Lie algebras
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`http://www.math.harvard.edu/~shlomo/docs/lie_algebras.pdf`
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Errata and addenda by Darij Grinberg

Errata

The numbering of pages in the following errata matches the numbering of pages in the notes (not the numbering of pages in the PDF file).

- **page 8, very first formula:** Replace

$$A + B + \frac{1}{2}A^2 + AB + \frac{1}{2}B^2 - \frac{1}{2}\left(A + B + \cdots\right)^2$$

  by

$$A + B + \frac{1}{2}A^2 + AB + \frac{1}{2}B^2 - \frac{1}{2}\left(A + B\right)^2 + \cdots$$

- **page 8, the formula after (1.1):** The left hand side,

$$\frac{1}{12}\left(A^2B + AB^2 + B^2A + BA^2 - 2ABA - BAB]\right)$$

  has an] bracket too much.

- Something that is generally confusing me in this chapter (at least its geometric parts) is the unclarity about whether we are considering linear Lie groups or general Lie groups. When you write "One of the important consequences of the mere existence of this formula is the following" on page 8, you suddenly switch from GL_n to general Lie groups; it would be helpful if you would tell when you do so.

- **page 8, third paragraph:** "closed" should be "close" in "two elements of G sufficiently closed to the identity".

- **page 8, third paragraph:** There is a "the" too much in "the the product of $a = \exp\xi$ and $b = \exp\eta$".

- **page 9, remark 1:** In the last formula of remark 1, I would add a bracket to the left hand side $(\log(\exp A)(\exp B))$.

- **page 9, remark 2:** You write: "On the right hand side, exponentiation takes place in the algebra of endomorphisms of the ring in question.". You probably mean "of the Lie algebra in question".

- **page 11, middle of the page:** Why do we have

  $$\frac{d}{dt}\operatorname{Ad}(\exp tA)\,X = (\exp tA)\,AX\,(\exp -tA) + (\exp tA)\,X\,(-A)\,(\exp -tA)$$

  for a *general* (non-matrix) Lie algebra? I can't apply the product rule in this case as things are happening in different tangent spaces...

- **page 11, middle of the page:** In $\frac{d}{dt}\operatorname{Ad}\exp tA$ you could add a bracket: $\frac{d}{dt}\operatorname{Ad}(\exp tA)$. (But it's clear anyway).

- **page 11, middle of the page:** In $\exp t \operatorname{ad} A$ you could add a bracket, lest it looks like $(\exp t)(\operatorname{ad} A)$.

- **page 12, example Sp(n):** Here you write:

  $$dAJa^{\dagger} + AJdA^{\dagger} = 0.$$

  The a should be an A here.

- **page 12, example Gl(n,C):** Here you write:

  $$dAJ = JdA = 0.$$

  Do you really mean this, or should the first equality sign be a minus sign actually?

- **page 19, before the long formula:** "On then other hand" has an obvious typo.

- **page 20, §1.8.1, Uniqueness:** You write: "By the universal property $t = \ell'_t \circ t'$, $t' = \ell'_t \circ t$ so $t = (\ell'_t \circ \ell_{t'}) \circ t$, but also $t = t \circ \mathrm{id}$." Two errors here: first, $t' = \ell'_t \circ t$ should be $t' = \ell_{t'} \circ t$. Second, $t = t \circ \mathrm{id}$ should be $t = \mathrm{id} \circ t$.

- **page 20, §1.8.1, Existence:** You write: "Let M be the free vector space on the symbols $x_1, \ldots, x_m,\ x_i \in E_i$". I'd rather say "on the symbols $(x_1, \ldots, x_m)$ with $x_i \in E_i$", as otherwise it sounds like you are defining the direct sum of the E_i.

- **page 20, last formula on this page:** There is one comma too much in "$(x_1, \ldots, , ax_i, \ldots, x_m)$".

- **page 21, first line of §1.8.2:** Replace "they are they are" by "then they are".

- **page 21, §1.8.2:** You hardly mean "isomorphism" in "We have an isomorphism of A into $A \otimes B$".

