A Solomon Mackey formula for graded bialgebras

Darij Grinberg

October 16, 2024

Abstract. Given a graded bialgebra H, we let $\Delta^{[k]} : H \to H^{\otimes k}$ and $m^{[k]} : H^{\otimes k} \to H$ be its iterated (co)multiplications for all $k \in \mathbb{N}$. For any k-tuple $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k) \in \mathbb{N}^k$ of nonnegative integers, and any permutation σ of $\{1, 2, \ldots, k\}$, we consider the map $p_{\alpha,\sigma} := m^{[k]} \circ P_\alpha \circ \sigma^{-1} \circ \Delta^{[k]} : H \to H$, where P_α denotes the projection of $H^{\otimes k}$ onto its multigraded component $H_{\alpha_1} \otimes H_{\alpha_2} \otimes \cdots \otimes H_{\alpha_k}$, and where $\sigma^{-1} : H \to H$ permutes the tensor factors.

We prove formulas for the composition $p_{\alpha,\sigma} \circ p_{\beta,\tau}$ and the convolution $p_{\alpha,\sigma} \star p_{\beta,\tau}$ of two such maps. When H is cocommutative, these generalize Patras's 1994 results (which, in turn, generalize Solomon's Mackey formula).

We also construct a combinatorial Hopf algebra PNSym ("permuted noncommutative symmetric functions") that governs the maps $p_{\alpha,\sigma}$ for arbitrary connected graded bialgebras H in the same way as the well-known NSym governs them in the cocommutative case. We end by outlining an application to checking identities for connected graded Hopf algebras.

Mathematics Subject Classifications: 16T30, 05E05.

Keywords: Hopf algebra, bialgebra, Adams operations, graded connected Hopf algebra, combinatorial Hopf algebra, noncommutative symmetric functions, stalactic equivalence.

This is a preliminary report on a project. Its goal is to classify the identities that hold between the natural (**k**-linear) operations on the category of graded **k**-bialgebras. The following approach is likely improvable (being entirely focused on unary operations $H \to H$, despite the multivalued operations $H^{\otimes k} \to H^{\otimes \ell}$ being probably a more natural object of study) and rather inchoate (the proofs lacking in elegance and readability), but the main result (Theorem 1.19) appears worthy of dissemination and – to my great surprise – new. Even more surprisingly, a combinatorial Hopf algebra (named PNSym for "permuted noncommutative symmetric functions") emerges from this study which, too, seems to have hitherto escaped the eyes of the algebraic combinatorics community. Thus I hope that this report will be of some use in the time before the right proofs are found and written up (hopefully without requiring a revision of the respective statements).

I would like to thank the organizers of the CATMI 2023 workshop in Bergen (Gunnar Fløystad, Håkon Gylterud, Hans Zanna Munthe-Kaas, Uwe Wolter) for the occasion to present these ideas, and Marcelo Aguiar, Amy Pang and Sarah Brauner for stimulating conversations. Two referees have furthermore given useful and constructive feedback that has improved the paper.

0.1. Introduction

On any bialgebra H, we can define the *Adams operations* (also known as *characteristic operations* or *dilations*) $\operatorname{id}^{\star k} = m^{[k]} \circ \Delta^{[k]}$ (where $\Delta^{[k]} : H \to H^{\otimes k}$ is an iterated comultiplication, and $m^{[k]} : H^{\otimes k} \to H$ is an iterated multiplication[1]). These operations are natural in H (that is, equivariant with respect to bialgebra morphisms) and have been studied for years (see, e.g., [Kashin19] and [AguLau14] for some recent work), as have been several similar operators (e.g., [Patras94], [Loday98, §4.5], [PatReu98]) on Hopf algebras and graded bialgebras. More generally, for any bialgebra H, we can define the "twisted Adams operations" $m^{[k]} \circ \sigma^{-1} \circ \Delta^{[k]}$ for any permutation $\sigma \in \mathfrak{S}_k$ (where σ^{-1} acts on $H^{\otimes k}$ by permuting the tensorands)[2]. When H is a graded bialgebra, one can further refine these operations by injecting a projection map P_α between the $m^{[k]}$ and the σ. To be specific: For any k-tuple $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k) \in \mathbb{N}^k$, we let $P_\alpha : H^{\otimes k} \to H^{\otimes k}$ be the projection on the α-th multigraded component (i.e., the tensor product of the projections $H \to H_{\alpha_1}$, $H \to H_{\alpha_2}$, …, $H \to H_{\alpha_k}$).

The resulting "twisted projecting Adams operations" $p_{\alpha,\sigma} := m^{[k]} \circ P_\alpha \circ \sigma^{-1} \circ \Delta^{[k]}$ (for $k \in \mathbb{N}$ and $\sigma \in \mathfrak{S}_k$ and $\alpha \in \mathbb{N}^k$) have an interesting history. When H is cocommutative, they have been studied under the name of "opérateurs de descente" by Patras in [Patras94] (see [CarPat21, §5.1] for a more recent exposition) and used by Reutenauer [Reuten93, §9.1] to prove Solomon's Mackey formula for the symmetric group. The dual case – when H is commutative – is essentially equivalent. In both of these cases, the permutation σ is immaterial, since it can be swallowed either by the $\Delta^{[k]}$ (when H is cocommutative) or by the $m^{[k]}$ (when H is commutative). Thus, in these cases, the operators depend only on the k-tuple α. Moreover, when the graded bialgebra H is connected, the k-tuple α can be compressed by removing all 0's from it, thus becoming a composition (a tuple of

[1]These are commonly known as $\Delta^{(k-1)}$ and $m^{(k-1)}$, but we prefer the superscripts to match the number of tensorands.

[2]The notation $\mathfrak{S}_k$ means the k-th symmetric group. Our use of σ^{-1} instead of σ is meant to simplify the formula for the action.

positive integers). One of Patras's key results ([Patras94, Théorème II,7], [Reuten93, Theorem 9.2], [CarPat21, Theorem 5.1.1]) is a formula for the composition $p_{\alpha,\mathrm{id}} \circ p_{\beta,\mathrm{id}}$ of two such operators: When H is cocommutative and $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k) \in \mathbb{N}^k$ and $\beta = (\beta_1, \beta_2, \ldots, \beta_\ell) \in \mathbb{N}^\ell$, it claims that

$$p_{\alpha,\mathrm{id}} \circ p_{\beta,\mathrm{id}} = \sum_{\substack{\gamma_{i,j} \in \mathbb{N} \text{ for all } i \in [k] \text{ and } j \in [\ell]; \\ \gamma_{i,1} + \gamma_{i,2} + \cdots + \gamma_{i,\ell} = \alpha_i \text{ for all } i \in [k]; \\ \gamma_{1,j} + \gamma_{2,j} + \cdots + \gamma_{k,j} = \beta_j \text{ for all } j \in [\ell]}} p_{(\gamma_{1,1}, \gamma_{1,2}, \ldots, \gamma_{k,\ell}),\mathrm{id}} \tag{1}$$

(where $[n] := \{1, 2, \ldots, n\}$ for each $n \in \mathbb{N}$, and where $(\gamma_{1,1}, \gamma_{1,2}, \ldots, \gamma_{k,\ell})$ denotes the list of all $k\ell$ numbers $\gamma_{i,j}$ listed in lexicographic order of their subscripts)[3]. When H is furthermore connected (i.e., when $H_0 \cong \mathbf{k}$), we can restrict ourselves to compositions by removing all 0's from our tuples. In that situation, the formula (1) is structurally identical to the expansion of the internal product of two complete homogeneous noncommutative symmetric functions in NSym (see [GKLLRT94, Proposition 5.1]), which reveals that the operators $p_{\alpha,\mathrm{id}}$ are images of the latter functions under an algebra morphism from NSym (under the internal product) to $\operatorname{End} H$. Thus, noncommutative symmetric functions govern many natural operations for cocommutative bialgebras. A dual theory exists for commutative H.

To my knowledge, no analogue of the formula (1) has been proposed for general H. This general case is somewhat complicated by the fact that the operators of the form $p_{\alpha,\mathrm{id}}$ are no longer closed under composition; but the ones of the more general form $p_{\alpha,\sigma}$ still are. The main result of this paper (Theorem 1.19) is a formula that generalizes (1) to this case. The rest of this paper is devoted to studying the operators $p_{\alpha,\sigma}$ further, particularly under connectedness assumptions, and to analyzing the formal structure behind their composition and convolution rules. This structure is encoded in a new combinatorial Hopf algebra I call PNSym ("permuted noncommutative symmetric functions"), whose standard basis is indexed by what I have christened "mopiscotions" (permuted compositions). Some properties of this Hopf algebra and its internal multiplication are outlined, and several questions are left open for further research. A connection to the work of Aguiar and Mahajan – in particular, their Janus monoid – is discussed. Finally, the results are applied to obtain an algorithm for verifying functorial identities for connected graded bialgebras.

0.2. Plan of this paper

In Section 1, I introduce the operators $p_{\alpha,\sigma}$ for an arbitrary graded $\mathbf{k}$-bialgebra H, and prove the main result (Theorem 1.19), which is a formula that expands a composition $p_{\alpha,\sigma} \circ p_{\beta,\tau}$ of two such operators as a linear combination of other such operators. This formula generalizes (1). In this section, I also discuss how the operators $p_{\alpha,\sigma}$ behave under convolution (Proposition 1.15), tensoring (Proposition 1.39)

[3] To be precise, all the above-cited sources ([Patras94, Théorème II,7], [Reuten93, Theorem 9.2], [CarPat21, Theorem 5.1.1]) are restricting themselves to the case when H is connected.

and bialgebra duality (Proposition 1.40), and show that they can – for an appropriate choice of H – be linearly independent (Theorems 1.36 and 1.37). I suspect that the operators $p_{\alpha,\sigma}$ and their infinite $\mathbf{k}$-linear combinations span all possible natural $\mathbf{k}$-linear endomorphisms on the category of connected graded bialgebras (i.e., all $\mathbf{k}$-linear maps $H \to H$ that are defined for every connected graded bialgebra H and are functorial with respect to graded bialgebra morphisms); while I have been unable to prove this, this conjecture is easily verified for all known natural endomorphisms I have found in the literature.

The remaining two sections are much terser and should be viewed as a preliminary report on a field in flux. Only the slightest indications of proofs are given, and several open questions are posed.

In Section 2, I define a combinatorial Hopf algebra PNSym (the "permuted noncommutative symmetric functions") that governs the "twisted projecting descent operations" $p_{\alpha,\sigma}$ on an arbitrary connected graded bialgebra H just like NSym does for the cocommutative case. This Hopf algebra appears to be new, and I attempt to uncover some of its structure. In particular, its internal multiplication has much in common with that of NSym (which it lifts: there is a projection $\mathfrak{p}$: PNSym $\to$ NSym that respects all structures). Several questions are left unanswered here.

In the final Section 3, I briefly discuss an application of the above results: Namely, they can be used to mechanically verify any identity between operators of the form $p_{\alpha,\sigma}$ (and their sums, convolutions and compositions) that hold for arbitrary (connected) graded bialgebras.

0.3. Notations

Let $\mathbb{N} = \{0, 1, 2, \ldots\}$.

We fix a commutative ring $\mathbf{k}$, which shall serve as our base ring throughout this paper. In particular, all modules, algebras, coalgebras, bialgebras and Hopf algebras will be over $\mathbf{k}$. Tensor products and hom spaces are defined over $\mathbf{k}$ as well. Algebras are understood to be unital and associative unless said otherwise.

We will use standard concepts – such as iterated (co)multiplications, tensor products, (co)commutativity, etc. – related to bialgebras (and occasionally Hopf algebras). The reader can find them explained, e.g., in [GriRei20, Chapter 1].

If H is any bialgebra, then the multiplication, the unit, the comultiplication, and the counit of H (regarded as linear maps) will be denoted by

$$m_H : H \otimes H \to H, \qquad u_H : \mathbf{k} \to H,$$
$$\Delta_H : H \to H \otimes H, \qquad \epsilon_H : H \to \mathbf{k},$$

respectively. If no ambiguity is to be feared, then we will abbreviate them as m, u, Δ and ϵ. We furthermore denote the unity of a ring R (viewed as an element of R) by 1_R.

Graded $\mathbf{k}$-modules are always understood to be $\mathbb{N}$-graded, i.e., to have direct sum decompositions $V = \bigoplus_{n \in \mathbb{N}} V_n$. The n-th graded component of a graded $\mathbf{k}$-

module V will be called V_n. If $n < 0$, then this is the zero submodule 0. Tensor products of graded **k**-modules are equipped with the usual grading:

$$\left(A_{(1)} \otimes A_{(2)} \otimes \cdots \otimes A_{(k)}\right)_n = \sum_{\substack{(i_1,i_2,\ldots,i_k)\in\mathbb{N}^k;\\ i_1+i_2+\cdots+i_k=n}} \left(A_{(1)}\right)_{i_1} \otimes \left(A_{(2)}\right)_{i_2} \otimes \cdots \otimes \left(A_{(k)}\right)_{i_k}.$$

A **k**-linear map $f : U \to V$ between two graded **k**-modules U and V is said to be *graded* if every $n \in \mathbb{N}$ satisfies $f(U_n) \subseteq V_n$.

Graded bialgebras are bialgebras whose operations (m, u, Δ and ϵ) are graded **k**-linear maps. We do **not** twist our tensor products by the grading (i.e., we do **not** follow the topologists' sign conventions); in particular, the tensor product $A \otimes B$ of two algebras has its multiplication defined by $(a \otimes b)(a' \otimes b') = aa' \otimes bb'$, regardless of any possible gradings on A and B.

We recall that a non-graded bialgebra can be viewed as a graded bialgebra concentrated in degree 0 (that is, a graded bialgebra H with $H = H_0$). Thus, all claims about graded bialgebras that we make below can be specialized to non-graded bialgebras.

A graded **k**-bialgebra H is said to be *connected* if $H_0 = \mathbf{k} \cdot 1_H$. A known fact (e.g., [GriRei20, Proposition 1.4.16]) says that any connected graded bialgebra is automatically a Hopf algebra, i.e., has an antipode. Hopf algebras are not at the center of our present work, but will occasionally appear in applications.

If H is any **k**-module, then $\operatorname{End} H$ shall denote the **k**-module of **k**-linear maps from H to H. This **k**-module $\operatorname{End} H$ becomes an algebra under composition of maps. As usual, we denote this composition operation by $\circ$ (so that $f \circ g$ means the composition of two maps f and g, sending each $x \in H$ to $f(g(x))$). The neutral element of this composition is the identity map $\operatorname{id}_H \in \operatorname{End} H$.

If H is a bialgebra, then the **k**-module $\operatorname{End} H$ has yet another canonical multiplication, known as *convolution* and denoted by $\star$; it is defined by[4]

$$f \star g := m_H \circ (f \otimes g) \circ \Delta_H \qquad \text{for all } f, g \in \operatorname{End} H.$$

This operation $\star$ is **k**-bilinear and associative and has the neutral element $u_H \circ \epsilon_H$; thus, $\operatorname{End} H$ becomes an algebra under this operation. See [GriRei20, Definition 1.4.1] for more about convolution.

Thus, when H is a bialgebra, $\operatorname{End} H$ becomes a **k**-algebra in two natural ways: once using the composition $\circ$, and once using the convolution $\star$.

1. The maps $p_{\alpha,\sigma}$ for a graded bialgebra H

1.1. Definitions

Let H be a graded **k**-bialgebra. We fix it for the rest of Section 1.

[4]Actually, convolution is defined in the same way for **k**-linear maps from any given coalgebra to any given algebra.

The set $\operatorname{End}_{\mathrm{gr}} H$ of all graded $\mathbf{k}$-module endomorphisms of H is a $\mathbf{k}$-submodule of $\operatorname{End} H$, and is preserved under both composition and convolution (and thus is a $\mathbf{k}$-subalgebra of both the composition algebra $\operatorname{End} H$ and the convolution algebra $\operatorname{End} H$). Furthermore, we can consider the $\mathbf{k}$-submodule $\mathbf{E}(H)$ of $\operatorname{End}_{\mathrm{gr}} H$ that consists only of those $f \in \operatorname{End}_{\mathrm{gr}} H$ that annihilate all but finitely many graded components of H (that is, that satisfy $f(H_n) = 0$ for all sufficiently high n). This submodule $\mathbf{E}(H)$ is itself graded, with the n-th graded component being canonically isomorphic to $\operatorname{End}(H_n)$. This submodule $\mathbf{E}(H)$, too, is preserved under both composition and convolution (but usually does not contain id_H, whence it is not a subalgebra of the composition algebra $\operatorname{End} H$).

This module $\mathbf{E}(H)$ has been studied, e.g., by Hazewinkel [Hazewi04]. Unlike him, we shall not consider $\mathbf{E}(H)$ for any specific H, but we shall instead focus on the "generic" $\mathbf{E}(H)$. In other words, we will consider the functorial endomorphisms of H that are defined for all graded bialgebras H and always belong to $\mathbf{E}(H)$. Such endomorphisms include

- the projections $p_0, p_1, p_2, \ldots$ from H onto the graded components $H_0, H_1, H_2, \ldots$ (regarded as endomorphisms of H);
- their convolutions $p_{(i_1,i_2,\ldots,i_k)} := p_{i_1} \star p_{i_2} \star \cdots \star p_{i_k}$ with $i_1, i_2, \ldots, i_k \in \mathbb{N}$;
- the (nonempty) compositions $p_\alpha \circ p_\beta \circ \cdots \circ p_\kappa$ of such convolutions.

However, we can define a broader class of such endomorphisms. To do so, we need some notations:

Definition 1.1. For each $n \in \mathbb{N}$, we let $p_n : H \to H$ denote the canonical projection of the graded module H onto its n-th graded component H_n.

Definition 1.2. For each $k \in \mathbb{N}$, we set $[k] := \{1, 2, \ldots, k\}$.

Definition 1.3. For each $k \in \mathbb{N}$, we let $\mathfrak{S}_k$ denote the k-th symmetric group, i.e., the group of all permutations of $[k]$.

Definition 1.4. For each $k \in \mathbb{N}$, we let the group $\mathfrak{S}_k$ act on $H^{\otimes k}$ from the left by permuting tensorands, according to the rule

$$\sigma \cdot (h_1 \otimes h_2 \otimes \cdots \otimes h_k) = h_{\sigma^{-1}(1)} \otimes h_{\sigma^{-1}(2)} \otimes \cdots \otimes h_{\sigma^{-1}(k)} \qquad \text{for all } \sigma \in \mathfrak{S}_k \text{ and } h_1, h_2, \ldots, h_k \in H.$$

This is an action by bialgebra endomorphisms.

Definition 1.5. For any $k \in \mathbb{N}$, we let $m^{[k]} : H^{\otimes k} \to H$ and $\Delta^{[k]} : H \to H^{\otimes k}$ be the iterated multiplication and iterated comultiplication maps of the bialgebra

H (denoted $m^{(k-1)}$ and $\Delta^{(k-1)}$ in [GriRei20, Exercises 1.4.19 and 1.4.20], and denoted $\Pi^{[k]}$ and $\Delta^{[k]}$ in [Patras94]). Note that $m^{[k]}$ sends each pure tensor $a_1 \otimes a_2 \otimes \cdots \otimes a_k \in H^{\otimes k}$ to the product $a_1 a_2 \cdots a_k$, whereas $\Delta^{[k]}$ sends each element $x \in H$ to $\sum_{(x)} x_{(1)} \otimes x_{(2)} \otimes \cdots \otimes x_{(k)}$ (using Sweedler notation).

We recall that these iterated multiplications and comultiplications satisfy the rules

$$m^{[u+v]} = m \circ \left(m^{[u]} \otimes m^{[v]}\right) \tag{2}$$

and

$$\Delta^{[u+v]} = \left(\Delta^{[u]} \otimes \Delta^{[v]}\right) \circ \Delta \tag{3}$$

for all $u, v \in \mathbb{N}$. (Indeed, these are the claims of [GriRei20, Exercise 1.4.19(a)] and [GriRei20, Exercise 1.4.20(a)], respectively, applied to $i = u - 1$ and $k = u + v - 1$.) Also, $m^{[1]} = \Delta^{[1]} = \mathrm{id}_H$ and $m^{[0]} = u : \mathbf{k} \to H$ and $\Delta^{[0]} = \epsilon : H \to \mathbf{k}$.

Definition 1.6. **(a)** A *weak composition* means a finite tuple of nonnegative integers.

(b) If $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$ is a weak composition, then its *size* $|\alpha|$ is defined to be the number $\alpha_1 + \alpha_2 + \cdots + \alpha_k \in \mathbb{N}$.

(c) A *composition* means a finite tuple of positive integers.

