=1}^{k-1}} g_{\nu'(i)} \right) \cdot \delta^{-1}. \tag{4}$$

But by the induction assumption, there exists a $\beta \in \langle g_1, g_2, \ldots, g_{k-2} \rangle$ such that

$$\overrightarrow{\prod_{i=1}^{k-1}} g_{\omega'(i)} = \beta \cdot \left(\overrightarrow{\prod_{i=1}^{k-1}} g_{\nu'(i)} \right) \cdot \beta^{-1}. \tag{5}$$

Now, since $\beta \in \langle g_1, g_2, \ldots, g_{k-2} \rangle$, we have $g_k \beta = \beta g_k$ (as an easy consequence of (1)). Thus, (3) becomes

$$\begin{aligned}
\overrightarrow{\prod_{i=1}^{k}} g_{\omega(i)} &= \gamma g_k \cdot \underbrace{\left(\overrightarrow{\prod_{i=1}^{k-1}} g_{\omega'(i)} \right)}_{\substack{= \beta \cdot \left(\overrightarrow{\prod_{i=1}^{k-1}} g_{\nu'(i)} \right) \cdot \beta^{-1} \\ \text{(by (5))}}} \cdot \gamma^{-1} \\
&= \gamma \underbrace{g_k \beta}_{= \beta g_k} \cdot \left(\overrightarrow{\prod_{i=1}^{k-1}} g_{\nu'(i)} \right) \cdot \beta^{-1} \gamma^{-1} = \gamma \underbrace{\beta}_{= \beta \delta^{-1} \delta} g_k \cdot \left(\overrightarrow{\prod_{i=1}^{k-1}} g_{\nu'(i)} \right) \cdot \underbrace{\beta^{-1}}_{= \delta^{-1} \delta \beta^{-1}} \gamma^{-1} \\
&= \gamma \beta \delta^{-1} \underbrace{\delta g_k \cdot \left(\overrightarrow{\prod_{i=1}^{k-1}} g_{\nu'(i)} \right) \cdot \delta^{-1}}_{\substack{= \overrightarrow{\prod_{i=1}^{k}} g_{\nu(i)} \\ \text{(by (4))}}} \delta \beta^{-1} \gamma^{-1} \\
&= \gamma \beta \delta^{-1} \cdot \left(\overrightarrow{\prod_{i=1}^{k}} g_{\nu(i)} \right) \cdot \underbrace{\delta \beta^{-1} \gamma^{-1}}_{= \left(\delta^{-1} \right)^{-1} \beta^{-1} \gamma^{-1} = \left(\gamma \beta \delta^{-1} \right)^{-1}} \\
&= \gamma \beta \delta^{-1} \cdot \left(\overrightarrow{\prod_{i=1}^{k}} g_{\nu(i)} \right) \cdot \left(\gamma \beta \delta^{-1} \right)^{-1}.
\end{aligned}$$

[1]Indeed, let $a = \omega^{-1}(k)$, set $\gamma = g_{\omega(1)} g_{\omega(2)} \cdots g_{\omega(a-1)}$, and let $\omega' \in \mathfrak{S}_{k-1}$ be defined by

$$(\omega'(1), \omega'(2), \ldots, \omega'(k-1)) = (\omega(a+1), \omega(a+2), \ldots, \omega(k), \omega(1), \omega(2), \ldots, \omega(a-1)).$$

Thus, there exists an $\alpha \in \langle g_1, g_2, \ldots, g_{k-1} \rangle$ such that $\overrightarrow{\prod_{i=1}^{k}} g_{\omega(i)} = \alpha \cdot \left(\overrightarrow{\prod_{i=1}^{k}} g_{\nu(i)} \right) \cdot \alpha^{-1}$ (namely, $\alpha = \gamma\beta\delta^{-1}$). This completes the induction step, and proves Lemma 5.1.