- **page 21, first line of §1.8.3:** Replace "The **tensor algebra** of a vector space" by "The **tensor algebra** of V".
- **page 21, second line of §1.8.3:** "the universal problem for maps" should be "the universal problem for linear maps".
- **page 22, second line from above:** "of the last subsection" should be "of Subsection 1.8.1".
- **page 22, second line of §1.8.4:** "generated the elements" should be "generated by the elements". Also, you might want to add "for $x, y \in L$" after $[x, y] - x \otimes y + y \otimes x$, lest it looks like "for $x, y \in TL$" or something like that.
- **page 22, last line of §1.8.4:** You write: "must vanish on I if it is to be a Lie algebra homomorphism". The "it" here refers to α, not ψ; probably better just to write α instead.
- **page 23, §1.8.7:** In the second formula of §1.8.7, you write:

  $$xy \otimes 1 + x \otimes y + y \otimes x + +1 \otimes xy,$$

  There is a + sign too much in here.
- **page 24, §1.8.7:** You write: "is the identity (on 1 and on) L and hence is the identity". The closing bracket is misplaced here; it should be after the first "and".
- **page 24, §1.8.7:** Replace D by C in "a map $\varepsilon : D \to k$ (called a **co-unit**)".
- **page 24, §1.8.7:** I was very surprised to see that you are using the word "co-algebra" to mean "not necessarily co-associative co-algebra". You might want to point out in a footnote that you are using this nonstandard notation.
- **page 24, §1.8.7, middle of the page:** You write: "Hence the comultiplication is is coassociative." Redundant "is" in this sentence.
- **page 24, §1.9:** One comma too much in "For example,,".
- **page 26, middle of the page:** There is one closing bracket too much in "Recall that if $M = (i_1, \ldots, i_n))$ we set $\ell(M) = n$ and call it the length of M".
- **page 26, proof of Lemma 1:** You write: "We will inductively define a map". I think "a k-bilinear map" would be slightly better here, as you define it on products of bases.

- **page 27, first paragraph:** You write: "Notice that the first of these two cases is consistent with (and forced on us) by (1.20)". Again, brackets are misplaced: It should be "[...] (and forced on us by) (1.20)".
- **page 27, middle of the page:** You write: "Furthermore (*) holds with j and N replaced by M". Should be "replaced by i and M'".
- **page 27, last paragraph:** I don't see what is happening beginning with "So we know that (1.21) holds for $x = x_i, y = x_k$ and $v = z_{(jP)}$ (if $j \leq P$) or $v = z_Q$ (otherwise)." How do you know this? I think you need a stronger inductive assumption for this.
  Also, it seems to me that you don't just use that I is totally ordered, but also that I is well-ordered, as you do transfinite induction over i (assuming that something is already done for all $j < i$ and then doing it for i). I may be seriously off track here.
- **page 28, proof of Lemma 1:** There are missing] brackets in here (three of them; each one in $[x_j, [x_i, x_k]]$).
- **page 28, §1.10:** Maybe point out that you are in characteristic 0 (otherwise, the claim that the elements of L are the only primitives in $U(L)$ is invalid).
- **page 29, §1.11.1:** You write: "We have a "multiplication" map $M_X \times M_X$ given by the inclusion". The "$M_X \times M_X$" should be "$M_X \times M_X \to M_X$" here.
- **page 29, §1.11.1:** A reminder that algebras are not necessarily assumed associative wouldn't be out of place here.
- **page 30, §1.11.2, middle of the page:** "each of the a_n also belong to I" should be "[...] belongs to I".
- **page 31, §1.11.3, third paragraph:** In "thus give rise to a Lie algebra homomorphism", replace "give" by "gives".
- **page 31, §1.11.3, third paragraph:** "both compositions $\Phi \circ \Psi$ and $\Psi \circ \Phi$ are the identity on X" should be "Both [...]".
- **page 31:** There is a point too much in (1.24).
- **page 32, §1.12:** Why is $\mathbf{L}_X$ the set of all $\alpha \in m$ satisfying $\Delta\alpha = \alpha \otimes 1 + 1 \otimes \alpha$? (It's not hard, but you might mention it as an exercise - it is tacitly used in §1.21.1.)
- **page 33, fourth line from above (not counting the formulas):** Please add a whitespace after "general" in "and in generalL_X^n is spanned by elements of the form".

- **page 34, fifth line of formulas:** Add a $\sum_m$ sign in front of
$$\sum_{p_i+q_i\geq 1} \frac{(-1)^{m+1}}{m} \frac{x^{p_1}y^{q_1}x^{p_2}y^{q_2}\cdots x^{p_m}y^{q_m}}{p_1!q_1!\cdots p_m!q_m!}.$$

- **page 34, definition of $z'_{p,q}$:** There is a mistake in here (which comes directly from Serre, I think): The condition $p_m \geq 1$ is wrong; for example it leads to a wrong formula for $n = 1$.

- I would really like to see a final paragraph about the relation between the algebraic BCH and the geometric BCH. The formulas both compute $\log((\exp A)(\exp B))$, but the A and B are different, the meanings of log and exp are different, and the right hand sides are different, too. This is probably obvious for a geometer, but how can one derive the geometric BCH from the algebraic one?