Definition 1.7. For any weak composition $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$, we define the projection map $P_\alpha : H^{\otimes k} \to H^{\otimes k}$ to be the tensor product $p_{\alpha_1} \otimes p_{\alpha_2} \otimes \cdots \otimes p_{\alpha_k}$ of the $\mathbf{k}$-linear maps $p_{\alpha_1}, p_{\alpha_2}, \ldots, p_{\alpha_k}$. (Thus, if we regard $H^{\otimes k}$ as an $\mathbb{N}^k$-graded $\mathbf{k}$-module, then P_α is its projection onto its degree-$(\alpha_1, \alpha_2, \ldots, \alpha_k)$ component.)

Definition 1.8. For any weak composition $\alpha \in \mathbb{N}^k$ and any permutation $\sigma \in \mathfrak{S}_k$, we define a map $p_{\alpha,\sigma} : H \to H$ by the formula

$$p_{\alpha,\sigma} := m^{[k]} \circ P_\alpha \circ \sigma^{-1} \circ \Delta^{[k]}, \tag{4}$$

where the "σ^{-1}" really means the action of $\sigma^{-1} \in \mathfrak{S}_k$ on $H^{\otimes k}$ (as in Definition 1.4).

We can rewrite the definition of $p_{\alpha,\sigma}$ in more concrete terms using the Sweedler notation:

Remark 1.9. For any weak composition $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k) \in \mathbb{N}^k$ and any permutation $\sigma \in \mathfrak{S}_k$, the map $p_{\alpha,\sigma} : H \to H$ is given by

$$p_{\alpha,\sigma}(x) = \sum_{(x)} p_{\alpha_1}\left(x_{(\sigma(1))}\right) p_{\alpha_2}\left(x_{(\sigma(2))}\right) \cdots p_{\alpha_k}\left(x_{(\sigma(k))}\right)$$

for every $x \in H$, where we are using the Sweedler notation $\sum_{(x)} x_{(1)} \otimes x_{(2)} \otimes \cdots \otimes x_{(k)}$ for the iterated coproduct $\Delta^{[k]}(x) \in H^{\otimes k}$.

The following is near-obvious:

Proposition 1.10. Let $\alpha \in \mathbb{N}^k$ be a weak composition, and $\sigma \in \mathfrak{S}_k$ a permutation. Then, the map $p_{\alpha,\sigma}$ is a graded $\mathbf{k}$-module endomorphism of H that sends $H_{|\alpha|}$ to $H_{|\alpha|}$ and sends all other graded components H_n to 0. Thus, $p_{\alpha,\sigma}$ lies in the $|\alpha|$-th graded component of $\mathbf{E}(H)$.

Proof. The gradedness of $p_{\alpha,\sigma}$ follows from the fact that all four maps $m^{[k]}$, P_α, σ^{-1} and $\Delta^{[k]}$ in (4) are graded. Thus, the map $p_{\alpha,\sigma}$ sends $H_{|\alpha|}$ to $H_{|\alpha|}$. To see that it sends all other H_n to 0, we only need to observe that for every $n \neq |\alpha|$, we have $\left(\sigma^{-1} \circ \Delta^{[k]}\right)(H_n) \subseteq \left(H^{\otimes k}\right)_n$ (since σ^{-1} and $\Delta^{[k]}$ are graded), and that P_α annihilates $\left(H^{\otimes k}\right)_n$ (since $n \neq |\alpha|$). □

1.2. Particular cases

Let us contrast our maps $p_{\alpha,\sigma}$ to Patras's descent operators p_α (denoted B_α in [Patras94, §II]). We recall how the latter are defined (generalizing slightly from compositions to weak compositions):

Definition 1.11. For any weak composition $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k) \in \mathbb{N}^k$, we define a map $p_\alpha : H \to H$ by the formula

$$p_\alpha := p_{\alpha_1} \star p_{\alpha_2} \star \cdots \star p_{\alpha_k}.$$

(Recall that $\star$ denotes convolution.)

Our $p_{\alpha,\sigma}$ recover these for $\sigma = \mathrm{id}$:

Proposition 1.12. Let $\alpha \in \mathbb{N}^k$ be a weak composition. Then, $p_{\alpha,\mathrm{id}} = p_\alpha$ (where id is the identity permutation in $\mathfrak{S}_k$).

Proof. Write α as $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$. Then, (4) yields

$$p_{\alpha,\mathrm{id}} = m^{[k]} \circ P_\alpha \circ \underbrace{\mathrm{id}^{-1}}_{=\mathrm{id}} \circ \Delta^{[k]} = m^{[k]} \circ P_\alpha \circ \Delta^{[k]}. \tag{5}$$

On the other hand, [GriRei20, Exercise 1.4.23] yields[5]

$$p_{\alpha_1} \star p_{\alpha_2} \star \cdots \star p_{\alpha_k} = m^{[k]} \circ \underbrace{\left(p_{\alpha_1} \otimes p_{\alpha_2} \otimes \cdots \otimes p_{\alpha_k}\right)}_{\substack{=P_\alpha \\ \text{(by the definition of } P_\alpha)}} \circ \Delta^{[k]} = m^{[k]} \circ P_\alpha \circ \Delta^{[k]}.$$

[5]Recall that the maps $m^{[k]}$ and $\Delta^{[k]}$ are called $m^{(k-1)}$ and $\Delta^{(k-1)}$ in [GriRei20].

Now, the definition of p_α yields

$$p_\alpha = p_{\alpha_1} \star p_{\alpha_2} \star \cdots \star p_{\alpha_k} = m^{[k]} \circ P_\alpha \circ \Delta^{[k]}. \tag{6}$$

Comparing this with (5), we obtain $p_{\alpha,\mathrm{id}} = p_\alpha$. This proves Proposition 1.12. □

If the bialgebra H is commutative or cocommutative, then we can bring all our $p_{\alpha,\sigma}$ to the form p_β for some weak composition β:

Proposition 1.13. Let $\alpha \in \mathbb{N}^k$ be a weak composition, and $\sigma \in \mathfrak{S}_k$ a permutation.

(a) If H is commutative, then

$$p_{\alpha,\sigma} = p_{\sigma\cdot\alpha},$$

where we are using the left action of $\mathfrak{S}_k$ on $\mathbb{N}^k$ defined by

$$\sigma \cdot (\alpha_1, \alpha_2, \ldots, \alpha_k) := \left(\alpha_{\sigma^{-1}(1)}, \alpha_{\sigma^{-1}(2)}, \ldots, \alpha_{\sigma^{-1}(k)}\right).$$

(b) If H is cocommutative, then

$$p_{\alpha,\sigma} = p_\alpha.$$

To prove this, we need a simple lemma:

Lemma 1.14. Let $f_1, f_2, \ldots, f_k$ be k arbitrary elements of $\operatorname{End} H$. Let $\sigma \in \mathfrak{S}_k$. Then,

$$\sigma \circ (f_1 \otimes f_2 \otimes \cdots \otimes f_k) = \left(f_{\sigma^{-1}(1)} \otimes f_{\sigma^{-1}(2)} \otimes \cdots \otimes f_{\sigma^{-1}(k)}\right) \circ \sigma$$

(where σ and $f_1 \otimes f_2 \otimes \cdots \otimes f_k$ and $f_{\sigma^{-1}(1)} \otimes f_{\sigma^{-1}(2)} \otimes \cdots \otimes f_{\sigma^{-1}(k)}$ are understood as endomorphisms of $H^{\otimes k}$).

Proof of Lemma 1.14. Both sides send any given pure tensor $h_1 \otimes h_2 \otimes \cdots \otimes h_k \in H^{\otimes k}$ to

$$f_{\sigma^{-1}(1)}\left(h_{\sigma^{-1}(1)}\right) \otimes f_{\sigma^{-1}(2)}\left(h_{\sigma^{-1}(2)}\right) \otimes \cdots \otimes f_{\sigma^{-1}(k)}\left(h_{\sigma^{-1}(k)}\right).$$

Thus, they agree on all pure tensors, and hence are identical. □

Proof of Proposition 1.13. Write α as $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$.

(a) Assume that H is commutative. Then, $m^{[k]} \circ \tau = m^{[k]}$ for any $\tau \in \mathfrak{S}_k$ (by [GriRei20, Exercise 1.5.10]). Thus, in particular, $m^{[k]} \circ \sigma^{-1} = m^{[k]}$. However, it is straightforward to see (and actually true for any $\mathbf{k}$-module in place of H) that

$$\sigma \circ P_\alpha = P_{\sigma\cdot\alpha} \circ \sigma$$

as maps from $H^{\otimes k}$ to $H^{\otimes k}$. (Indeed, this follows from Lemma 1.14 (applied to $f_i = p_{\alpha_i}$), since $P_\alpha = p_{\alpha_1} \otimes p_{\alpha_2} \otimes \cdots \otimes p_{\alpha_k}$ and $P_{\sigma \cdot \alpha} = p_{\alpha_{\sigma^{-1}(1)}} \otimes p_{\alpha_{\sigma^{-1}(2)}} \otimes \cdots \otimes p_{\alpha_{\sigma^{-1}(k)}}$.)

Now, the definition of $p_{\alpha,\sigma}$ yields

$$p_{\alpha,\sigma} = m^{[k]} \circ \underbrace{P_\alpha}_{\substack{=\sigma^{-1} \circ P_{\sigma \cdot \alpha} \circ \sigma \\ (\text{since } \sigma \circ P_\alpha = P_{\sigma \cdot \alpha} \circ \sigma)}} \circ \sigma^{-1} \circ \Delta^{[k]}$$
$$= \underbrace{m^{[k]} \circ \sigma^{-1}}_{=m^{[k]}} \circ P_{\sigma \cdot \alpha} \circ \underbrace{\sigma \circ \sigma^{-1}}_{=\mathrm{id}} \circ \Delta^{[k]} = m^{[k]} \circ P_{\sigma \cdot \alpha} \circ \Delta^{[k]}.$$

Comparing this with

$$p_{\sigma \cdot \alpha} = m^{[k]} \circ P_{\sigma \cdot \alpha} \circ \Delta^{[k]} \qquad \text{(by (6), applied to } \sigma \cdot \alpha \text{ instead of } \alpha),$$

we obtain $p_{\alpha,\sigma} = p_{\sigma \cdot \alpha}$. Proposition 1.13 **(a)** is thus proved.

(b) Assume that H is cocommutative. Then, $\tau \circ \Delta^{[k]} = \Delta^{[k]}$ for any $\tau \in \mathfrak{S}_k$ (by [GriRei20, Exercise 1.5.10]). Thus, in particular, $\sigma^{-1} \circ \Delta^{[k]} = \Delta^{[k]}$. Now, the definition of $p_{\alpha,\sigma}$ yields

$$p_{\alpha,\sigma} = m^{[k]} \circ P_\alpha \circ \underbrace{\sigma^{-1} \circ \Delta^{[k]}}_{=\Delta^{[k]}} = m^{[k]} \circ P_\alpha \circ \Delta^{[k]}.$$

Comparing this with (6), we find $p_{\alpha,\sigma} = p_\alpha$. This proves Proposition 1.13 **(b)**. □

However, in general, when H is neither commutative nor cocommutative, we cannot "simplify" $p_{\alpha,\sigma}$. (See Theorem 1.36 for a concretization of this claim.)

If the graded bialgebra H is connected, then each $p_{\alpha,\sigma}$ for a weak composition α and a permutation σ can be rewritten in the form $p_{\beta,\tau}$ for a composition (not just weak composition) β and a permutation τ. (Indeed, since H is connected, we can remove all $p_0\left(x_{(i)}\right)$ factors from the product in Remark 1.9.) See Proposition 2.5 below for an explicit statement of this claim.

1.3. The convolution formula

It is easy to see that any convolution of two maps of the form $p_{\alpha,\sigma}$ is again a map of such form:

Proposition 1.15. Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$ be a weak composition, and let $\sigma \in \mathfrak{S}_k$ be a permutation.

Let $\beta = (\beta_1, \beta_2, \ldots, \beta_\ell)$ be a weak composition, and let $\tau \in \mathfrak{S}_\ell$ be a permutation.

Let $\alpha\beta$ be the concatenation of α and β; this is the weak composition $(\alpha_1, \alpha_2, \ldots, \alpha_k, \beta_1, \beta_2, \ldots, \beta_\ell)$.

Let $\sigma \oplus \tau$ be the image of (σ, τ) under the obvious map $\mathfrak{S}_k \times \mathfrak{S}_\ell \to \mathfrak{S}_{k+\ell}$. Explicitly, this is the permutation of $[k+\ell]$ that sends each $i \leq k$ to $\sigma(i)$ and sends each $j > k$ to $k + \tau(j-k)$.

Then,

$$p_{\alpha,\sigma} \star p_{\beta,\tau} = p_{\alpha\beta,\sigma\oplus\tau}. \tag{7}$$

Proof. Let us first observe that

$$\underbrace{\sigma^{-1}}_{\substack{\text{this means}\\ \text{the action}\\ \text{of } \sigma^{-1} \text{ on } H^{\otimes k}}} \otimes \underbrace{\tau^{-1}}_{\substack{\text{this means}\\ \text{the action}\\ \text{of } \tau^{-1} \text{ on } H^{\otimes \ell}}} = \underbrace{(\sigma \oplus \tau)^{-1}}_{\substack{\text{this means}\\ \text{the action}\\ \text{of } (\sigma\oplus\tau)^{-1} \text{ on } H^{\otimes(k+\ell)}}} .$$

(This can be shown, e.g., by acting on a pure tensor: Any pure tensor

$$h_1 \otimes h_2 \otimes \cdots \otimes h_k \otimes g_1 \otimes g_2 \otimes \cdots \otimes g_\ell \in H^{\otimes(k+\ell)}$$

is sent by both $\sigma^{-1} \otimes \tau^{-1}$ and $(\sigma \oplus \tau)^{-1}$ to the same image

$$h_{\sigma(1)} \otimes h_{\sigma(2)} \otimes \cdots \otimes h_{\sigma(k)} \otimes g_{\tau(1)} \otimes g_{\tau(2)} \otimes \cdots \otimes g_{\tau(\ell)}.$$

Thus, the two $\mathbf{k}$-linear maps $\sigma^{-1} \otimes \tau^{-1}$ and $(\sigma \oplus \tau)^{-1}$ agree on each pure tensor, and thus are identical.)

Furthermore, we have $P_\alpha \otimes P_\beta = P_{\alpha\beta}$ (as maps from $H^{\otimes(k+\ell)}$ to $H^{\otimes(k+\ell)}$). (This follows easily from the definitions of P_α, P_β and $P_{\alpha\beta}$.)

The definition of convolution yields $p_{\alpha,\sigma} \star p_{\beta,\tau} = m \circ (p_{\alpha,\sigma} \otimes p_{\beta,\tau}) \circ \Delta$. In view of

$$\begin{aligned}
&\underbrace{p_{\alpha,\sigma}}_{\substack{= m^{[k]} \circ P_\alpha \circ \sigma^{-1} \circ \Delta^{[k]} \\ \text{(by (4))}}} \otimes \underbrace{p_{\beta,\tau}}_{\substack{= m^{[\ell]} \circ P_\beta \circ \tau^{-1} \circ \Delta^{[\ell]} \\ \text{(by (4))}}} \\
&= \left(m^{[k]} \circ P_\alpha \circ \sigma^{-1} \circ \Delta^{[k]}\right) \otimes \left(m^{[\ell]} \circ P_\beta \circ \tau^{-1} \circ \Delta^{[\ell]}\right) \\
&= \left(m^{[k]} \otimes m^{[\ell]}\right) \circ \underbrace{(P_\alpha \otimes P_\beta)}_{= P_{\alpha\beta}} \circ \underbrace{\left(\sigma^{-1} \otimes \tau^{-1}\right)}_{= (\sigma\oplus\tau)^{-1}} \circ \left(\Delta^{[k]} \otimes \Delta^{[\ell]}\right) \\
&= \left(m^{[k]} \otimes m^{[\ell]}\right) \circ P_{\alpha\beta} \circ (\sigma \oplus \tau)^{-1} \circ \left(\Delta^{[k]} \otimes \Delta^{[\ell]}\right),
\end{aligned}$$

we can rewrite this as

$$\begin{aligned}
p_{\alpha,\sigma} \star p_{\beta,\tau} &= \underbrace{m \circ \left(m^{[k]} \otimes m^{[\ell]}\right)}_{\substack{= m^{[k+\ell]} \\ \text{(by (2))}}} \circ P_{\alpha\beta} \circ (\sigma \oplus \tau)^{-1} \circ \underbrace{\left(\Delta^{[k]} \otimes \Delta^{[\ell]}\right) \circ \Delta}_{\substack{= \Delta^{[k+\ell]} \\ \text{(by (3))}}} \\
&= m^{[k+\ell]} \circ P_{\alpha\beta} \circ (\sigma \oplus \tau)^{-1} \circ \Delta^{[k+\ell]} = p_{\alpha\beta,\sigma\oplus\tau}
\end{aligned}$$

(since $p_{\alpha\beta,\sigma\oplus\tau}$ is defined as $m^{[k+\ell]} \circ P_{\alpha\beta} \circ (\sigma \oplus \tau)^{-1} \circ \Delta^{[k+\ell]}$). This proves Proposition 1.15. □

1.4. The composition formula: statement

It is more interesting to compute the composition of two maps of the form $p_{\alpha,\sigma}$. It turns out that this is again a $\mathbf{k}$-linear combination of maps of such form, and the explicit formula is similar to Solomon's Mackey formula for the descent algebra (or, even more closely related, [Reuten93, Theorem 9.2 and §9.5.1]). Before we state the formula, let us introduce one more operation on permutations:

Definition 1.16. Let $\sigma \in \mathfrak{S}_k$ and $\tau \in \mathfrak{S}_\ell$ be two permutations. Then, $\tau[\sigma] \in \mathfrak{S}_{k\ell}$ shall denote the permutation of $[k\ell]$ that sends each $\ell(i-1)+j$ (with $i \in [k]$ and $j \in [\ell]$) to $k(\tau(j)-1)+\sigma(i)$. This permutation is well-defined because of Remark 1.17 below.

Remark 1.17. Let $\sigma \in \mathfrak{S}_k$ and $\tau \in \mathfrak{S}_\ell$ be two permutations. Why is the permutation $\tau[\sigma] \in \mathfrak{S}_{k\ell}$ in Definition 1.16 well-defined? To see that it is well-defined as a map, it suffices to observe that each element $p \in [k\ell]$ can be written uniquely in the form $\ell(i-1)+j$ with $i \in [k]$ and $j \in [\ell]$. (This observation follows from the fact that the $k+1$ numbers $0\ell, 1\ell, 2\ell, \ldots, k\ell$ subdivide the set $[k\ell]$ into k intervals of length ℓ each.)

But it remains to show that the map $\tau[\sigma]$ is indeed a permutation in $\mathfrak{S}_{k\ell}$. The best way to show this is by factoring it as a composition of three bijections.

Namely, define two maps λ and ρ from $[k] \times [\ell]$ to $[k\ell]$ by setting

$$\lambda(i,j) := k(j-1)+i \in [k\ell] \qquad \text{and} \qquad \rho(i,j) := \ell(i-1)+j \in [k\ell]$$

for all $(i,j) \in [k] \times [\ell]$. It is easy to see that both of these maps λ and ρ are well-defined. Moreover, these maps λ and ρ are bijections. In fact, ρ sends each pair $(i,j) \in [k] \times [\ell]$ to the position of (i,j) in the lexicographically ordered Cartesian product $[k] \times [\ell]$, whereas λ sends each pair $(i,j) \in [k] \times [\ell]$ to the position of (j,i) in the lexicographically ordered Cartesian product $[\ell] \times [k]$. Thus, ρ is the order-isomorphism between the lexicographically ordered Cartesian product $[k] \times [\ell]$ and the chain $[k\ell]$, whereas λ is the order-isomorphism between the lexicographically ordered Cartesian product $[\ell] \times [k]$ and the chain $[k\ell]$ composed with the standard swap map $[k] \times [\ell] \to [\ell] \times [k]$, $(i,j) \mapsto (j,i)$.

Next, we consider the map

$$\begin{aligned} \sigma \times \tau : [k] \times [\ell] &\to [k] \times [\ell], \\ (i,j) &\mapsto (\sigma(i), \tau(j)). \end{aligned}$$

This is the Cartesian product of the two maps σ and τ, and thus is a bijection (since σ and τ are bijections). Clearly, the composition $\lambda \circ (\sigma \times \tau) \circ \rho^{-1}$ is a well-defined bijection from $[k\ell]$ to $[k\ell]$ (since it is a composition of bijections and their inverses), i.e., a permutation in $\mathfrak{S}_{k\ell}$.

Now, we claim that the permutation $\tau[\sigma]$ in Definition 1.16 is simply $\lambda \circ (\sigma \times \tau) \circ \rho^{-1}$. Indeed, for each $i \in [k]$ and $j \in [\ell]$, we have

$$\begin{aligned}\left(\lambda \circ (\sigma \times \tau) \circ \rho^{-1}\right) \underbrace{(\ell(i-1)+j)}_{\substack{=\rho(i,j) \\ \text{(by the definition of } \rho)}} &= \left(\lambda \circ (\sigma \times \tau) \circ \rho^{-1}\right) (\rho(i,j)) \\ &= \lambda \left(\underbrace{(\sigma \times \tau)(i,j)}_{=(\sigma(i),\tau(j))} \right) = \lambda(\sigma(i), \tau(j)) \\ &= k(\tau(j)-1) + \sigma(i)\end{aligned}$$

(by the definition of λ). Thus, the permutation $\lambda \circ (\sigma \times \tau) \circ \rho^{-1} \in \mathfrak{S}_{k\ell}$ sends each $\ell(i-1)+j$ (with $i \in [k]$ and $j \in [\ell]$) to $k(\tau(j)-1)+\sigma(i)$. In other words, $\lambda \circ (\sigma \times \tau) \circ \rho^{-1}$ satisfies the condition imposed on $\tau[\sigma]$ in Definition 1.16. Thus, $\tau[\sigma] = \lambda \circ (\sigma \times \tau) \circ \rho^{-1} \in \mathfrak{S}_{k\ell}$. In other words, $\tau[\sigma]$ is really a permutation in $\mathfrak{S}_{k\ell}$. This completes our proof.

Example 1.18. Let $s_1 \in \mathfrak{S}_2$ be the transposition that swaps 1 with 2. Let $\mathrm{id}_3 \in \mathfrak{S}_3$ be the identity permutation. Then, $s_1[\mathrm{id}_3] \in \mathfrak{S}_6$ is the permutation that sends $1, 2, 3, 4, 5, 6$ to $4, 1, 5, 2, 6, 3$, respectively.

It can be shown that the operation that sends two permutations $\tau \in \mathfrak{S}_\ell$ and $\sigma \in \mathfrak{S}_k$ to $\tau[\sigma]$ is associative (see Claim 1 in the Second proof idea for Theorem 2.7 below for an outline of a proof), but we will not use this fact in the present section. Furthermore, the identity permutation $\mathrm{id}_{[1]} \in \mathfrak{S}_1$ is neutral for this operation (i.e., satisfies $\mathrm{id}_{[1]}[\sigma] = \sigma\left[\mathrm{id}_{[1]}\right] = \sigma$ for all $\sigma \in \mathfrak{S}_k$). However, if two positive integers k and ℓ are both larger than 1, then $\mathrm{id}_{[k]}\left[\mathrm{id}_{[\ell]}\right] \neq \mathrm{id}_{[k\ell]}$. Another description of $\tau[\sigma]$ can be found in Lemma 1.27 below.

We can now state the explicit formula for composition of $p_{\alpha,\sigma}$'s:

Theorem 1.19. Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$ be a weak composition, and let $\sigma \in \mathfrak{S}_k$ be a permutation.

Let $\beta = (\beta_1, \beta_2, \ldots, \beta_\ell)$ be a weak composition, and let $\tau \in \mathfrak{S}_\ell$ be a permutation.

Then,

$$p_{\alpha,\sigma} \circ p_{\beta,\tau} = \sum_{\substack{\gamma_{i,j} \in \mathbb{N} \text{ for all } i \in [k] \text{ and } j \in [\ell]; \\ \gamma_{i,1}+\gamma_{i,2}+\cdots+\gamma_{i,\ell}=\alpha_i \text{ for all } i \in [k]; \\ \gamma_{1,j}+\gamma_{2,j}+\cdots+\gamma_{k,j}=\beta_j \text{ for all } j \in [\ell]}} p_{(\gamma_{1,1},\gamma_{1,2},\ldots,\gamma_{k,\ell}),\tau[\sigma]}.$$

Here, $(\gamma_{1,1}, \gamma_{1,2}, \ldots, \gamma_{k,\ell})$ denotes the $k\ell$-tuple consisting of all $k\ell$ numbers $\gamma_{i,j}$ (for all $i \in [k]$ and $j \in [\ell]$) listed in the order of lexicographically increasing pairs (i, j) (i.e., the number $\gamma_{i,j}$ comes before $\gamma_{u,v}$ if and only if either $i < u$ or $(i = u$ and $j < v)$).

We note that the sum in Theorem 1.19 can be viewed as a sum over all $k \times \ell$-matrices $\left(\gamma_{i,j}\right)_{i\in[k],\ j\in[\ell]} \in \mathbb{N}^{k\times\ell}$ with row sums α (that is, the sum of all entries in the i-th row of the matrix equals α_i) and column sums β. Such matrices are known as *contingency tables* with marginal distributions α and β (see, e.g., [LyuPak20, §1]), and have found ample uses in combinatorics (e.g., [JamKer81, Corollary 1.3.11] or [GriRei20, Definition 4.3.4]).

Example 1.20. Let $s_1 \in \mathfrak{S}_2$ be the transposition that swaps 1 with 2. Then, the permutation $s_1 [s_1] \in \mathfrak{S}_4$ sends $1, 2, 3, 4$ to $4, 2, 3, 1$.

Let (a, b) and (c, d) be two weak compositions in $\mathbb{N}^2$. Then, Theorem 1.19 says that

$$p_{(a,b),s_1} \circ p_{(c,d),s_1} = \sum_{\substack{\gamma_{1,1},\gamma_{1,2},\gamma_{2,1},\gamma_{2,2}\in\mathbb{N};\\ \gamma_{1,1}+\gamma_{1,2}=a;\ \gamma_{2,1}+\gamma_{2,2}=b;\\ \gamma_{1,1}+\gamma_{2,1}=c;\ \gamma_{1,2}+\gamma_{2,2}=d}} p_{(\gamma_{1,1},\gamma_{1,2},\gamma_{2,1},\gamma_{2,2}),s_1[s_1]}.$$

This is a sum over all 2×2-matrices $\left(\gamma_{i,j}\right)_{i\in[2],\ j\in[2]} \in \mathbb{N}^{2\times 2}$ with row sums (a, b) and column sums (c, d). How this sum looks like depends on whether $a + b$ equals $c + d$ or not:

- If $a + b \neq c + d$, then there are no such matrices, and therefore the sum is empty. Thus,
$$p_{(a,b),s_1} \circ p_{(c,d),s_1} = 0 \qquad \text{in this case.}$$
This is not surprising, since the image of the map $p_{(c,d),s_1}$ is contained in the graded component H_{c+d} of H, whereas the map $p_{(a,b),s_1}$ is 0 on this component.

- If $a + b = c + d$, then these matrices are precisely the matrices of the form $\begin{pmatrix} i & a-i \\ c-i & i-g \end{pmatrix}$, where $g = c - b = a - d$ and where $i \in \mathbb{Z}$ satisfies $\max\{0, g\} \leq i \leq \min\{a, c\}$. Thus, our above formula becomes
$$p_{(a,b),s_1} \circ p_{(c,d),s_1} = \sum_{i=\max\{0,g\}}^{\min\{a,c\}} p_{(i,\ a-i,\ c-i,\ i-g),\ s_1[s_1]} \qquad \text{where } g = c - b = a - d.$$
For example, for $a = b = c = d = 1$, this simplifies to
$$p_{(1,1),s_1} \circ p_{(1,1),s_1} = p_{(0,1,1,0),s_1[s_1]} + p_{(1,0,0,1),s_1[s_1]}.$$

This all is not hard to check by hand.

Remark 1.21. A bialgebra without a grading can always be interpreted as a graded bialgebra H concentrated in degree 0 (that is, satisfying $H_0 = H$ and $H_k = 0$ for all $k > 0$). Thus, Theorem 1.19 can be applied to bialgebras without a grading. If H is such a bialgebra, than $P_{\alpha,\sigma} = 0$ whenever the weak composition α has any nonzero entry. Thus, the claim of Theorem 1.19 can be greatly simplified in this case. The resulting claim is

$$p_{\mathbf{0},\sigma} \circ p_{\mathbf{0},\tau} = p_{\mathbf{0},\tau[\sigma]},$$

where $\mathbf{0}$ means the tuple $(0, 0, \ldots, 0)$ of appropriate length. This is precisely the equality $\Psi^{(n,\sigma)} \circ \Psi^{(m,\tau)} = \Psi^{(nm,\Phi(\sigma,\tau))}$ in Pirashvili's [Pirash02, §1] and the Proposition in [KaSoZh06, §1.3] by Kashina, Sommerhäuser and Zhu. (Note that the $\Phi(\sigma, \tau)$ in [Pirash02, Proposition 5.3] is exactly our $\sigma[\tau]$. I do not know what the apparent reversal in the roles of σ and τ is due to, but I have verified my version of the result.

1.5. The composition formula: lemmas on tensors

As preparation for the proof of Theorem 1.19, we shall first work out how the different kinds of operators composed in (4) (iterated multiplications, comultiplications, projections, permutations) can be "commuted" past each other. We begin with the simplest operators: projections and permutations. These commutation formulas have nothing to do with bialgebras, and would equally make sense for any graded module instead of H.

Recall that $\mathbb{N}^k$ (for any given $k \in \mathbb{N}$) is the set of all weak compositions of length k. The symmetric group $\mathfrak{S}_k$ acts from the right on this set $\mathbb{N}^k$ by permuting the entries: For any $(\gamma_1, \gamma_2, \ldots, \gamma_k) \in \mathbb{N}^k$ and $\pi \in \mathfrak{S}_k$, we have

$$(\gamma_1, \gamma_2, \ldots, \gamma_k) \cdot \pi = \left(\gamma_{\pi(1)}, \gamma_{\pi(2)}, \ldots, \gamma_{\pi(k)}\right). \tag{8}$$

This action has the following property:

Lemma 1.22. For any $\pi \in \mathfrak{S}_k$ and $\gamma \in \mathbb{N}^k$, we have

$$P_\gamma \circ \pi = \pi \circ P_{\gamma \cdot \pi} \tag{9}$$

and

$$\pi \circ P_\gamma = P_{\gamma \cdot \pi^{-1}} \circ \pi. \tag{10}$$

Proof. Let $\pi \in \mathfrak{S}_k$ and $\gamma \in \mathbb{N}^k$. Write γ as $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_k)$. Then, the map

$P_\gamma \circ \pi : H^{\otimes k} \to H^{\otimes k}$ sends each pure tensor $h_1 \otimes h_2 \otimes \cdots \otimes h_k$ to

$$\begin{aligned} & P_\gamma \left(\pi \left(h_1 \otimes h_2 \otimes \cdots \otimes h_k\right)\right) \\ & = P_\gamma \left(h_{\pi^{-1}(1)} \otimes h_{\pi^{-1}(2)} \otimes \cdots \otimes h_{\pi^{-1}(k)}\right) \\ & = p_{\gamma_1}\left(h_{\pi^{-1}(1)}\right) \otimes p_{\gamma_2}\left(h_{\pi^{-1}(2)}\right) \otimes \cdots \otimes p_{\gamma_k}\left(h_{\pi^{-1}(k)}\right), \end{aligned}$$

whereas the map $\pi \circ P_{\gamma \cdot \pi} : H^{\otimes k} \to H^{\otimes k}$ sends it to

$$\begin{aligned} & \pi \left(P_{\gamma\cdot\pi} \left(h_1 \otimes h_2 \otimes \cdots \otimes h_k\right)\right) \\ & = \pi \left(p_{\gamma_{\pi(1)}}\left(h_1\right) \otimes p_{\gamma_{\pi(2)}}\left(h_2\right) \otimes \cdots \otimes p_{\gamma_{\pi(k)}}\left(h_k\right)\right) \\ & \qquad \left(\begin{array}{c} \text{since } \gamma = \left(\gamma_1, \gamma_2, \ldots, \gamma_k\right) \\ \text{entails } \gamma \cdot \pi = \left(\gamma_{\pi(1)}, \gamma_{\pi(2)}, \ldots, \gamma_{\pi(k)}\right) \end{array}\right) \\ & = p_{\gamma_{\pi(\pi^{-1}(1))}}\left(h_{\pi^{-1}(1)}\right) \otimes p_{\gamma_{\pi(\pi^{-1}(2))}}\left(h_{\pi^{-1}(2)}\right) \otimes \cdots \otimes p_{\gamma_{\pi(\pi^{-1}(k))}}\left(h_{\pi^{-1}(k)}\right) \\ & = p_{\gamma_1}\left(h_{\pi^{-1}(1)}\right) \otimes p_{\gamma_2}\left(h_{\pi^{-1}(2)}\right) \otimes \cdots \otimes p_{\gamma_k}\left(h_{\pi^{-1}(k)}\right), \end{aligned}$$

which is the same result. Thus, the linear maps $P_\gamma \circ \pi$ and $\pi \circ P_{\gamma\cdot\pi}$ agree on each pure tensor, and therefore are identical. This proves (9).

Applying (9) to $\gamma \cdot \pi^{-1}$ instead of γ, we obtain

$$P_{\gamma\cdot\pi^{-1}} \circ \pi = \pi \circ \underbrace{P_{\gamma\cdot\pi^{-1}\cdot\pi}}_{=P_\gamma} = \pi \circ P_\gamma.$$

This proves (10). □

Moreover, the following holds:

Lemma 1.23. Let $\alpha, \beta \in \mathbb{N}^k$ be two weak compositions. Then,

$$P_\alpha \circ P_\beta = \begin{cases} P_\alpha, & \text{if } \alpha = \beta; \\ 0, & \text{if } \alpha \neq \beta. \end{cases}$$

Proof. This follows easily from the fact that the projections $p_k : H \to H$ satisfy $p_a \circ p_a = p_a$ and $p_a \circ p_b = 0$ for all $a \neq b$. □

The remainder of this section is devoted to lemmas whose purpose is to factor the permutation $\tau[\sigma]$ (which appears in Theorem 1.19) into simpler permutations (Lemma 1.27) and to establish how these simpler permutations interact with tensors and linear maps. We begin by introducing one of these simpler permutations, which we call the *Zolotarev shuffle*:

Lemma 1.24. Let $k, \ell \in \mathbb{N}$. Let $\zeta \in \mathfrak{S}_{k\ell}$ be the permutation of $[k\ell]$ that sends each $k(j-1)+i$ (with $i \in [k]$ and $j \in [\ell]$) to $\ell(i-1)+j$. This is called the *Zolotarev shuffle* (and appears, e.g., as $\nu^{-1}\mu$ in [Rousse94]).[6]

Let $(h_{i,j})_{i\in[\ell],\ j\in[k]} \in H^{\ell\times k}$ be any $\ell \times k$-matrix over H. Then,

$$\begin{aligned}
&\zeta(h_{1,1} \otimes h_{1,2} \otimes \cdots \otimes h_{1,k} \\
&\qquad \otimes h_{2,1} \otimes h_{2,2} \otimes \cdots \otimes h_{2,k} \\
&\qquad \otimes \cdots \\
&\qquad \otimes h_{\ell,1} \otimes h_{\ell,2} \otimes \cdots \otimes h_{\ell,k}) \\
&= h_{1,1} \otimes h_{2,1} \otimes \cdots \otimes h_{\ell,1} \\
&\qquad \otimes h_{1,2} \otimes h_{2,2} \otimes \cdots \otimes h_{\ell,2} \\
&\qquad \otimes \cdots \\
&\qquad \otimes h_{1,k} \otimes h_{2,k} \otimes \cdots \otimes h_{\ell,k}.
\end{aligned} \tag{11}$$

(Here, the $h_{i,j}$ on the left hand side appear in the order of lexicographically increasing pairs (i,j), whereas the $h_{i,j}$ on the right hand side appear in the order of lexicographically increasing pairs (j,i).)

We note that the Zolotarev shuffle ζ in Lemma 1.24 can also be described as $\left(\mathrm{id}_{[\ell]}\left[\mathrm{id}_{[k]}\right]\right)^{-1}$ using the notation $\tau[\sigma]$ from Definition 1.16. This follows easily from Lemma 1.27 proved further below.

Proof of Lemma 1.24. Define two maps λ and ρ from $[k]\times[\ell]$ to $[k\ell]$ by setting

$$\lambda(i,j) := k(j-1)+i \in [k\ell] \qquad \text{and} \qquad \rho(i,j) := \ell(i-1)+j \in [k\ell]$$

for all $(i,j) \in [k]\times[\ell]$. These maps λ and ρ are bijections. In fact, ρ sends each pair (i,j) to the position of (i,j) in the lexicographically ordered Cartesian product $[k]\times[\ell]$, whereas λ sends each pair (i,j) to the position of (j,i) in the lexicographically ordered Cartesian product $[\ell]\times[k]$. (We already observed this all in Remark 1.17.)

Recall that all $i \in [k]$ and $j \in [\ell]$ satisfy $\zeta(k(j-1)+i) = \ell(i-1)+j$ (by the definition of ζ). In other words, all $i \in [k]$ and $j \in [\ell]$ satisfy $\zeta(\lambda(i,j)) = \rho(i,j)$ (since $\lambda(i,j) = k(j-1)+i$ and $\rho(i,j) = \ell(i-1)+j$). In other words,

$$\zeta \circ \lambda = \rho. \tag{12}$$

[6] To see that this ζ is well-defined, we can argue as follows:

- Each element of $[k\ell]$ can be uniquely written as $k(j-1)+i$ with $i \in [k]$ and $j \in [\ell]$. Moreover, $\ell(i-1)+j \in [k\ell]$ for any such i and j. Thus, ζ is a well-defined map from $[k\ell]$ to $[k\ell]$.
- Each element of $[k\ell]$ can be uniquely written as $\ell(i-1)+j$ with $i \in [k]$ and $j \in [\ell]$. Thus, ζ is bijective, i.e., a permutation of $[k\ell]$.

Set $g_{(j,i)} := h_{i,j}$ for all $(j,i) \in [k] \times [\ell]$. Recall that the bijection λ sends each pair (i,j) to the position of (j,i) in the lexicographically ordered Cartesian product $[\ell] \times [k]$. Hence, the list $\left(\lambda^{-1}(1), \lambda^{-1}(2), \ldots, \lambda^{-1}(k\ell)\right)$ consists of all the $k\ell$ pairs $(i,j) \in [k] \times [\ell]$ in the order of (lexicographically) increasing pairs (j,i). In other words,

$$\begin{aligned}\left(\lambda^{-1}(1), \lambda^{-1}(2), \ldots, \lambda^{-1}(k\ell)\right) = (&(1,1), (2,1), \ldots, (k,1), \\ &(1,2), (2,2), \ldots, (k,2), \\ &\ldots, \\ &(1,\ell), (2,\ell), \ldots, (k,\ell)).\end{aligned}$$

Thus,

$$\begin{aligned}g_{\lambda^{-1}(1)} \otimes g_{\lambda^{-1}(2)} \otimes \cdots \otimes g_{\lambda^{-1}(k\ell)} &= g_{(1,1)} \otimes g_{(2,1)} \otimes \cdots \otimes g_{(k,1)} \\ &\quad \otimes g_{(1,2)} \otimes g_{(2,2)} \otimes \cdots \otimes g_{(k,2)} \\ &\quad \otimes \cdots \\ &\quad \otimes g_{(1,\ell)} \otimes g_{(2,\ell)} \otimes \cdots \otimes g_{(k,\ell)} \\ &= h_{1,1} \otimes h_{1,2} \otimes \cdots \otimes h_{1,k} \\ &\quad \otimes h_{2,1} \otimes h_{2,2} \otimes \cdots \otimes h_{2,k} \\ &\quad \otimes \cdots \\ &\quad \otimes h_{\ell,1} \otimes h_{\ell,2} \otimes \cdots \otimes h_{\ell,k}\end{aligned}$$

(since $g_{(j,i)} = h_{i,j}$ for all j and i). By a similar argument, we obtain

$$\begin{aligned}g_{\rho^{-1}(1)} \otimes g_{\rho^{-1}(2)} \otimes \cdots \otimes g_{\rho^{-1}(k\ell)} &= h_{1,1} \otimes h_{2,1} \otimes \cdots \otimes h_{\ell,1} \\ &\quad \otimes h_{1,2} \otimes h_{2,2} \otimes \cdots \otimes h_{\ell,2} \\ &\quad \otimes \cdots \\ &\quad \otimes h_{1,k} \otimes h_{2,k} \otimes \cdots \otimes h_{\ell,k}.\end{aligned}$$

In view of these two equalities, we must show that

$$\zeta\left(g_{\lambda^{-1}(1)} \otimes g_{\lambda^{-1}(2)} \otimes \cdots \otimes g_{\lambda^{-1}(k\ell)}\right) = g_{\rho^{-1}(1)} \otimes g_{\rho^{-1}(2)} \otimes \cdots \otimes g_{\rho^{-1}(k\ell)}.$$

Since

$$\zeta\left(g_{\lambda^{-1}(1)} \otimes g_{\lambda^{-1}(2)} \otimes \cdots \otimes g_{\lambda^{-1}(k\ell)}\right) = g_{\lambda^{-1}\left(\zeta^{-1}(1)\right)} \otimes g_{\lambda^{-1}\left(\zeta^{-1}(2)\right)} \otimes \cdots \otimes g_{\lambda^{-1}\left(\zeta^{-1}(k\ell)\right)},$$

this is equivalent to showing that

$$g_{\lambda^{-1}\left(\zeta^{-1}(1)\right)} \otimes g_{\lambda^{-1}\left(\zeta^{-1}(2)\right)} \otimes \cdots \otimes g_{\lambda^{-1}\left(\zeta^{-1}(k\ell)\right)} = g_{\rho^{-1}(1)} \otimes g_{\rho^{-1}(2)} \otimes \cdots \otimes g_{\rho^{-1}(k\ell)}.$$

Thus, we need to check that $\lambda^{-1}\left(\zeta^{-1}(q)\right) = \rho^{-1}(q)$ for each $q \in [k\ell]$. In other words, we need to check that $\lambda^{-1} \circ \zeta^{-1} = \rho^{-1}$. But this follows from (12), since $\lambda^{-1} \circ \zeta^{-1} = (\zeta \circ \lambda)^{-1} = \rho^{-1}$ (by (12)). Hence, Lemma 1.24 is proven. □

Lemma 1.25. Let $k, \ell \in \mathbb{N}$. Let $\sigma \in \mathfrak{S}_k$. Let $\sigma^{\times \ell}$ denote the permutation in $\mathfrak{S}_{k\ell}$ that sends each $\ell(i-1)+j$ (with $i \in [k]$ and $j \in [\ell]$) to $\ell(\sigma(i)-1)+j$.

Let $f : H^{\otimes \ell} \to H$ be any $\mathbf{k}$-linear map. Then,

$$\sigma \circ f^{\otimes k} = f^{\otimes k} \circ \sigma^{\times \ell}$$

(as maps from $H^{\otimes k\ell}$ to $H^{\otimes k}$).

Proof. We begin with an even more basic claim:

Claim 1: Let $a_1, a_2, \ldots, a_k \in H^{\otimes \ell}$. Then,

$$\sigma^{\times \ell}(a_1 \otimes a_2 \otimes \cdots \otimes a_k) = a_{\sigma^{-1}(1)} \otimes a_{\sigma^{-1}(2)} \otimes \cdots \otimes a_{\sigma^{-1}(k)}$$

in $H^{\otimes k\ell}$. (Here, we are identifying $(H^{\otimes \ell})^{\otimes k}$ with $H^{\otimes k\ell}$ in the obvious way.)

Proof of Claim 1. Let $\omega = \sigma^{\times \ell}$.

The equality we need to prove depends linearly on each of $a_1, a_2, \ldots, a_k$. Hence, we can WLOG assume that each of $a_1, a_2, \ldots, a_k$ is a pure tensor. Assume this. Thus,

$$a_1 = g_1 \otimes g_2 \otimes \cdots \otimes g_\ell, \tag{13}$$

$$a_2 = g_{\ell+1} \otimes g_{\ell+2} \otimes \cdots \otimes g_{2\ell}, \tag{14}$$

$$\ldots,$$

$$a_k = g_{(k-1)\ell+1} \otimes g_{(k-1)\ell+2} \otimes \cdots \otimes g_{k\ell} \tag{15}$$

for some $g_1, g_2, \ldots, g_{k\ell} \in H$. Consider these $g_1, g_2, \ldots, g_{k\ell}$. Thus,

$$a_1 \otimes a_2 \otimes \cdots \otimes a_k = g_1 \otimes g_2 \otimes \cdots \otimes g_{k\ell}$$

(as we can see by tensoring together the equalities (13), (14), ..., (15)). Applying the permutation $\sigma^{\times \ell} \in \mathfrak{S}_{k\ell}$ to both sides of this equality, we obtain

$$\begin{aligned} &\sigma^{\times \ell}(a_1 \otimes a_2 \otimes \cdots \otimes a_k) \\ &= \underbrace{\sigma^{\times \ell}}_{=\omega}(g_1 \otimes g_2 \otimes \cdots \otimes g_{k\ell}) \\ &= \omega(g_1 \otimes g_2 \otimes \cdots \otimes g_{k\ell}) \\ &= g_{\omega^{-1}(1)} \otimes g_{\omega^{-1}(2)} \otimes \cdots \otimes g_{\omega^{-1}(k\ell)}. \end{aligned} \tag{16}$$

We shall now prove that

$$\begin{aligned} &a_{\sigma^{-1}(1)} \otimes a_{\sigma^{-1}(2)} \otimes \cdots \otimes a_{\sigma^{-1}(k)} \\ &= g_{\omega^{-1}(1)} \otimes g_{\omega^{-1}(2)} \otimes \cdots \otimes g_{\omega^{-1}(k\ell)}. \end{aligned} \tag{17}$$

Indeed, in order to prove this identity, it clearly suffices to show that

$$a_{\sigma^{-1}(i)} = g_{\omega^{-1}(\ell(i-1)+1)} \otimes g_{\omega^{-1}(\ell(i-1)+2)} \otimes \cdots \otimes g_{\omega^{-1}(\ell i)} \tag{18}$$

for each $i \in [k]$ (because then, tensoring the equalities (18) together for all $i \in [k]$ will yield (17)). But this is easy: Let $i \in [k]$. Then, the definition of $a_{\sigma^{-1}(i)}$ yields

$$a_{\sigma^{-1}(i)} = g_{\ell\left(\sigma^{-1}(i)-1\right)+1} \otimes g_{\ell\left(\sigma^{-1}(i)-1\right)+2} \otimes \cdots \otimes g_{\ell\left(\sigma^{-1}(i)\right)}. \tag{19}$$

But each $j \in [\ell]$ satisfies

$$\ell\left(\sigma^{-1}(i)-1\right)+j = \omega^{-1}\left(\ell\left(i-1\right)+j\right)$$

(since $\omega = \sigma^{\times \ell}$ was defined to send $\ell\left(\sigma^{-1}(i)-1\right)+j$ to $\ell\left(\underbrace{\sigma\left(\sigma^{-1}(i)\right)}_{=i}-1\right)+j = \ell\left(i-1\right)+j$) and thus $g_{\ell\left(\sigma^{-1}(i)-1\right)+j} = g_{\omega^{-1}(\ell(i-1)+j)}$. Hence, the right hand side of (19) equals the right hand side of (18). Therefore, (18) follows from (19) (since these two equalities have the same left hand side).

Forget that we fixed i. We thus have proved (18) for each $i \in [k]$. As explained, this proves (17).

Comparing (16) with (17), we find

$$\sigma^{\times \ell}\left(a_1 \otimes a_2 \otimes \cdots \otimes a_k\right) = a_{\sigma^{-1}(1)} \otimes a_{\sigma^{-1}(2)} \otimes \cdots \otimes a_{\sigma^{-1}(k)}.$$

This proves Claim 1. □

The rest is easy: Let $\mathbf{a} = a_1 \otimes a_2 \otimes \cdots \otimes a_k$ be a pure tensor in $\left(H^{\otimes \ell}\right)^{\otimes k}$ (with $a_1, a_2, \ldots, a_k \in H^{\otimes \ell}$). Then,

$$\begin{aligned}
\left(\sigma \circ f^{\otimes k}\right)(\mathbf{a}) &= \left(\sigma \circ f^{\otimes k}\right)\left(a_1 \otimes a_2 \otimes \cdots \otimes a_k\right) \\
&= \sigma\left(\underbrace{f^{\otimes k}\left(a_1 \otimes a_2 \otimes \cdots \otimes a_k\right)}_{=f(a_1)\otimes f(a_2)\otimes\cdots\otimes f(a_k)}\right) \\
&= \sigma\left(f\left(a_1\right) \otimes f\left(a_2\right) \otimes \cdots \otimes f\left(a_k\right)\right) \\
&= f\left(a_{\sigma^{-1}(1)}\right) \otimes f\left(a_{\sigma^{-1}(2)}\right) \otimes \cdots \otimes f\left(a_{\sigma^{-1}(k)}\right)
\end{aligned}$$

and

$$\begin{aligned}
\left(f^{\otimes k} \circ \sigma^{\times \ell}\right)(\mathbf{a}) &= \left(f^{\otimes k} \circ \sigma^{\times \ell}\right)(a_1 \otimes a_2 \otimes \cdots \otimes a_k) \\
&= f^{\otimes k}\left(\underbrace{\sigma^{\times \ell}(a_1 \otimes a_2 \otimes \cdots \otimes a_k)}_{\substack{= a_{\sigma^{-1}(1)} \otimes a_{\sigma^{-1}(2)} \otimes \cdots \otimes a_{\sigma^{-1}(k)} \\ \text{(by Claim 1)}}}\right) \\
&= f^{\otimes k}\left(a_{\sigma^{-1}(1)} \otimes a_{\sigma^{-1}(2)} \otimes \cdots \otimes a_{\sigma^{-1}(k)}\right) \\
&= f\left(a_{\sigma^{-1}(1)}\right) \otimes f\left(a_{\sigma^{-1}(2)}\right) \otimes \cdots \otimes f\left(a_{\sigma^{-1}(k)}\right).
\end{aligned}$$

Comparing these two equalities, we find $\left(\sigma \circ f^{\otimes k}\right)(\mathbf{a}) = \left(f^{\otimes k} \circ \sigma^{\times \ell}\right)(\mathbf{a})$.

Forget that we fixed $\mathbf{a}$. We thus have proved that $\left(\sigma \circ f^{\otimes k}\right)(\mathbf{a}) = \left(f^{\otimes k} \circ \sigma^{\times \ell}\right)(\mathbf{a})$ for each pure tensor $\mathbf{a}$ in $\left(H^{\otimes \ell}\right)^{\otimes k}$. In other words, the two maps $\sigma \circ f^{\otimes k}$ and $f^{\otimes k} \circ \sigma^{\times \ell}$ agree on all pure tensors in $\left(H^{\otimes \ell}\right)^{\otimes k}$. Since these two maps are $\mathbf{k}$-linear (and since the pure tensors span $\left(H^{\otimes \ell}\right)^{\otimes k}$), this entails that they must be identical. In other words, $\sigma \circ f^{\otimes k} = f^{\otimes k} \circ \sigma^{\times \ell}$. This proves Lemma 1.25. □

Lemma 1.26. Let $k, \ell \in \mathbb{N}$. Let $\tau \in \mathfrak{S}_\ell$. Let $\tau^{k\times}$ denote the permutation in $\mathfrak{S}_{k\ell}$ that sends each $k(j-1)+i$ (with $i \in [k]$ and $j \in [\ell]$) to $k(\tau(j)-1)+i$.

Let $f : H \to H^{\otimes k}$ be any $\mathbf{k}$-linear map. Then,

$$f^{\otimes \ell} \circ \tau = \tau^{k\times} \circ f^{\otimes \ell}$$

(as maps from $H^{\otimes \ell}$ to $H^{\otimes k\ell}$).

Proof. We begin with an even more basic claim:

Claim 1: Let $a_1, a_2, \ldots, a_\ell \in H^{\otimes k}$. Then,

$$\tau^{k\times}(a_1 \otimes a_2 \otimes \cdots \otimes a_\ell) = a_{\tau^{-1}(1)} \otimes a_{\tau^{-1}(2)} \otimes \cdots \otimes a_{\tau^{-1}(\ell)}$$

in $H^{\otimes k\ell}$. (Here, we are identifying $\left(H^{\otimes k}\right)^{\otimes \ell}$ with $H^{\otimes k\ell}$ in the obvious way.)

Proof of Claim 1. Analogous to the Claim 1 in our above proof of Lemma 1.25. □

Now, let $\mathbf{a} = a_1 \otimes a_2 \otimes \cdots \otimes a_\ell$ be a pure tensor in $H^{\otimes \ell}$ (with $a_1, a_2, \ldots, a_\ell \in H$).

Then,

$$\begin{aligned}\left(f^{\otimes\ell}\circ\tau\right)(\mathbf{a}) &= \left(f^{\otimes\ell}\circ\tau\right)(a_1\otimes a_2\otimes\cdots\otimes a_\ell)\\ &= f^{\otimes\ell}\left(\underbrace{\tau(a_1\otimes a_2\otimes\cdots\otimes a_\ell)}_{=a_{\tau^{-1}(1)}\otimes a_{\tau^{-1}(2)}\otimes\cdots\otimes a_{\tau^{-1}(\ell)}}\right)\\ &= f^{\otimes\ell}\left(a_{\tau^{-1}(1)}\otimes a_{\tau^{-1}(2)}\otimes\cdots\otimes a_{\tau^{-1}(\ell)}\right)\\ &= f\left(a_{\tau^{-1}(1)}\right)\otimes f\left(a_{\tau^{-1}(2)}\right)\otimes\cdots\otimes f\left(a_{\tau^{-1}(\ell)}\right)\end{aligned}$$

and

$$\begin{aligned}\left(\tau^{k\times}\circ f^{\otimes\ell}\right)(\mathbf{a}) &= \left(\tau^{k\times}\circ f^{\otimes\ell}\right)(a_1\otimes a_2\otimes\cdots\otimes a_\ell)\\ &= \tau^{k\times}\left(\underbrace{f^{\otimes\ell}(a_1\otimes a_2\otimes\cdots\otimes a_\ell)}_{=f(a_1)\otimes f(a_2)\otimes\cdots\otimes f(a_\ell)}\right)\\ &= \tau^{k\times}\left(f(a_1)\otimes f(a_2)\otimes\cdots\otimes f(a_\ell)\right)\\ &= f\left(a_{\tau^{-1}(1)}\right)\otimes f\left(a_{\tau^{-1}(2)}\right)\otimes\cdots\otimes f\left(a_{\tau^{-1}(\ell)}\right)\end{aligned}$$

(by Claim 1, applied to $f(a_i)$ instead of a_i). Comparing these two equalities, we find $\left(f^{\otimes\ell}\circ\tau\right)(\mathbf{a}) = \left(\tau^{k\times}\circ f^{\otimes\ell}\right)(\mathbf{a})$.

Forget that we fixed $\mathbf{a}$. We thus have proved that $\left(f^{\otimes\ell}\circ\tau\right)(\mathbf{a}) = \left(\tau^{k\times}\circ f^{\otimes\ell}\right)(\mathbf{a})$ for each pure tensor $\mathbf{a}$ in $H^{\otimes\ell}$. In other words, the two maps $f^{\otimes\ell}\circ\tau$ and $\tau^{k\times}\circ f^{\otimes\ell}$ agree on all pure tensors in $H^{\otimes\ell}$. Since these two maps are $\mathbf{k}$-linear (and since the pure tensors span $H^{\otimes\ell}$), this entails that they must be identical. In other words, $f^{\otimes\ell}\circ\tau = \tau^{k\times}\circ f^{\otimes\ell}$. This proves Lemma 1.26. □

Lemma 1.27. Let $k,\ell\in\mathbb{N}$. Let $\sigma\in\mathfrak{S}_k$ and $\tau\in\mathfrak{S}_\ell$. Define a permutation $\sigma^{\times\ell}\in\mathfrak{S}_{k\ell}$ as in Lemma 1.25, and define a permutation $\tau^{k\times}\in\mathfrak{S}_{k\ell}$ as in Lemma 1.26. Define a permutation $\zeta\in\mathfrak{S}_{k\ell}$ as in Lemma 1.24. Recall also the permutation $\tau[\sigma]$ defined in Definition 1.16. Then,

$$\tau^{k\times}\circ\zeta^{-1}\circ\sigma^{\times\ell} = \tau[\sigma] \tag{20}$$

and

$$\left(\sigma^{\times\ell}\right)^{-1}\circ\zeta\circ\left(\tau^{k\times}\right)^{-1} = (\tau[\sigma])^{-1}. \tag{21}$$

Proof. Let us first prove (20). Let $n\in[k\ell]$. Write n in the form $n=\ell(i-1)+j$ for some $i\in[k]$ and $j\in[\ell]$. (This is clearly possible.) Thus,

$$\sigma^{\times\ell}(n) = \sigma^{\times\ell}(\ell(i-1)+j) = \ell(\sigma(i)-1)+j$$

(by the definition of $\sigma^{\times\ell}$). Hence,

$$\zeta^{-1}\left(\sigma^{\times\ell}(n)\right)=\zeta^{-1}\left(\ell\left(\sigma\left(i\right)-1\right)+j\right)=k\left(j-1\right)+\sigma\left(i\right)$$

(since the definition of ζ yields $\zeta\left(k\left(j-1\right)+\sigma\left(i\right)\right)=\ell\left(\sigma\left(i\right)-1\right)+j$). Hence,

$$\tau^{k\times}\left(\zeta^{-1}\left(\sigma^{\times\ell}(n)\right)\right)=\tau^{k\times}\left(k\left(j-1\right)+\sigma\left(i\right)\right)=k\left(\tau\left(j\right)-1\right)+\sigma\left(i\right)$$

(by the definition of $\tau^{k\times}$). Comparing this with

$$\begin{aligned}(\tau[\sigma])(n)&=(\tau[\sigma])\left(\ell\left(i-1\right)+j\right) && \text{(since } n=\ell\left(i-1\right)+j\text{)}\\ &=k\left(\tau\left(j\right)-1\right)+\sigma\left(i\right) && \text{(by the definition of } \tau[\sigma]\text{)},\end{aligned}$$

we obtain $\tau^{k\times}\left(\zeta^{-1}\left(\sigma^{\times\ell}(n)\right)\right)=(\tau[\sigma])(n)$.

Forget that we fixed n. We thus have shown that $\tau^{k\times}\left(\zeta^{-1}\left(\sigma^{\times\ell}(n)\right)\right)=(\tau[\sigma])(n)$ for each $n\in[k\ell]$. In other words, $\tau^{k\times}\circ\zeta^{-1}\circ\sigma^{\times\ell}=\tau[\sigma]$. This proves (20).

Now, taking inverses on both sides of (20), we obtain $\left(\tau^{k\times}\circ\zeta^{-1}\circ\sigma^{\times\ell}\right)^{-1}=(\tau[\sigma])^{-1}$. In other words, $\left(\sigma^{\times\ell}\right)^{-1}\circ\zeta\circ\left(\tau^{k\times}\right)^{-1}=(\tau[\sigma])^{-1}$ (since $\left(\tau^{k\times}\circ\zeta^{-1}\circ\sigma^{\times\ell}\right)^{-1}=\left(\sigma^{\times\ell}\right)^{-1}\circ\zeta\circ\left(\tau^{k\times}\right)^{-1}$). This proves (21). □

Recall again that each symmetric group $\mathfrak{S}_k$ acts on the corresponding set $\mathbb{N}^k$ of k-tuples from the right. For this action, we have two further elementary combinatorial properties:

Lemma 1.28. Let $k,\ell\in\mathbb{N}$. Define a permutation $\zeta\in\mathfrak{S}_{k\ell}$ as in Lemma 1.24.
Let $\theta_{i,j}\in\mathbb{N}$ be a nonnegative integer for each $i\in[k]$ and $j\in[\ell]$. Then,

$$(\theta_{1,1},\theta_{1,2},\ldots,\theta_{k,\ell})\cdot\zeta=(\theta_{1,1},\theta_{2,1},\ldots,\theta_{k,\ell}).$$

(Here, $(\theta_{1,1},\theta_{1,2},\ldots,\theta_{k,\ell})$ denotes the list of all $k\ell$ numbers $\theta_{i,j}$ in the order of lexicographically increasing pairs (i,j), whereas $(\theta_{1,1},\theta_{2,1},\ldots,\theta_{k,\ell})$ denotes the list of all $k\ell$ numbers $\theta_{i,j}$ in the order of lexicographically increasing pairs (j,i).)

Proof. Let us write $\theta_{(i,j)}$ for $\theta_{i,j}$. Thus, θ_p is defined for any pair $p\in[k]\times[\ell]$.
Define the two bijections λ and ρ as in the proof of Lemma 1.24. Then,

$$\begin{aligned}(\theta_{1,1},\theta_{1,2},\ldots,\theta_{k,\ell})&=\left(\theta_{\rho^{-1}(1)},\theta_{\rho^{-1}(2)},\ldots,\theta_{\rho^{-1}(k\ell)}\right) \qquad\text{and}\\ (\theta_{1,1},\theta_{2,1},\ldots,\theta_{k,\ell})&=\left(\theta_{\lambda^{-1}(1)},\theta_{\lambda^{-1}(2)},\ldots,\theta_{\lambda^{-1}(k\ell)}\right).\end{aligned}$$

Thus,

$$\begin{aligned}
\underbrace{\left(\theta_{1,1},\theta_{1,2},\ldots,\theta_{k,\ell}\right)}_{=\left(\theta_{\rho^{-1}(1)},\theta_{\rho^{-1}(2)},\ldots,\theta_{\rho^{-1}(k\ell)}\right)} \cdot \zeta &= \left(\theta_{\rho^{-1}(1)},\theta_{\rho^{-1}(2)},\ldots,\theta_{\rho^{-1}(k\ell)}\right) \cdot \zeta \\
&= \left(\theta_{\rho^{-1}(\zeta(1))},\theta_{\rho^{-1}(\zeta(2))},\ldots,\theta_{\rho^{-1}(\zeta(k\ell))}\right) \\
&= \left(\theta_{\lambda^{-1}(1)},\theta_{\lambda^{-1}(2)},\ldots,\theta_{\lambda^{-1}(k\ell)}\right) \\
&\qquad \left(\begin{array}{c} \text{since each } i \in [k\ell] \\ \text{satisfies } \rho^{-1}\left(\zeta\left(i\right)\right) = \lambda^{-1}\left(i\right) \\ \text{(since (12) yields } \rho^{-1}\circ\zeta = \lambda^{-1}\text{)} \\ \text{and thus } \theta_{\rho^{-1}(\zeta(i))} = \theta_{\lambda^{-1}(i)} \end{array}\right) \\
&= \left(\theta_{1,1},\theta_{2,1},\ldots,\theta_{k,\ell}\right),
\end{aligned}$$

and this proves Lemma 1.28. □

Lemma 1.29. Let $k,\ell \in \mathbb{N}$. Let $\sigma \in \mathfrak{S}_k$ be any permutation. Define a permutation $\sigma^{\times \ell} \in \mathfrak{S}_{k\ell}$ as in Lemma 1.25.

Let $\theta_{i,j} \in \mathbb{N}$ be a nonnegative integer for each $i \in [k]$ and $j \in [\ell]$. Then,

$$\left(\theta_{1,1},\theta_{1,2},\ldots,\theta_{k,\ell}\right) \cdot \sigma^{\times \ell} = \left(\theta_{\sigma(1),1},\theta_{\sigma(1),2},\ldots,\theta_{\sigma(k),\ell}\right).$$

(Here, $\left(\theta_{1,1},\theta_{1,2},\ldots,\theta_{k,\ell}\right)$ denotes the list of all $k\ell$ numbers $\theta_{i,j}$ in the order of lexicographically increasing pairs (i,j), whereas $\left(\theta_{\sigma(1),1},\theta_{\sigma(1),2},\ldots,\theta_{\sigma(k),\ell}\right)$ denotes the list of all $k\ell$ numbers $\theta_{\sigma(i),j}$ in the same order.)

Proof. Let us denote the $k\ell$-tuples $\left(\theta_{1,1},\theta_{1,2},\ldots,\theta_{k,\ell}\right)$ and $\left(\theta_{\sigma(1),1},\theta_{\sigma(1),2},\ldots,\theta_{\sigma(k),\ell}\right)$ as $\left(\alpha_1,\alpha_2,\ldots,\alpha_{k\ell}\right)$ and $\left(\beta_1,\beta_2,\ldots,\beta_{k\ell}\right)$, respectively. Then, each $i \in [k]$ and $j \in [\ell]$ satisfy

$$\alpha_{\ell(i-1)+j} = \theta_{i,j} \tag{22}$$

(since $\left(\alpha_1,\alpha_2,\ldots,\alpha_{k\ell}\right) = \left(\theta_{1,1},\theta_{1,2},\ldots,\theta_{k,\ell}\right)$) and

$$\beta_{\ell(i-1)+j} = \theta_{\sigma(i),j} \tag{23}$$

(since $\left(\beta_1,\beta_2,\ldots,\beta_{k\ell}\right) = \left(\theta_{\sigma(1),1},\theta_{\sigma(1),2},\ldots,\theta_{\sigma(k),\ell}\right)$).

Let ω be the permutation $\sigma^{\times \ell}$. Then, for each $i \in [k]$ and $j \in [\ell]$, we have

$$\begin{aligned}
\omega\left(\ell\left(i-1\right)+j\right) &= \sigma^{\times \ell}\left(\ell\left(i-1\right)+j\right) \\
&= \ell\left(\sigma\left(i\right)-1\right)+j
\end{aligned} \tag{24}$$

(by the definition of $\sigma^{\times \ell}$).

Now, let $n \in [k\ell]$. Write n in the form $n = \ell(i-1) + j$ for some $i \in [k]$ and $j \in [\ell]$. (This is clearly possible.) Thus, $\omega(n) = \omega(\ell(i-1)+j) = \ell(\sigma(i)-1)+j$ (by (24)). This yields

$$\begin{aligned}
\alpha_{\omega(n)} &= \alpha_{\ell(\sigma(i)-1)+j} \\
&= \theta_{\sigma(i),j} \qquad \text{(by (22), applied to } \sigma(i) \text{ instead of } i\text{)} \\
&= \beta_{\ell(i-1)+j} \qquad \text{(by (23))}. \\
&= \beta_n \qquad (\text{since } \ell(i-1)+j = n).
\end{aligned}$$

Forget that we fixed n. We thus have shown that $\alpha_{\omega(n)} = \beta_n$ for each $n \in [k\ell]$. Now,

$$\begin{aligned}
&\underbrace{(\theta_{1,1}, \theta_{1,2}, \ldots, \theta_{k,\ell})}_{=(\alpha_1,\alpha_2,\ldots,\alpha_{k\ell})} \cdot \underbrace{\sigma^{\times \ell}}_{=\omega} \\
&= (\alpha_1, \alpha_2, \ldots, \alpha_{k\ell}) \cdot \omega \\
&= \left(\alpha_{\omega(1)}, \alpha_{\omega(2)}, \ldots, \alpha_{\omega(k\ell)}\right) \\
&= (\beta_1, \beta_2, \ldots, \beta_{k\ell}) \qquad \left(\text{since } \alpha_{\omega(n)} = \beta_n \text{ for each } n \in [k\ell]\right) \\
&= \left(\theta_{\sigma(1),1}, \theta_{\sigma(1),2}, \ldots, \theta_{\sigma(k),\ell}\right).
\end{aligned}$$

This proves Lemma 1.29. □

1.6. The composition formula: lemmas on bialgebras

Now, we step to some lemmas that rely on the bialgebra structure on H.

Lemma 1.30. Let $k, \ell \in \mathbb{N}$. Then, $m^{[k]} \circ \left(m^{[\ell]}\right)^{\otimes k} = m^{[k\ell]}$.

Proof. We induct on k. The *base case* ($k = 0$) is trivial (since $\left(m^{[\ell]}\right)^{\otimes 0} = \mathrm{id}_{\mathbf{k}}$ and $0 = 0\ell$). For the *induction step*, we fix a $k \in \mathbb{N}$, and we assume (as the induction hypothesis) that $m^{[k]} \circ \left(m^{[\ell]}\right)^{\otimes k} = m^{[k\ell]}$. We must now prove that $m^{[k+1]} \circ$

$\left(m^{[\ell]}\right)^{\otimes(k+1)} = m^{[(k+1)\ell]}$. But $(k+1)\,\ell = k\ell + \ell$, and thus

$$m^{[(k+1)\ell]} = m^{[k\ell+\ell]} = m \circ \left(\underbrace{m^{[k\ell]}}_{\substack{= m^{[k]} \circ \left(m^{[\ell]}\right)^{\otimes k} \\ \text{(by the induction hypothesis)}}} \otimes \underbrace{m^{[\ell]}}_{= \operatorname{id} \circ m^{[\ell]}} \right) \qquad \text{(by (2))}$$

$$= m \circ \underbrace{\left(\left(m^{[k]} \circ \left(m^{[\ell]}\right)^{\otimes k} \right) \otimes \left(\operatorname{id} \circ m^{[\ell]} \right) \right)}_{= \left(m^{[k]} \otimes \operatorname{id}\right) \circ \left(\left(m^{[\ell]}\right)^{\otimes k} \otimes m^{[\ell]} \right)}$$

$$= m \circ \left(m^{[k]} \otimes \underbrace{\operatorname{id}}_{= m^{[1]}} \right) \circ \underbrace{\left(\left(m^{[\ell]}\right)^{\otimes k} \otimes m^{[\ell]} \right)}_{= \left(m^{[\ell]}\right)^{\otimes(k+1)}}$$

$$= \underbrace{m \circ \left(m^{[k]} \otimes m^{[1]} \right)}_{\substack{= m^{[k+1]} \\ \text{(by (2))}}} \circ \left(m^{[\ell]}\right)^{\otimes(k+1)} = m^{[k+1]} \circ \left(m^{[\ell]}\right)^{\otimes(k+1)}.$$

Thus, $m^{[k+1]} \circ \left(m^{[\ell]}\right)^{\otimes(k+1)} = m^{[(k+1)\ell]}$ is proved, so that the induction is complete. This proves Lemma 1.30. □

Lemma 1.31. Let $k, \ell \in \mathbb{N}$. Then, $\left(\Delta^{[k]}\right)^{\otimes \ell} \circ \Delta^{[\ell]} = \Delta^{[k\ell]}$.

Proof. Upon swapping k and ℓ, this becomes the dual claim to Lemma 1.30, so the same proof can be used (with all arrows reversed). □

Lemma 1.32. Let $k, \ell \in \mathbb{N}$. Define a permutation $\zeta \in \mathfrak{S}_{k\ell}$ as in Lemma 1.24. Then,

$$\left(m^{[\ell]}\right)^{\otimes k} \circ \zeta \circ \left(\Delta^{[k]}\right)^{\otimes \ell} = \Delta^{[k]} \circ m^{[\ell]}.$$

Proof. From [GriRei20, Exercise 1.4.22(c)] (applied to $k-1$ and $\ell-1$ instead of k and ℓ), we obtain

$$m^{[\ell]}_{H^{\otimes k}} \circ \left(\Delta^{[k]}\right)^{\otimes \ell} = \Delta^{[k]} \circ m^{[\ell]}, \qquad (25)$$

where $m^{[\ell]}_{H^{\otimes k}}$ is the map defined just as $m^{[\ell]}$ but for the algebra $H^{\otimes k}$ instead of H. However, it is easy to see (and should be known) that

$$m^{[\ell]}_{H^{\otimes k}} = \left(m^{[\ell]}\right)^{\otimes k} \circ \zeta. \qquad (26)$$

Indeed, this can be checked on pure tensors, using Lemma 1.24.[7]

Using (26), we can rewrite (25) as

$$\left(m^{[\ell]}\right)^{\otimes k} \circ \zeta \circ \left(\Delta^{[k]}\right)^{\otimes \ell} = \Delta^{[k]} \circ m^{[\ell]}.$$

This proves Lemma 1.32. □

The next two lemmas combine the algebra and coalgebra structures with the grading:

---

[7] *Proof.* If $(h_{i,j})_{i \in [\ell],\ j \in [k]} \in H^{\ell \times k}$ is any $\ell \times k$-matrix over H, then

$$\begin{aligned}
&\left(\left(m^{[\ell]}\right)^{\otimes k} \circ \zeta\right) (h_{1,1} \otimes h_{1,2} \otimes \cdots \otimes h_{1,k} \\
&\qquad \otimes h_{2,1} \otimes h_{2,2} \otimes \cdots \otimes h_{2,k} \\
&\qquad \otimes \ \cdots \\
&\qquad \otimes h_{\ell,1} \otimes h_{\ell,2} \otimes \cdots \otimes h_{\ell,k}) \\
&= \left(m^{[\ell]}\right)^{\otimes k} (h_{1,1} \otimes h_{2,1} \otimes \cdots \otimes h_{\ell,1} \\
&\qquad \otimes h_{1,2} \otimes h_{2,2} \otimes \cdots \otimes h_{\ell,2} \\
&\qquad \otimes \ \cdots \\
&\qquad \otimes h_{1,k} \otimes h_{2,k} \otimes \cdots \otimes h_{\ell,k}) \\
&\qquad\qquad \left(\begin{array}{c}\text{here, we have applied the map } \left(m^{[\ell]}\right)^{\otimes k} \\ \text{to both sides of the equality (11)}\end{array}\right) \\
&= \underbrace{m^{[\ell]} (h_{1,1} \otimes h_{2,1} \otimes \cdots \otimes h_{\ell,1})}_{= h_{1,1} h_{2,1} \cdots h_{\ell,1}} \\
&\qquad \otimes \underbrace{m^{[\ell]} (h_{1,2} \otimes h_{2,2} \otimes \cdots \otimes h_{\ell,2})}_{= h_{1,2} h_{2,2} \cdots h_{\ell,2}} \\
&\qquad \otimes \cdots \\
&\qquad \otimes \underbrace{m^{[\ell]} (h_{1,k} \otimes h_{2,k} \otimes \cdots \otimes h_{\ell,k})}_{= h_{1,k} h_{2,k} \cdots h_{\ell,k}} \\
&= h_{1,1} h_{2,1} \cdots h_{\ell,1} \otimes h_{1,2} h_{2,2} \cdots h_{\ell,2} \otimes \cdots \otimes h_{1,k} h_{2,k} \cdots h_{\ell,k} \\
&= (h_{1,1} \otimes h_{1,2} \otimes \cdots \otimes h_{1,k}) \cdot (h_{2,1} \otimes h_{2,2} \otimes \cdots \otimes h_{2,k}) \cdot \cdots \cdot (h_{\ell,1} \otimes h_{\ell,2} \otimes \cdots \otimes h_{\ell,k}) \\
&= m^{[\ell]}_{H^{\otimes k}} (h_{1,1} \otimes h_{1,2} \otimes \cdots \otimes h_{1,k} \\
&\qquad \otimes h_{2,1} \otimes h_{2,2} \otimes \cdots \otimes h_{2,k} \\
&\qquad \otimes \ \cdots \\
&\qquad \otimes h_{\ell,1} \otimes h_{\ell,2} \otimes \cdots \otimes h_{\ell,k}) .
\end{aligned}$$

In other words, the two maps $\left(m^{[\ell]}\right)^{\otimes k} \circ \zeta$ and $m^{[\ell]}_{H^{\otimes k}}$ agree on each pure tensor. Since these two maps are **k**-linear, they must therefore be identical (since the pure tensors span $H^{\otimes k \ell}$). In other words, $\left(m^{[\ell]}\right)^{\otimes k} \circ \zeta = m^{[\ell]}_{H^{\otimes k}}$. This proves (26).

Lemma 1.33. Let $k, \ell \in \mathbb{N}$. Let $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_k) \in \mathbb{N}^k$. Then,

$$P_\gamma \circ \left(m^{[\ell]}\right)^{\otimes k} = \sum_{\substack{\gamma_{i,j} \in \mathbb{N} \text{ for all } i \in [k] \text{ and } j \in [\ell]; \\ \gamma_{i,1}+\gamma_{i,2}+\cdots+\gamma_{i,\ell}=\gamma_i \text{ for all } i \in [k]}} \left(m^{[\ell]}\right)^{\otimes k} \circ P_{(\gamma_{1,1},\gamma_{1,2},\ldots,\gamma_{k,\ell})}.$$

Here, $(\gamma_{1,1}, \gamma_{1,2}, \ldots, \gamma_{k,\ell})$ denotes the $k\ell$-tuple consisting of all $k\ell$ numbers $\gamma_{i,j}$ (for all $i \in [k]$ and $j \in [\ell]$) listed in the order of lexicographically increasing pairs (i, j).

Proof. Let $i \in \mathbb{N}$. Recall that $p_i : H \to H$ denotes the projection of the graded $\mathbf{k}$-module H onto its i-th graded component. Likewise, let $p_i' : H^{\otimes \ell} \to H^{\otimes \ell}$ denote the projection of the graded $\mathbf{k}$-module $H^{\otimes \ell}$ onto its i-th graded component. The map $m^{[\ell]} : H^{\otimes \ell} \to H$ is graded[8]. Thus, it commutes with the projection onto the i-th graded component. In other words,

$$p_i \circ m^{[\ell]} = m^{[\ell]} \circ p_i'. \tag{27}$$

However, the definition of the grading on $H^{\otimes \ell}$ yields that the i-th graded component of $H^{\otimes \ell}$ is $\bigoplus_{\substack{(i_1,i_2,\ldots,i_\ell) \in \mathbb{N}^\ell; \\ i_1+i_2+\cdots+i_\ell=i}} H_{i_1} \otimes H_{i_2} \otimes \cdots \otimes H_{i_\ell}$. Hence, it is easy to see that the projection p_i' onto this component is

$$\sum_{\substack{(i_1,i_2,\ldots,i_\ell) \in \mathbb{N}^\ell; \\ i_1+i_2+\cdots+i_\ell=i}} p_{i_1} \otimes p_{i_2} \otimes \cdots \otimes p_{i_\ell}.$$

In view of this, we can rewrite (27) as

$$\begin{aligned} p_i \circ m^{[\ell]} &= m^{[\ell]} \circ \left(\sum_{\substack{(i_1,i_2,\ldots,i_\ell) \in \mathbb{N}^\ell; \\ i_1+i_2+\cdots+i_\ell=i}} p_{i_1} \otimes p_{i_2} \otimes \cdots \otimes p_{i_\ell} \right) \\ &= \sum_{\substack{(i_1,i_2,\ldots,i_\ell) \in \mathbb{N}^\ell; \\ i_1+i_2+\cdots+i_\ell=i}} m^{[\ell]} \circ \left(p_{i_1} \otimes p_{i_2} \otimes \cdots \otimes p_{i_\ell}\right) \end{aligned} \tag{28}$$

(here, we have distributed the summation sign to the beginning of the product, since all the maps involved are linear).

Forget that we fixed i. We have thus proved the equality (28) for each $i \in \mathbb{N}$.

Now, from $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_k)$, we obtain $P_\gamma = p_{\gamma_1} \otimes p_{\gamma_2} \otimes \cdots \otimes p_{\gamma_k}$ (by the

[8]This follows by a straightforward induction on ℓ from the gradedness of the map $m : H \otimes H \to H$.

definition of P_γ). Hence,

$$\begin{aligned}
&\underbrace{P_\gamma}_{=p_{\gamma_1}\otimes p_{\gamma_2}\otimes\cdots\otimes p_{\gamma_k}} \circ \underbrace{\left(m^{[\ell]}\right)^{\otimes k}}_{=m^{[\ell]}\otimes m^{[\ell]}\otimes\cdots\otimes m^{[\ell]}} \\
&= \left(p_{\gamma_1}\otimes p_{\gamma_2}\otimes\cdots\otimes p_{\gamma_k}\right)\circ\left(m^{[\ell]}\otimes m^{[\ell]}\otimes\cdots\otimes m^{[\ell]}\right) \\
&= \left(p_{\gamma_1}\circ m^{[\ell]}\right)\otimes\left(p_{\gamma_2}\circ m^{[\ell]}\right)\otimes\cdots\otimes\left(p_{\gamma_k}\circ m^{[\ell]}\right) \\
&= \bigotimes_{s=1}^{k}\left(p_{\gamma_s}\circ m^{[\ell]}\right) \qquad \left(\text{here, we use the symbol } \bigotimes_{s=1}^{k} f_s \text{ for } f_1\otimes f_2\otimes\cdots\otimes f_k\right) \\
&= \bigotimes_{s=1}^{k}\left(\sum_{\substack{(i_1,i_2,\ldots,i_\ell)\in\mathbb{N}^\ell;\\ i_1+i_2+\cdots+i_\ell=\gamma_s}} m^{[\ell]}\circ\left(p_{i_1}\otimes p_{i_2}\otimes\cdots\otimes p_{i_\ell}\right)\right) \\
&\qquad\qquad (\text{here, we have applied (28) to } i=\gamma_s \text{ for each } s\in[k]) \\
&= \bigotimes_{s=1}^{k}\left(\sum_{\substack{(\gamma_{s,1},\gamma_{s,2},\ldots,\gamma_{s,\ell})\in\mathbb{N}^\ell;\\ \gamma_{s,1}+\gamma_{s,2}+\cdots+\gamma_{s,\ell}=\gamma_s}} m^{[\ell]}\circ\left(p_{\gamma_{s,1}}\otimes p_{\gamma_{s,2}}\otimes\cdots\otimes p_{\gamma_{s,\ell}}\right)\right) \\
&\qquad\qquad \left(\begin{array}{c}\text{here, we have renamed the index } (i_1,i_2,\ldots,i_\ell)\\ \text{as } (\gamma_{s,1},\gamma_{s,2},\ldots,\gamma_{s,\ell}) \text{ in the sum}\end{array}\right) \\
&= \sum_{\substack{\gamma_{s,j}\in\mathbb{N} \text{ for all } s\in[k] \text{ and } j\in[\ell];\\ \gamma_{s,1}+\gamma_{s,2}+\cdots+\gamma_{s,\ell}=\gamma_s \text{ for all } s\in[k]}} \bigotimes_{s=1}^{k}\left(m^{[\ell]}\circ\left(p_{\gamma_{s,1}}\otimes p_{\gamma_{s,2}}\otimes\cdots\otimes p_{\gamma_{s,\ell}}\right)\right) \qquad (29)
\end{aligned}$$

(by the product rule for tensor products, which says

$$\bigotimes_{s=1}^{k}\sum_{a\in A_s} f_{s,a} = \sum_{(a_1,a_2,\ldots,a_k)\in A_1\times A_2\times\cdots\times A_k}\bigotimes_{s=1}^{k} f_{s,a_s},$$

and which we here have applied to
$A_s = \left\{(\gamma_{s,1},\gamma_{s,2},\ldots,\gamma_{s,\ell})\in\mathbb{N}^\ell \mid \gamma_{s,1}+\gamma_{s,2}+\cdots+\gamma_{s,\ell}=\gamma_s\right\}$ and $f_{s,\left(\gamma_{s,1},\gamma_{s,2},\ldots,\gamma_{s,\ell}\right)} = m^{[\ell]}\circ\left(p_{\gamma_{s,1}}\otimes p_{\gamma_{s,2}}\otimes\cdots\otimes p_{\gamma_{s,\ell}}\right)$). However, if $\gamma_{s,j}$ is a nonnegative integer for all

$s \in [k]$ and all $j \in [\ell]$, then

$$\begin{aligned}
&\bigotimes_{s=1}^{k} \left(m^{[\ell]} \circ \left(p_{\gamma_{s,1}} \otimes p_{\gamma_{s,2}} \otimes \cdots \otimes p_{\gamma_{s,\ell}} \right) \right) \\
&= \underbrace{\left(\bigotimes_{s=1}^{k} m^{[\ell]} \right)}_{=\left(m^{[\ell]}\right)^{\otimes k}} \circ \underbrace{\left(\bigotimes_{s=1}^{k} \left(p_{\gamma_{s,1}} \otimes p_{\gamma_{s,2}} \otimes \cdots \otimes p_{\gamma_{s,\ell}} \right) \right)}_{\substack{=P_{\left(\gamma_{1,1},\gamma_{1,2},\ldots,\gamma_{k,\ell}\right)} \\ \text{(by the definition of } P_{\left(\gamma_{1,1},\gamma_{1,2},\ldots,\gamma_{k,\ell}\right)}\text{)}}} \\
&= \left(m^{[\ell]} \right)^{\otimes k} \circ P_{\left(\gamma_{1,1},\gamma_{1,2},\ldots,\gamma_{k,\ell}\right)}.
\end{aligned}$$

Thus, (29) can be rewritten as

$$\begin{aligned}
P_\gamma \circ \left(m^{[\ell]} \right)^{\otimes k} &= \sum_{\substack{\gamma_{s,j} \in \mathbb{N} \text{ for all } s \in [k] \text{ and } j \in [\ell]; \\ \gamma_{s,1}+\gamma_{s,2}+\cdots+\gamma_{s,\ell}=\gamma_s \text{ for all } s \in [k]}} \left(m^{[\ell]} \right)^{\otimes k} \circ P_{\left(\gamma_{1,1},\gamma_{1,2},\ldots,\gamma_{k,\ell}\right)} \\
&= \sum_{\substack{\gamma_{i,j} \in \mathbb{N} \text{ for all } i \in [k] \text{ and } j \in [\ell]; \\ \gamma_{i,1}+\gamma_{i,2}+\cdots+\gamma_{i,\ell}=\gamma_i \text{ for all } i \in [k]}} \left(m^{[\ell]} \right)^{\otimes k} \circ P_{\left(\gamma_{1,1},\gamma_{1,2},\ldots,\gamma_{k,\ell}\right)}.
\end{aligned}$$

This proves Lemma 1.33. □

Lemma 1.34. Let $k, \ell \in \mathbb{N}$. Let $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_k) \in \mathbb{N}^k$. Then,

$$\left(\Delta^{[\ell]} \right)^{\otimes k} \circ P_\gamma = \sum_{\substack{\gamma_{i,j} \in \mathbb{N} \text{ for all } i \in [k] \text{ and } j \in [\ell]; \\ \gamma_{i,1}+\gamma_{i,2}+\cdots+\gamma_{i,\ell}=\gamma_i \text{ for all } i \in [k]}} P_{\left(\gamma_{1,1},\gamma_{1,2},\ldots,\gamma_{k,\ell}\right)} \circ \left(\Delta^{[\ell]} \right)^{\otimes k}.$$

Here, $(\gamma_{1,1}, \gamma_{1,2}, \ldots, \gamma_{k,\ell})$ denotes the $k\ell$-tuple consisting of all $k\ell$ numbers $\gamma_{i,j}$ (for all $i \in [k]$ and $j \in [\ell]$) listed in the order of lexicographically increasing pairs (i, j).

Proof. This is the dual statement to Lemma 1.33, and is proved by the same argument "with all arrows reversed" (meaning that any composition $f_1 \circ f_2 \circ \cdots \circ f_\ell$ of several maps should be reversed to become $f_\ell \circ f_{\ell-1} \circ \cdots \circ f_1$) and with all m's replaced by Δ's. □

1.7. The composition formula: proof

We are finally able to prove Theorem 1.19. The proof follows the same paradigm as Patras's proof of the commutative and cocommutative particular cases ([Patras94, preuve de Théorème II,7] and [CarPat21, proof of Theorem 5.11]), but involves more complexity due to the presence of permutations.

Proof of Theorem 1.19. The formula (4) yields

$$p_{\alpha,\sigma} = m^{[k]} \circ P_\alpha \circ \sigma^{-1} \circ \Delta^{[k]} \qquad \text{and} \tag{30}$$
$$p_{\beta,\tau} = m^{[\ell]} \circ P_\beta \circ \tau^{-1} \circ \Delta^{[\ell]}. \tag{31}$$

Multiplying these two equalities (using the operation $\circ$), we obtain

$$p_{\alpha,\sigma} \circ p_{\beta,\tau} = m^{[k]} \circ P_\alpha \circ \sigma^{-1} \circ \Delta^{[k]} \circ m^{[\ell]} \circ P_\beta \circ \tau^{-1} \circ \Delta^{[\ell]}.$$

Our main task now is to "commute" the operators in this equality past each other, moving both m factors to the left, both Δ factors to the right, and ideally obtaining some expression of the form $m^{[k\ell]} \circ P_\gamma \circ \rho^{-1} \circ \Delta^{[k\ell]}$ (more precisely, it will be a sum of such expressions). For this purpose, we shall derive some "commutation-like" formulas.

Lemma 1.33 (applied to α and α_i instead of γ and γ_i) yields

$$P_\alpha \circ \left(m^{[\ell]}\right)^{\otimes k} = \sum_{\substack{\gamma_{i,j} \in \mathbb{N} \text{ for all } i \in [k] \text{ and } j \in [\ell]; \\ \gamma_{i,1}+\gamma_{i,2}+\cdots+\gamma_{i,\ell}=\alpha_i \text{ for all } i \in [k]}} \left(m^{[\ell]}\right)^{\otimes k} \circ P_{\left(\gamma_{1,1},\gamma_{1,2},\ldots,\gamma_{k,\ell}\right)}. \tag{32}$$

Lemma 1.34 (applied to k, ℓ, β and β_i instead of ℓ, k, γ and γ_i) yields

$$\begin{aligned}
&\left(\Delta^{[k]}\right)^{\otimes \ell} \circ P_\beta \\
&= \sum_{\substack{\gamma_{i,j} \in \mathbb{N} \text{ for all } i \in [\ell] \text{ and } j \in [k]; \\ \gamma_{i,1}+\gamma_{i,2}+\cdots+\gamma_{i,k}=\beta_i \text{ for all } i \in [\ell]}} P_{\left(\gamma_{1,1},\gamma_{1,2},\ldots,\gamma_{\ell,k}\right)} \circ \left(\Delta^{[k]}\right)^{\otimes \ell} \\
&= \sum_{\substack{\theta_{j,i} \in \mathbb{N} \text{ for all } i \in [\ell] \text{ and } j \in [k]; \\ \theta_{1,i}+\theta_{2,i}+\cdots+\theta_{k,i}=\beta_i \text{ for all } i \in [\ell]}} P_{\left(\theta_{1,1},\theta_{2,1},\ldots,\theta_{k,\ell}\right)} \circ \left(\Delta^{[k]}\right)^{\otimes \ell} \\
&\qquad \left(\text{here, we have renamed the } \gamma_{i,j} \text{ as } \theta_{j,i}\right) \\
&= \sum_{\substack{\theta_{i,j} \in \mathbb{N} \text{ for all } j \in [\ell] \text{ and } i \in [k]; \\ \theta_{1,j}+\theta_{2,j}+\cdots+\theta_{k,j}=\beta_j \text{ for all } j \in [\ell]}} P_{\left(\theta_{1,1},\theta_{2,1},\ldots,\theta_{k,\ell}\right)} \circ \left(\Delta^{[k]}\right)^{\otimes \ell} \\
&\qquad \left(\text{here, we have renamed the indices } i \text{ and } j \text{ as } j \text{ and } i\right) \\
&= \sum_{\substack{\theta_{i,j} \in \mathbb{N} \text{ for all } i \in [k] \text{ and } j \in [\ell]; \\ \theta_{1,j}+\theta_{2,j}+\cdots+\theta_{k,j}=\beta_j \text{ for all } j \in [\ell]}} P_{\left(\theta_{1,1},\theta_{2,1},\ldots,\theta_{k,\ell}\right)} \circ \left(\Delta^{[k]}\right)^{\otimes \ell}
\end{aligned} \tag{33}$$

(here, we have just rewritten the "$\theta_{i,j} \in \mathbb{N}$ for all $j \in [\ell]$ and $i \in [k]$" under the summation sign as "$\theta_{i,j} \in \mathbb{N}$ for all $i \in [k]$ and $j \in [\ell]$").

Next, we shall use a nearly trivial fact: If f, g, u, v are four maps such that the compositions $f \circ u$ and $g \circ v$ are well-defined (i.e., the target of u is the domain of f, and the target of v is the domain of g) and satisfy $f \circ u = g \circ v$, and if the maps f and v are invertible, then

$$u \circ v^{-1} = f^{-1} \circ g \tag{34}$$

(since $\underbrace{f \circ u}_{= g \circ v} \circ v^{-1} = g \circ \underbrace{v \circ v^{-1}}_{= \mathrm{id}} = g$ and thus $g = f \circ u \circ v^{-1}$, so that $f^{-1} \circ g = \underbrace{f^{-1} \circ f}_{= \mathrm{id}} \circ u \circ v^{-1} = u \circ v^{-1}$).

Define a permutation $\sigma^{\times \ell} \in \mathfrak{S}_{k\ell}$ as in Lemma 1.25. Define a permutation $\tau^{k\times} \in \mathfrak{S}_{k\ell}$ as in Lemma 1.26. Define a permutation $\zeta \in \mathfrak{S}_{k\ell}$ as in Lemma 1.24.

Lemma 1.25 (applied to $f = m^{[\ell]}$) yields $\sigma \circ \left(m^{[\ell]}\right)^{\otimes k} = \left(m^{[\ell]}\right)^{\otimes k} \circ \sigma^{\times \ell}$. Hence, (34) (applied to $f = \sigma$ and $u = \left(m^{[\ell]}\right)^{\otimes k}$ and $g = \left(m^{[\ell]}\right)^{\otimes k}$ and $v = \sigma^{\times \ell}$) yields

$$\left(m^{[\ell]}\right)^{\otimes k} \circ \left(\sigma^{\times \ell}\right)^{-1} = \sigma^{-1} \circ \left(m^{[\ell]}\right)^{\otimes k}. \tag{35}$$

Lemma 1.26 (applied to $f = \Delta^{[k]}$) yields $\left(\Delta^{[k]}\right)^{\otimes \ell} \circ \tau = \tau^{k\times} \circ \left(\Delta^{[k]}\right)^{\otimes \ell}$. In other words, $\tau^{k\times} \circ \left(\Delta^{[k]}\right)^{\otimes \ell} = \left(\Delta^{[k]}\right)^{\otimes \ell} \circ \tau$. Hence, (34) (applied to $f = \tau^{k\times}$ and $u = \left(\Delta^{[k]}\right)^{\otimes \ell}$ and $g = \left(\Delta^{[k]}\right)^{\otimes \ell}$ and $v = \tau$) yields

$$\left(\Delta^{[k]}\right)^{\otimes \ell} \circ \tau^{-1} = \left(\tau^{k\times}\right)^{-1} \circ \left(\Delta^{[k]}\right)^{\otimes \ell}. \tag{36}$$

Furthermore, if $\theta_{i,j} \in \mathbb{N}$ is a nonnegative integer for each $i \in [k]$ and $j \in [\ell]$, then we have

$$\begin{aligned} & P_{\left(\theta_{1,1}, \theta_{1,2}, \ldots, \theta_{k,\ell}\right)} \circ \zeta \\ & = \zeta \circ P_{\left(\theta_{1,1}, \theta_{1,2}, \ldots, \theta_{k,\ell}\right) \cdot \zeta} \\ & \qquad \left(\text{by (9), applied to } \pi = \zeta \text{ and } \gamma = \left(\theta_{1,1}, \theta_{1,2}, \ldots, \theta_{k,\ell}\right)\right) \\ & = \zeta \circ P_{\left(\theta_{1,1}, \theta_{2,1}, \ldots, \theta_{k,\ell}\right)} \end{aligned} \tag{37}$$

(since Lemma 1.28 yields $\left(\theta_{1,1}, \theta_{1,2}, \ldots, \theta_{k,\ell}\right) \cdot \zeta = \left(\theta_{1,1}, \theta_{2,1}, \ldots, \theta_{k,\ell}\right)$) and

$$\begin{aligned} & P_{\left(\theta_{1,1}, \theta_{1,2}, \ldots, \theta_{k,\ell}\right)} \circ \sigma^{\times \ell} \\ & = \sigma^{\times \ell} \circ P_{\left(\theta_{1,1}, \theta_{1,2}, \ldots, \theta_{k,\ell}\right) \cdot \sigma^{\times \ell}} \\ & \qquad \left(\text{by (9), applied to } \pi = \sigma^{\times \ell} \text{ and } \gamma = \left(\theta_{1,1}, \theta_{1,2}, \ldots, \theta_{k,\ell}\right)\right) \\ & = \sigma^{\times \ell} \circ P_{\left(\theta_{\sigma(1),1}, \theta_{\sigma(1),2}, \ldots, \theta_{\sigma(k),\ell}\right)} \end{aligned}$$

(since Lemma 1.29 yields $(\theta_{1,1}, \theta_{1,2}, \ldots, \theta_{k,\ell}) \cdot \sigma^{\times \ell} = \left(\theta_{\sigma(1),1}, \theta_{\sigma(1),2}, \ldots, \theta_{\sigma(k),\ell}\right)$) and therefore $\sigma^{\times \ell} \circ P_{\left(\theta_{\sigma(1),1}, \theta_{\sigma(1),2}, \ldots, \theta_{\sigma(k),\ell}\right)} = P_{\left(\theta_{1,1}, \theta_{1,2}, \ldots, \theta_{k,\ell}\right)} \circ \sigma^{\times \ell}$, so that

$$P_{\left(\theta_{\sigma(1),1}, \theta_{\sigma(1),2}, \ldots, \theta_{\sigma(k),\ell}\right)} \circ \left(\sigma^{\times \ell}\right)^{-1} = \left(\sigma^{\times \ell}\right)^{-1} \circ P_{\left(\theta_{1,1}, \theta_{1,2}, \ldots, \theta_{k,\ell}\right)} \qquad (38)$$

(by (34), applied to $f = \sigma^{\times \ell}$ and $u = P_{\left(\theta_{\sigma(1),1}, \theta_{\sigma(1),2}, \ldots, \theta_{\sigma(k),\ell}\right)}$ and $g = P_{\left(\theta_{1,1}, \theta_{1,2}, \ldots, \theta_{k,\ell}\right)}$ and $v = \sigma^{\times \ell}$).

Now, (30) and (31) yield

$$p_{\alpha,\sigma} \circ p_{\beta,\tau}$$
$$= m^{[k]} \circ P_\alpha \circ \sigma^{-1} \circ \underbrace{\Delta^{[k]} \circ m^{[\ell]}}_{\substack{=\left(m^{[\ell]}\right)^{\otimes k} \circ \zeta \circ \left(\Delta^{[k]}\right)^{\otimes \ell} \\ \text{(by Lemma 1.32)}}} \circ P_\beta \circ \tau^{-1} \circ \Delta^{[\ell]}$$
$$= m^{[k]} \circ P_\alpha \circ \underbrace{\sigma^{-1} \circ \left(m^{[\ell]}\right)^{\otimes k}}_{\substack{=\left(m^{[\ell]}\right)^{\otimes k} \circ \left(\sigma^{\times \ell}\right)^{-1} \\ \text{(by (35))}}} \circ \zeta$$
$$\circ \underbrace{\left(\Delta^{[k]}\right)^{\otimes \ell} \circ P_\beta}_{\substack{= \sum_{\substack{\theta_{i,j} \in \mathbb{N} \text{ for all } i \in [k] \text{ and } j \in [\ell]; \\ \theta_{1,j} + \theta_{2,j} + \cdots + \theta_{k,j} = \beta_j \text{ for all } j \in [\ell]}} P_{\left(\theta_{1,1}, \theta_{2,1}, \ldots, \theta_{k,\ell}\right)} \circ \left(\Delta^{[k]}\right)^{\otimes \ell} \\ \text{(by (33))}}} \circ \tau^{-1} \circ \Delta^{[\ell]}$$
$$= \sum_{\substack{\theta_{i,j} \in \mathbb{N} \text{ for all } i \in [k] \text{ and } j \in [\ell]; \\ \theta_{1,j} + \theta_{2,j} + \cdots + \theta_{k,j} = \beta_j \text{ for all } j \in [\ell]}} m^{[k]} \circ \underbrace{P_\alpha \circ \left(m^{[\ell]}\right)^{\otimes k}}_{\substack{= \sum_{\substack{\gamma_{i,j} \in \mathbb{N} \text{ for all } i \in [k] \text{ and } j \in [\ell]; \\ \gamma_{i,1} + \gamma_{i,2} + \cdots + \gamma_{i,\ell} = \alpha_i \text{ for all } i \in [k]}} \left(m^{[\ell]}\right)^{\otimes k} \circ P_{\left(\gamma_{1,1}, \gamma_{1,2}, \ldots, \gamma_{k,\ell}\right)} \\ \text{(by (32))}}}$$
$$\circ \left(\sigma^{\times \ell}\right)^{-1} \circ \zeta \circ P_{\left(\theta_{1,1}, \theta_{2,1}, \ldots, \theta_{k,\ell}\right)} \circ \underbrace{\left(\Delta^{[k]}\right)^{\otimes \ell} \circ \tau^{-1}}_{\substack{=\left(\tau^{k\times}\right)^{-1} \circ \left(\Delta^{[k]}\right)^{\otimes \ell} \\ \text{(by (36))}}} \circ \Delta^{[\ell]}$$
$$\left(\begin{array}{c}\text{here, we have distributed the summation sign to the} \\ \text{beginning of the product, since all our maps are } \mathbf{k}\text{-linear}\end{array}\right)$$
$$= \sum_{\substack{\gamma_{i,j} \in \mathbb{N} \text{ for all } i \in [k] \text{ and } j \in [\ell]; \\ \gamma_{i,1} + \gamma_{i,2} + \cdots + \gamma_{i,\ell} = \alpha_i \text{ for all } i \in [k]}} \sum_{\substack{\theta_{i,j} \in \mathbb{N} \text{ for all } i \in [k] \text{ and } j \in [\ell]; \\ \theta_{1,j} + \theta_{2,j} + \cdots + \theta_{k,j} = \beta_j \text{ for all } j \in [\ell]}}$$
$$m^{[k]} \circ \left(m^{[\ell]}\right)^{\otimes k} \circ P_{\left(\gamma_{1,1}, \gamma_{1,2}, \ldots, \gamma_{k,\ell}\right)} \circ \left(\sigma^{\times \ell}\right)^{-1}$$
$$\circ \zeta \circ P_{\left(\theta_{1,1}, \theta_{2,1}, \ldots, \theta_{k,\ell}\right)} \circ \left(\tau^{k\times}\right)^{-1} \circ \left(\Delta^{[k]}\right)^{\otimes \ell} \circ \Delta^{[\ell]} \tag{39}$$
$$\left(\begin{array}{c}\text{here, again, we have distributed the summation sign to the} \\ \text{beginning of the product, since all our maps are } \mathbf{k}\text{-linear}\end{array}\right).$$

However, if $\gamma_{i,j}$ and $\theta_{i,j}$ are nonnegative integers for all $i \in [k]$ and all $j \in [\ell]$, then

$$\begin{aligned}
&\underbrace{m^{[k]} \circ \left(m^{[\ell]}\right)^{\otimes k}}_{\substack{=m^{[k\ell]} \\ \text{(by Lemma 1.30)}}} \circ P_{\left(\gamma_{1,1},\gamma_{1,2},\ldots,\gamma_{k,\ell}\right)} \circ \left(\sigma^{\times \ell}\right)^{-1} \\
&\qquad \circ \underbrace{\zeta \circ P_{\left(\theta_{1,1},\theta_{2,1},\ldots,\theta_{k,\ell}\right)}}_{\substack{=P_{\left(\theta_{1,1},\theta_{1,2},\ldots,\theta_{k,\ell}\right)}\circ\zeta \\ \text{(by (37))}}} \circ \left(\tau^{k\times}\right)^{-1} \circ \underbrace{\left(\Delta^{[k]}\right)^{\otimes \ell} \circ \Delta^{[\ell]}}_{\substack{=\Delta^{[k\ell]} \\ \text{(by Lemma 1.31)}}} \\
&= m^{[k\ell]} \circ P_{\left(\gamma_{1,1},\gamma_{1,2},\ldots,\gamma_{k,\ell}\right)} \circ \underbrace{\left(\sigma^{\times \ell}\right)^{-1} \circ P_{\left(\theta_{1,1},\theta_{1,2},\ldots,\theta_{k,\ell}\right)}}_{\substack{=P_{\left(\theta_{\sigma(1),1},\theta_{\sigma(1),2},\ldots,\theta_{\sigma(k),\ell}\right)}\circ\left(\sigma^{\times\ell}\right)^{-1} \\ \text{(by (38))}}} \circ \zeta \circ \left(\tau^{k\times}\right)^{-1} \circ \Delta^{[k\ell]} \\
&= m^{[k\ell]} \circ \underbrace{P_{\left(\gamma_{1,1},\gamma_{1,2},\ldots,\gamma_{k,\ell}\right)} \circ P_{\left(\theta_{\sigma(1),1},\theta_{\sigma(1),2},\ldots,\theta_{\sigma(k),\ell}\right)}}_{\substack{=\begin{cases} P_{\left(\gamma_{1,1},\gamma_{1,2},\ldots,\gamma_{k,\ell}\right)}, & \text{if } \left(\gamma_{1,1},\gamma_{1,2},\ldots,\gamma_{k,\ell}\right) = \left(\theta_{\sigma(1),1},\theta_{\sigma(1),2},\ldots,\theta_{\sigma(k),\ell}\right); \\ 0, & \text{otherwise} \end{cases} \\ \text{(by Lemma 1.23)}}} \\
&\qquad \circ \underbrace{\left(\sigma^{\times \ell}\right)^{-1} \circ \zeta \circ \left(\tau^{k\times}\right)^{-1}}_{\substack{=\left(\tau[\sigma]\right)^{-1} \\ \text{(by (21))}}} \circ \Delta^{[k\ell]} \\
&= m^{[k\ell]} \circ \begin{cases} P_{\left(\gamma_{1,1},\gamma_{1,2},\ldots,\gamma_{k,\ell}\right)}, & \text{if } \left(\gamma_{1,1},\gamma_{1,2},\ldots,\gamma_{k,\ell}\right) = \left(\theta_{\sigma(1),1},\theta_{\sigma(1),2},\ldots,\theta_{\sigma(k),\ell}\right); \\ 0, & \text{otherwise} \end{cases} \\
&\qquad \circ \left(\tau\left[\sigma\right]\right)^{-1} \circ \Delta^{[k\ell]} \\
&= m^{[k\ell]} \circ \begin{cases} P_{\left(\gamma_{1,1},\gamma_{1,2},\ldots,\gamma_{k,\ell}\right)}, & \text{if } \gamma_{i,j} = \theta_{\sigma(i),j} \text{ for all } i,j; \\ 0, & \text{otherwise} \end{cases} \\
&\qquad \circ \left(\tau\left[\sigma\right]\right)^{-1} \circ \Delta^{[k\ell]}
\end{aligned}$$

(since the condition "$\left(\gamma_{1,1},\gamma_{1,2},\ldots,\gamma_{k,\ell}\right) = \left(\theta_{\sigma(1),1},\theta_{\sigma(1),2},\ldots,\theta_{\sigma(k),\ell}\right)$" is equivalent to "$\gamma_{i,j} = \theta_{\sigma(i),j}$ for all i, j"). Thus, we can rewrite (39) as

$$\begin{aligned} p_{\alpha,\sigma} \circ p_{\beta,\tau} &= \sum_{\substack{\gamma_{i,j} \in \mathbb{N} \text{ for all } i \in [k] \text{ and } j \in [\ell]; \\ \gamma_{i,1}+\gamma_{i,2}+\cdots+\gamma_{i,\ell}=\alpha_i \text{ for all } i \in [k]}} \ \sum_{\substack{\theta_{i,j} \in \mathbb{N} \text{ for all } i \in [k] \text{ and } j \in [\ell]; \\ \theta_{1,j}+\theta_{2,j}+\cdots+\theta_{k,j}=\beta_j \text{ for all } j \in [\ell]}} \\ &\qquad m^{[k\ell]} \circ \begin{cases} P_{(\gamma_{1,1},\gamma_{1,2},\ldots,\gamma_{k,\ell})}, & \text{if } \gamma_{i,j} = \theta_{\sigma(i),j} \text{ for all } i, j; \\ 0, & \text{otherwise} \end{cases} \\ &\qquad \circ \left(\tau\left[\sigma\right]\right)^{-1} \circ \Delta^{[k\ell]} \\ &= \sum_{\substack{\gamma_{i,j} \in \mathbb{N} \text{ for all } i \in [k] \text{ and } j \in [\ell]; \\ \gamma_{i,1}+\gamma_{i,2}+\cdots+\gamma_{i,\ell}=\alpha_i \text{ for all } i \in [k]}} \ \sum_{\substack{\theta_{i,j} \in \mathbb{N} \text{ for all } i \in [k] \text{ and } j \in [\ell]; \\ \theta_{1,j}+\theta_{2,j}+\cdots+\theta_{k,j}=\beta_j \text{ for all } j \in [\ell]; \\ \gamma_{i,j}=\theta_{\sigma(i),j} \text{ for all } i,j}} \\ &\qquad m^{[k\ell]} \circ P_{(\gamma_{1,1},\gamma_{1,2},\ldots,\gamma_{k,\ell})} \circ \left(\tau\left[\sigma\right]\right)^{-1} \circ \Delta^{[k\ell]} \end{aligned}$$

(here, we have discarded all the addends that do **not** satisfy $\left(\gamma_{i,j} = \theta_{\sigma(i),j} \text{ for all } i, j\right)$, because these addends are 0). Thus,

$$\begin{aligned} p_{\alpha,\sigma} \circ p_{\beta,\tau} &= \sum_{\substack{\gamma_{i,j} \in \mathbb{N} \text{ for all } i \in [k] \text{ and } j \in [\ell]; \\ \gamma_{i,1}+\gamma_{i,2}+\cdots+\gamma_{i,\ell}=\alpha_i \text{ for all } i \in [k]}} \ \sum_{\substack{\theta_{i,j} \in \mathbb{N} \text{ for all } i \in [k] \text{ and } j \in [\ell]; \\ \theta_{1,j}+\theta_{2,j}+\cdots+\theta_{k,j}=\beta_j \text{ for all } j \in [\ell]; \\ \gamma_{i,j}=\theta_{\sigma(i),j} \text{ for all } i,j}} \underbrace{m^{[k\ell]} \circ P_{(\gamma_{1,1},\gamma_{1,2},\ldots,\gamma_{k,\ell})} \circ \left(\tau\left[\sigma\right]\right)^{-1} \circ \Delta^{[k\ell]}}_{\substack{= p_{(\gamma_{1,1},\gamma_{1,2},\ldots,\gamma_{k,\ell}),\tau[\sigma]} \\ \text{(by (4))}}} \\ &= \sum_{\substack{\gamma_{i,j} \in \mathbb{N} \text{ for all } i \in [k] \text{ and } j \in [\ell]; \\ \gamma_{i,1}+\gamma_{i,2}+\cdots+\gamma_{i,\ell}=\alpha_i \text{ for all } i \in [k]}} \ \sum_{\substack{\theta_{i,j} \in \mathbb{N} \text{ for all } i \in [k] \text{ and } j \in [\ell]; \\ \theta_{1,j}+\theta_{2,j}+\cdots+\theta_{k,j}=\beta_j \text{ for all } j \in [\ell]; \\ \gamma_{i,j}=\theta_{\sigma(i),j} \text{ for all } i,j}} p_{(\gamma_{1,1},\gamma_{1,2},\ldots,\gamma_{k,\ell}),\tau[\sigma]}. \end{aligned} \tag{40}$$

Now, let us simplify the inner sum in (40). If $\delta_{i,j}$ and $\theta_{i,j}$ are nonnegative integers for all $i \in [k]$ and $j \in [\ell]$ that satisfy $\left(\gamma_{i,j} = \theta_{\sigma(i),j} \text{ for all } i, j\right)$, then each $j \in [\ell]$ satisfies

$$\begin{aligned} \theta_{1,j} + \theta_{2,j} + \cdots + \theta_{k,j} &= \theta_{\sigma(1),j} + \theta_{\sigma(2),j} + \cdots + \theta_{\sigma(k),j} \\ &\qquad \left(\text{since } \sigma \in \mathfrak{S}_k \text{ is a bijection from } [k] \text{ to } [k]\right) \\ &= \gamma_{1,j} + \gamma_{2,j} + \cdots + \gamma_{k,j} \end{aligned}$$

(since the condition $\left(\gamma_{i,j} = \theta_{\sigma(i),j} \text{ for all } i, j\right)$ allows us to rewrite each $\theta_{\sigma(i),j}$ as $\gamma_{i,j}$). Thus, the "$\theta_{1,j} + \theta_{2,j} + \cdots + \theta_{k,j}$" under the second summation sign on the right hand side of (40) can be rewritten as "$\gamma_{1,j} + \gamma_{2,j} + \cdots + \gamma_{k,j}$". As a consequence, we can rewrite (40) as

$$\begin{aligned}
& p_{\alpha,\sigma} \circ p_{\beta,\tau} \\
&= \sum_{\substack{\gamma_{i,j} \in \mathbb{N} \text{ for all } i \in [k] \text{ and } j \in [\ell]; \\ \gamma_{i,1}+\gamma_{i,2}+\cdots+\gamma_{i,\ell}=\alpha_i \text{ for all } i \in [k]}} \ \sum_{\substack{\theta_{i,j} \in \mathbb{N} \text{ for all } i \in [k] \text{ and } j \in [\ell]; \\ \gamma_{1,j}+\gamma_{2,j}+\cdots+\gamma_{k,j}=\beta_j \text{ for all } j \in [\ell]; \\ \gamma_{i,j}=\theta_{\sigma(i),j} \text{ for all } i,j}} p_{\left(\gamma_{1,1},\gamma_{1,2},\ldots,\gamma_{k,\ell}\right),\tau[\sigma]} \\
&= \sum_{\substack{\gamma_{i,j} \in \mathbb{N} \text{ for all } i \in [k] \text{ and } j \in [\ell]; \\ \gamma_{i,1}+\gamma_{i,2}+\cdots+\gamma_{i,\ell}=\alpha_i \text{ for all } i \in [k]; \\ \gamma_{1,j}+\gamma_{2,j}+\cdots+\gamma_{k,j}=\beta_j \text{ for all } j \in [\ell];}} \underbrace{\sum_{\substack{\theta_{i,j} \in \mathbb{N} \text{ for all } i \in [k] \text{ and } j \in [\ell]; \\ \gamma_{i,j}=\theta_{\sigma(i),j} \text{ for all } i,j}} p_{\left(\gamma_{1,1},\gamma_{1,2},\ldots,\gamma_{k,\ell}\right),\tau[\sigma]}}_{\substack{=p_{\left(\gamma_{1,1},\gamma_{1,2},\ldots,\gamma_{k,\ell}\right),\tau[\sigma]} \\ \text{(since the condition "}\gamma_{i,j}=\theta_{\sigma(i),j} \text{ for all } i,j\text{"} \\ \text{ensures that } \theta_{i,j}=\gamma_{\sigma^{-1}(i),j} \text{ for all } i,j, \\ \text{and thus uniquely determines the } \theta_{i,j}; \\ \text{hence, the sum has exactly one addend)}}} \\
&= \sum_{\substack{\gamma_{i,j} \in \mathbb{N} \text{ for all } i \in [k] \text{ and } j \in [\ell]; \\ \gamma_{i,1}+\gamma_{i,2}+\cdots+\gamma_{i,\ell}=\alpha_i \text{ for all } i \in [k]; \\ \gamma_{1,j}+\gamma_{2,j}+\cdots+\gamma_{k,j}=\beta_j \text{ for all } j \in [\ell]}} p_{\left(\gamma_{1,1},\gamma_{1,2},\ldots,\gamma_{k,\ell}\right),\tau[\sigma]}.
\end{aligned}$$

This proves Theorem 1.19. □

As we mentioned above, (1) is a particular case of Theorem 1.19 using Proposition 1.13 **(b)**. Likewise, the analogue of (1) for cocommutative bialgebras can be recovered from Theorem 1.19 using Proposition 1.13 **(a)**.

Question 1.35. In [Pang21, Proposition 12], Pang generalized (1) to Hopf algebras with involutions (as long as they are commutative or cocommutative). Is a similar generalization possible for Theorem 1.19?

1.8. Linear independence of $p_{\alpha,\sigma}$'s

Both Theorem 1.19 and Proposition 1.15 hold for arbitrary graded (not necessarily connected) bialgebras. Let us now focus our view on the connected graded bialgebras (which, as we recall, are automatically Hopf algebras).

Using Theorem 1.19 and (7), we can expand any nested convolution-and-composition of $p_{\alpha,\sigma}$'s as a $\mathbf{k}$-linear combination of single $p_{\alpha,\sigma}$'s. Using Proposition 2.5 further below, we can furthermore transform the α's in the resulting combination into actual compositions, not just weak compositions. This allows us to mechanically prove equalities for $p_{\alpha,\sigma}$'s that involve only convolution, composition and $\mathbf{k}$-linear combination and that are supposed to be valid for any connected graded Hopf algebra H. The reason why this works is the following "generic linear independence" theorem:

Theorem 1.36. **(a)** There is a connected graded Hopf algebra H such that the family

$$(p_{\alpha,\sigma})_{\substack{k\in\mathbb{N};\\ \alpha \text{ is a composition of length } k;\\ \sigma\in\mathfrak{S}_k}}$$

(of endomorphisms of H) is $\mathbf{k}$-linearly independent, and such that each H_n is a free $\mathbf{k}$-module.

(b) Let $n \in \mathbb{N}$. Then, there is a connected graded Hopf algebra H such that the family

$$(p_{\alpha,\sigma})_{\substack{k\in\mathbb{N};\\ \alpha \text{ is a composition of length } k \text{ and size } <n;\\ \sigma\in\mathfrak{S}_k}}$$

(of endomorphisms of H) is $\mathbf{k}$-linearly independent, and such that each H_n is a free $\mathbf{k}$-module with a finite basis.

Proof of Theorem 1.36 (sketched). **(b)** Let H be the free $\mathbf{k}$-algebra with generators

$$x_{i,j} \qquad \text{with } i,j \in \mathbb{Z} \text{ satisfying } 1 \leq i < j \leq n.$$

We also set

$$x_{k,k} := 1_H \qquad \text{for each } k \in \{1,2,\ldots,n\}.$$

We make the $\mathbf{k}$-algebra H graded by declaring each $x_{i,j}$ to be homogeneous of degree $j-i$. We define a comultiplication $\Delta : H \to H \otimes H$ on H to be the $\mathbf{k}$-algebra homomorphism that satisfies

$$\Delta\left(x_{i,j}\right) = \sum_{k=i}^{j} x_{i,k} \otimes x_{k,j} \qquad \text{for each } i,j \in \mathbb{Z} \text{ satisfying } 1 \leq i < j \leq n.$$

We define a counit $\epsilon : H \to \mathbf{k}$ in the obvious way to preserve the grading (so that $\epsilon\left(x_{i,j}\right) = 0$ whenever $i < j$). It is easy to see that H is a connected graded $\mathbf{k}$-bialgebra, thus a connected graded Hopf algebra.[9]

It is easy to see that

$$\Delta^{[k]}\left(x_{i,j}\right) = \sum_{i=u_0\leq u_1\leq\cdots\leq u_k=j} x_{u_0,u_1} \otimes x_{u_1,u_2} \otimes \cdots \otimes x_{u_{k-1},u_k}$$

for any $i \leq j$ and any $k \in \mathbb{N}$. Thus, for any $i \leq j$ and any $k \in \mathbb{N}$ and any permutation $\sigma \in \mathfrak{S}_k$, we have

$$\begin{aligned}\left(\sigma^{-1}\circ\Delta^{[k]}\right)\left(x_{i,j}\right) &= \sigma^{-1}\cdot\sum_{i=u_0\leq u_1\leq\cdots\leq u_k=j} x_{u_0,u_1} \otimes x_{u_1,u_2} \otimes \cdots \otimes x_{u_{k-1},u_k} \\ &= \sum_{i=u_0\leq u_1\leq\cdots\leq u_k=j} x_{u_{\sigma(1)-1},u_{\sigma(1)}} \otimes x_{u_{\sigma(2)-1},u_{\sigma(2)}} \otimes \cdots \otimes x_{u_{\sigma(k)-1},u_{\sigma(k)}}\end{aligned}$$

[9]This H resembles certain bialgebras that appear in the literature. In particular, it can be regarded either as a noncommutative version of a reduced incidence algebra of the chain poset with n elements, or as a noncommutative variant of the subalgebra of the Schur algebra corresponding to the upper-triangular matrices with equal numbers on the diagonal. But these connections are tangential for our purpose.

and therefore (by applying the projection P_α to both sides)

$$\left(P_\alpha \circ \sigma^{-1} \circ \Delta^{[k]}\right)\left(x_{i,j}\right) = \sum_{\substack{i=u_0 \leq u_1 \leq \cdots \leq u_k = j; \\ u_{\sigma(i)} - u_{\sigma(i)-1} = \alpha_i \text{ for each } i \in [k]}} x_{u_{\sigma(1)-1}, u_{\sigma(1)}} \otimes x_{u_{\sigma(2)-1}, u_{\sigma(2)}} \otimes \cdots \otimes x_{u_{\sigma(k)-1}, u_{\sigma(k)}}.$$

Hence, for any composition $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k) \in \mathbb{N}^k$ and any $\sigma \in \mathfrak{S}_k$ and any $1 \leq i < j \leq n$ satisfying $j - i = |\alpha|$, we have

$$p_{\alpha,\sigma}\left(x_{i,j}\right) = x_{u_{\sigma(1)-1}, u_{\sigma(1)}} x_{u_{\sigma(2)-1}, u_{\sigma(2)}} \cdots x_{u_{\sigma(k)-1}, u_{\sigma(k)}},$$

where $(u_0 < u_1 < \cdots < u_k)$ is the unique strictly increasing sequence of integers satisfying $u_0 = i$ and $u_k = j$ and $u_{\sigma(i)} - u_{\sigma(i)-1} = \alpha_i$ for all $i \in [k]$ [10]. Hence, for any choice of $1 \leq i < j \leq n$, the images $p_{\alpha,\sigma}\left(x_{i,j}\right)$ as α runs over all compositions of $j - i$ and σ runs over all permutations of $[\ell(\alpha)]$ are distinct monomials[11] and

[10] Why is there a unique sequence with these properties? The easiest way to see this is as follows: Rewrite the condition "$u_{\sigma(i)} - u_{\sigma(i)-1} = \alpha_i$ for all $i \in [k]$" as "$u_j - u_{j-1} = \alpha_{\sigma^{-1}(j)}$ for all $j \in [k]$" (here we just substituted $\sigma^{-1}(j)$ for i). In the latter form, the condition simply dictates the differences between consecutive entries of the sequence $(u_0, u_1, \ldots, u_k)$. Thus, clearly, there exists a unique sequence $(u_0, u_1, \ldots, u_k)$ of integers satisfying this condition and having starting entry $u_0 = i$. Moreover, this sequence will be strictly increasing (since the differences $u_j - u_{j-1} = \alpha_{\sigma^{-1}(j)}$ are positive) and will satisfy

$$u_k - u_0 = \sum_{j=1}^{k} \underbrace{\left(u_j - u_{j-1}\right)}_{=\alpha_{\sigma^{-1}(j)}} \qquad \text{(by the telescope principle)}$$
$$= \sum_{j=1}^{k} \alpha_{\sigma^{-1}(j)} = \alpha_{\sigma^{-1}(1)} + \alpha_{\sigma^{-1}(2)} + \cdots + \alpha_{\sigma^{-1}(k)} = \alpha_1 + \alpha_2 + \cdots + \alpha_k = |\alpha| = j - i$$

and therefore $u_k = j - i + \underbrace{u_0}_{=i} = j - i + i = j$. Hence, it will be a strictly increasing sequence $(u_0 < u_1 < \cdots < u_k)$ of integers satisfying $u_0 = i$ and $u_k = j$ and $u_{\sigma(i)} - u_{\sigma(i)-1} = \alpha_i$ for all $i \in [k]$.

[11] Why are they distinct? Because the noncommutative monomial

$$p_{\alpha,\sigma}\left(x_{i,j}\right) = x_{u_{\sigma(1)-1}, u_{\sigma(1)}} x_{u_{\sigma(2)-1}, u_{\sigma(2)}} \cdots x_{u_{\sigma(k)-1}, u_{\sigma(k)}}$$

allows us to uniquely reconstruct α and σ as follows: Recall that all the indeterminates $x_{p,q}$ with $p < q$ are distinct. The noncommutative monomial $p_{\alpha,\sigma}\left(x_{i,j}\right) = x_{u_{\sigma(1)-1}, u_{\sigma(1)}} x_{u_{\sigma(2)-1}, u_{\sigma(2)}} \cdots x_{u_{\sigma(k)-1}, u_{\sigma(k)}}$ is a product of k such indeterminates (since $u_0 < u_1 < \cdots < u_k$, so that $u_{\sigma(i)-1} < u_{\sigma(i)}$ for each i). Hence, from this product, we can recover the pairs $\left(u_{\sigma(1)-1}, u_{\sigma(1)}\right)$, $\left(u_{\sigma(2)-1}, u_{\sigma(2)}\right)$, $\ldots$, $\left(u_{\sigma(k)-1}, u_{\sigma(k)}\right)$, and thus reconstruct the values $u_{\sigma(1)}, u_{\sigma(2)}, \ldots, u_{\sigma(k)}$. These, in turn, allow us to reconstruct the permutation σ (since $u_0 < u_1 < \cdots < u_k$). Recalling the condition $u_{\sigma(i)} - u_{\sigma(i)-1} = \alpha_i$ once again, we then recover the α_i and thus the composition α.

therefore are **k**-linearly independent. This yields the **k**-linear independence of the family

$$\left(p_{\alpha,\sigma}\right)_{\substack{k\in\mathbb{N};\\ \alpha \text{ is a composition of length } k \text{ and size } s;\\ \sigma\in\mathfrak{S}_k}}$$

for any given $s \in \{0,1,\ldots,n-1\}$. Since each $p_{\alpha,\sigma}$ lies in $\operatorname{End}_{\mathbf{k}}\left(H_{|\alpha|}\right)$, we thus obtain the **k**-linear independence of the entire family

$$\left(p_{\alpha,\sigma}\right)_{\substack{k\in\mathbb{N};\\ \alpha \text{ is a composition of length } k \text{ and size } <n;\\ \sigma\in\mathfrak{S}_k}}$$

(since the sum $\sum\limits_{s=0}^{n-1} \operatorname{End}_{\mathbf{k}}\left(H_s\right)$ is a direct sum). This proves Theorem 1.36 **(b)**.

(a) As for part **(b)**, but remove "$\leq n$" everywhere. □

Theorem 1.36 also has an analogue for graded bialgebras and Hopf algebras without the connectedness requirement:

Theorem 1.37. **(a)** There is a graded Hopf algebra H such that the family

$$\left(p_{\alpha,\sigma}\right)_{\substack{k\in\mathbb{N};\\ \alpha \text{ is a weak composition of length } k;\\ \sigma\in\mathfrak{S}_k}}$$

(of endomorphisms of H) is **k**-linearly independent, and such that each H_n is a free **k**-module.

(b) Let $n \in \mathbb{N}$. Then, there is a graded Hopf algebra H such that the family

$$\left(p_{\alpha,\sigma}\right)_{\substack{k\in\{0,1,\ldots,n-1\};\\ \alpha \text{ is a weak composition of length } k \text{ and size } <n;\\ \sigma\in\mathfrak{S}_k}}$$

(of endomorphisms of H) is **k**-linearly independent, and such that each H_n is a free **k**-module with a finite basis.

Proof idea. **(b)** This is similar to Theorem 1.36 **(b)**, but there are more indices involved. We shall only give a rough outline of the proof.

Let H be the free **k**-algebra with generators

$$x_{i,j}^{p,q} \qquad \text{with } (i,p), (j,q) \in [n]^2 \text{ satisfying } i \leq j \text{ and } p \leq q.$$

(The "p,q" superscript is not an exponent!) Note that we don't set any of the $x_{i,j}^{p,q}$ equal to 1, unlike in the previous proof.

We make the **k**-algebra H graded by declaring each $x_{i,j}^{p,q}$ to be homogeneous of degree $j-i$. We define a comultiplication $\Delta : H \to H \otimes H$ on H to be the **k**-algebra homomorphism that satisfies

$$\Delta\left(x_{i,j}^{p,q}\right) = \sum_{k=i}^{j} \sum_{r=p}^{q} x_{i,k}^{p,r} \otimes x_{k,j}^{r,q} \qquad \text{for all generators } x_{i,j}^{p,q}.$$

We define a counit $\epsilon : H \to \mathbf{k}$ by

$$\epsilon\left(x_{i,j}^{p,q}\right) = \begin{cases} 1, & \text{if } (i,p) = (j,q); \\ 0, & \text{if } (i,p) \neq (j,q). \end{cases}$$

It is easy to see that H is a graded $\mathbf{k}$-bialgebra. It is not connected (with our grading) and not a Hopf algebra (since its grouplike elements $x_{i,i}^{p,p}$ have no inverses); we will say more about this later.

For any generator $x_{i,j}^{p,q}$ of H and any $k \in \mathbb{N}$, we can show that

$$\Delta^{[k]}\left(x_{i,j}^{p,q}\right) = \sum_{\substack{i=u_0 \leq u_1 \leq \cdots \leq u_k = j; \\ p = v_0 \leq v_1 \leq \cdots \leq v_k = q}} x_{u_0,u_1}^{v_0,v_1} \otimes x_{u_1,u_2}^{v_1,v_2} \otimes \cdots \otimes x_{u_{k-1},u_k}^{v_{k-1},v_k}.$$

Now, fix $i \leq j$ in $[n]$. As we just showed, we have

$$\Delta^{[k]}\left(x_{i,j}^{1,n}\right) = \sum_{\substack{i=u_0 \leq u_1 \leq \cdots \leq u_k = j; \\ 1 = v_0 \leq v_1 \leq \cdots \leq v_k = n}} x_{u_0,u_1}^{v_0,v_1} \otimes x_{u_1,u_2}^{v_1,v_2} \otimes \cdots \otimes x_{u_{k-1},u_k}^{v_{k-1},v_k}$$

for any $k \in \mathbb{N}$. Thus, by a similar argument as in the proof of Theorem 1.36 **(b)**, we can show that for any weak composition $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k) \in \mathbb{N}^k$ and any $\sigma \in \mathfrak{S}_k$ and any $1 \leq i \leq j \leq n$ satisfying $j - i = |\alpha|$, we have

$$p_{\alpha,\sigma}\left(x_{i,j}^{1,n}\right) = \sum_{\substack{i=u_0 \leq u_1 \leq \cdots \leq u_k = j; \\ 1 = v_0 \leq v_1 \leq \cdots \leq v_k = n; \\ u_{\sigma(i)} - u_{\sigma(i)-1} = \alpha_i \text{ for each } i \in [k]}} \left(x_{u_0,u_1}^{v_0,v_1} x_{u_1,u_2}^{v_1,v_2} \cdots x_{u_{k-1},u_k}^{v_{k-1},v_k}\right) \leftharpoonup \sigma,$$

where the "$\leftharpoonup$" arrow denotes the action of a permutation on the positions in a monomial (i.e., we set $(y_1 y_2 \cdots y_k) \leftharpoonup \sigma := y_{\sigma(1)} y_{\sigma(2)} \cdots y_{\sigma(k)}$ whenever $\sigma \in \mathfrak{S}_k$ and whenever $y_1, y_2, \ldots, y_k$ are some of the generators of H).

We now claim that these images $p_{\alpha,\sigma}\left(x_{i,j}^{1,n}\right)$ – where k ranges over $\{0, 1, \ldots, n-1\}$, where α ranges over the weak compositions of length k and size $j - i$, and where σ ranges over $\mathfrak{S}_k$ – are $\mathbf{k}$-linearly independent. Unfortunately, they no longer are single monomials, but hope is not lost: Each of them contains a monomial (with coefficient 1) that is not contained in any of the others (which ensures that any nontrivial $\mathbf{k}$-linear combination of these images will have nonzero coefficients in front of some of these monomials). To wit, $p_{\alpha,\sigma}\left(x_{i,j}^{1,n}\right)$ contains (with coefficient 1) the monomial

$$\left(x_{u_0,u_1}^{1,2} x_{u_1,u_2}^{2,3} \cdots x_{u_{k-1},u_{k-2}}^{k-1,k} x_{u_{k-1},u_k}^{k,n}\right) \leftharpoonup \sigma$$

for the unique weakly increasing sequence $(u_0 \leq u_1 \leq \cdots \leq u_k)$ of integers satisfying $u_0 = i$ and $u_k = j$ and $u_{\sigma(i)} - u_{\sigma(i)-1} = \alpha_i$ for all $i \in [k]$ (this monomial is obtained by taking $v_i = i + 1$ for each $i \in \{0, 1, \ldots, k-1\}$ and $v_k = n$), and this monomial is not contained in any other $p_{\beta,\tau}\left(x_{i,j}^{1,n}\right)$ with $(\beta, \tau) \neq (\alpha, \sigma)$ (because reading its superscripts in order reveals the permutation σ, and then undoing the σ-action and reading the subscripts in order reveals the u_i's and thus the weak composition α). Thus, the images $p_{\alpha,\sigma}\left(x_{i,j}^{1,n}\right)$ are $\mathbf{k}$-linearly independent. Hence, so are the maps $p_{\alpha,\sigma}$ themselves. From here, we can finish the proof of Theorem 1.37 **(b)** just as in the proof of Theorem 1.36, except that our H is just a bialgebra, not a Hopf algebra.

Now how do we make H into a Hopf algebra? The simplest way would be to factor out the ideal generated by all differences of the form $x_{i,i}^{p,p} - 1$; in other words, we replace the indeterminates $x_{i,i}^{p,p}$ by copies of 1. With this change, our graded $\mathbf{k}$-bialgebra is still not connected; however, it would be connected with a different grading (namely, with the grading in which each $x_{i,j}^{p,q}$ is homogeneous of degree $(j-i) + (q-p)$), and thus is a Hopf algebra. The antipode is graded with respect to both gradings (this follows, e.g., from [GriRei20, Exercise 1.4.29(e)]). Our above argument for the linear independence of the $p_{\alpha,\sigma}$'s still applies with minor changes, which we leave to the reader to perform.

(a) As for part **(b)**, but remove "$\leq n$" everywhere and replace "$[n]$" by "$\mathbb{Z}$". $\square$

1.9. The tensor product formula

We shall now connect the $p_{\alpha,\sigma}$ operators on different graded bialgebras. For this, we need a piece of notation:

Convention 1.38. The endomorphism $p_{\alpha,\sigma} : H \to H$ defined for a given graded $\mathbf{k}$-bialgebra H shall be denoted by "$p_{\alpha,\sigma}$ for H" when we want to stress its dependence on H. Thus, for instance, if K is another graded $\mathbf{k}$-bialgebra, then the analogous endomorphism $p_{\alpha,\sigma} : K \to K$ will be denoted by "$p_{\alpha,\sigma}$ for K".

Using this convention, we can now describe the endomorphism $p_{\alpha,\sigma}$ for a tensor product of two $\mathbf{k}$-bialgebras:

Proposition 1.39. Let H and G be two graded bialgebras. Let $\alpha \in \mathbb{N}^k$ be a weak composition of length k, and let $\sigma \in \mathfrak{S}_k$ be a permutation. Then,

$$(p_{\alpha,\sigma} \text{ for } H \otimes G) = \sum_{\substack{\beta,\gamma \in \mathbb{N}^k; \\ \beta + \gamma = \alpha}} \left(p_{\beta,\sigma} \text{ for } H\right) \otimes \left(p_{\gamma,\sigma} \text{ for } G\right)$$

as endomorphisms of $H \otimes G$. Here, $\beta + \gamma$ denotes the entrywise sum of β and γ (that is, if $\beta = (\beta_1, \beta_2, \ldots, \beta_k)$ and $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_k)$, then $\beta + \gamma := (\beta_1 + \gamma_1,\ \beta_2 + \gamma_2,\ \ldots,\ \beta_k + \gamma_k)$).

Proof sketch. We shall use the Sweedler notation again.

The definition of the grading on $H \otimes G$ shows that $(H \otimes G)_a = \bigoplus_{\substack{b,c \in \mathbb{N}; \\ b+c=a}} H_b \otimes G_c$ for any $a \in \mathbb{N}$. Hence,

$$p_a (u \otimes v) = \sum_{\substack{b,c \in \mathbb{N}; \\ b+c=a}} p_b (u) \otimes p_c (v) \tag{41}$$

for any $u \in H$, any $v \in G$ and any $a \in \mathbb{N}$.

Write the weak composition $\alpha \in \mathbb{N}^k$ as $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$. Consider a pure tensor $h \otimes g \in H \otimes G$. Then, Remark 1.9 yields

$$\begin{aligned}
& p_{\alpha,\sigma}(h \otimes g) \\
&= \sum_{(h\otimes g)} p_{\alpha_1}\left((h\otimes g)_{(\sigma(1))}\right) p_{\alpha_2}\left((h\otimes g)_{(\sigma(2))}\right) \cdots p_{\alpha_k}\left((h\otimes g)_{(\sigma(k))}\right) \\
&= \sum_{(h\otimes g)} \prod_{i=1}^{k} p_{\alpha_i}\left(\underbrace{(h\otimes g)_{(\sigma(i))}}_{=h_{(\sigma(i))}\otimes g_{(\sigma(i))}}\right) \\
&= \sum_{(h),\ (g)} \prod_{i=1}^{k} \underbrace{p_{\alpha_i}\left(h_{(\sigma(i))} \otimes g_{(\sigma(i))}\right)}_{\substack{=\sum_{\substack{b,c\in\mathbb{N};\\ b+c=\alpha_i}} p_b\left(h_{(\sigma(i))}\right)\otimes p_c\left(g_{(\sigma(i))}\right) \\ \text{(by (41))}}} \\
&= \sum_{(h),\ (g)} \prod_{i=1}^{k} \sum_{\substack{b,c\in\mathbb{N};\\ b+c=\alpha_i}} p_b\left(h_{(\sigma(i))}\right) \otimes p_c\left(g_{(\sigma(i))}\right) \\
&= \sum_{(h),\ (g)} \sum_{\substack{\beta_i,\gamma_i\in\mathbb{N} \text{ for all } i\in[k];\\ \beta_i+\gamma_i=\alpha_i \text{ for all } i\in[k]}} \prod_{i=1}^{k} \left(p_{\beta_i}\left(h_{(\sigma(i))}\right) \otimes p_{\gamma_i}\left(g_{(\sigma(i))}\right)\right) \qquad \text{(by the product rule)} \\
&= \underbrace{\sum_{\substack{\beta_i,\gamma_i\in\mathbb{N} \text{ for all } i\in[k];\\ \beta_i+\gamma_i=\alpha_i \text{ for all } i\in[k]}}}_{=\sum_{\substack{\beta=(\beta_1,\beta_2,\ldots,\beta_k)\in\mathbb{N}^k \text{ and}\\ \gamma=(\gamma_1,\gamma_2,\ldots,\gamma_k)\in\mathbb{N}^k;\\ \beta+\gamma=\alpha}}} \sum_{(h),\ (g)} \underbrace{\prod_{i=1}^{k} \left(p_{\beta_i}\left(h_{(\sigma(i))}\right) \otimes p_{\gamma_i}\left(g_{(\sigma(i))}\right)\right)}_{=\left(\prod_{i=1}^{k} p_{\beta_i}\left(h_{(\sigma(i))}\right)\right)\otimes\left(\prod_{i=1}^{k} p_{\gamma_i}\left(g_{(\sigma(i))}\right)\right)} \\
&= \sum_{\substack{\beta=(\beta_1,\beta_2,\ldots,\beta_k)\in\mathbb{N}^k \text{ and}\\ \gamma=(\gamma_1,\gamma_2,\ldots,\gamma_k)\in\mathbb{N}^k;\\ \beta+\gamma=\alpha}} \sum_{(h),\ (g)} \left(\prod_{i=1}^{k} p_{\beta_i}\left(h_{(\sigma(i))}\right)\right) \otimes \left(\prod_{i=1}^{k} p_{\gamma_i}\left(g_{(\sigma(i))}\right)\right) \\
&= \sum_{\substack{\beta=(\beta_1,\beta_2,\ldots,\beta_k)\in\mathbb{N}^k \text{ and}\\ \gamma=(\gamma_1,\gamma_2,\ldots,\gamma_k)\in\mathbb{N}^k;\\ \beta+\gamma=\alpha}} \underbrace{\left(\sum_{(h)} \prod_{i=1}^{k} p_{\beta_i}\left(h_{(\sigma(i))}\right)\right)}_{\substack{=p_{\beta,\sigma}(h)\\ \text{(by Remark 1.9)}}} \otimes \underbrace{\left(\sum_{(g)} \prod_{i=1}^{k} p_{\gamma_i}\left(g_{(\sigma(i))}\right)\right)}_{\substack{=p_{\gamma,\sigma}(g)\\ \text{(by Remark 1.9)}}}
\end{aligned}$$

$$= \sum_{\substack{\beta=(\beta_1,\beta_2,\ldots,\beta_k)\in\mathbb{N}^k \text{ and} \\ \gamma=(\gamma_1,\gamma_2,\ldots,\gamma_k)\in\mathbb{N}^k; \\ \beta+\gamma=\alpha}} p_{\beta,\sigma}(h) \otimes p_{\gamma,\sigma}(g) = \sum_{\substack{\beta,\gamma\in\mathbb{N}^k; \\ \beta+\gamma=\alpha}} p_{\beta,\sigma}(h) \otimes p_{\gamma,\sigma}(g)$$

$$= \left(\sum_{\substack{\beta,\gamma\in\mathbb{N}^k; \\ \beta+\gamma=\alpha}} \left(p_{\beta,\sigma} \text{ for } H\right) \otimes \left(p_{\gamma,\sigma} \text{ for } G\right) \right) (h \otimes g).$$

Hence, the two **k**-linear maps

$$\left(p_{\alpha,\sigma} \text{ for } H \otimes G\right) \text{ and } \sum_{\substack{\beta,\gamma\in\mathbb{N}^k; \\ \beta+\gamma=\alpha}} \left(p_{\beta,\sigma} \text{ for } H\right) \otimes \left(p_{\gamma,\sigma} \text{ for } G\right)$$

agree on each pure tensor. Therefore, they are identical. This proves Proposition 1.39. □

1.10. The dual formula

Graded bialgebras often (not always) have duals. Indeed, if $H = \bigoplus_{n\in\mathbb{N}} H_n$ is a graded **k**-bialgebra whose all graded components H_n are finite free[12] **k**-modules, then its graded dual $H^o := \bigoplus_{n\in\mathbb{N}} (H_n)^*$ itself becomes a graded **k**-bialgebra (see [GriRei20, §1.6]). Now we claim (using Convention 1.38 again):

Proposition 1.40. Let H be a graded **k**-bialgebra whose all graded components are finite free **k**-modules. Consider its graded dual H^o. Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$ be a weak composition of length k, and let $\sigma \in \mathfrak{S}_k$ be a permutation. Then,

$$\left(p_{\alpha,\sigma} \text{ for } H^o\right) = \underbrace{\left(p_{\alpha\cdot\sigma^{-1},\ \sigma^{-1}} \text{ for } H\right)^*}_{\text{restricted to } H^o}$$

as endomorphisms of H^o. Here, "$\alpha \cdot \sigma^{-1}$" is understood using the right action of $\mathfrak{S}_k$ on $\mathbb{N}^k$ defined in (8).

Proof sketch. By definition, $\left(p_{\alpha\cdot\sigma^{-1},\ \sigma^{-1}} \text{ for } H\right) = m_H^{[k]} \circ P_{\alpha\cdot\sigma^{-1}} \circ \underbrace{\left(\sigma^{-1}\right)^{-1}}_{=\sigma} \circ \Delta_H^{[k]} = m_H^{[k]} \circ P_{\alpha\cdot\sigma^{-1}} \circ \sigma \circ \Delta_H^{[k]}$, so that

$$\begin{aligned} \left(p_{\alpha\cdot\sigma^{-1},\ \sigma^{-1}} \text{ for } H\right)^* &= \left(m_H^{[k]} \circ P_{\alpha\cdot\sigma^{-1}} \circ \sigma \circ \Delta_H^{[k]}\right)^* \\ &= \left(\Delta_H^{[k]}\right)^* \circ \sigma^* \circ \left(P_{\alpha\cdot\sigma^{-1}}\right)^* \circ \left(m_H^{[k]}\right)^*. \end{aligned} \tag{42}$$

[12] "Finite free" means "free of finite rank", i.e., "has a finite basis".

However, the definition of the bialgebra structure on the graded dual H^o shows that $\left(\Delta_H^{[k]}\right)^* = m_{H^o}^{[k]}$ and $\left(m_H^{[k]}\right)^* = \Delta_{H^o}^{[k]}$. Moreover, basic linear algebra (really just combinatorics of indices in tensor products) shows that $\sigma^* = \sigma^{-1}$ and that $\left(P_{\alpha\cdot\sigma^{-1}}\right)^* = P_{\alpha\cdot\sigma^{-1}}$ (since $\left(P_\beta\right)^* = P_\beta$ for any weak composition β). Using these four equalities, we can rewrite (42) as

$$\begin{aligned}\left(p_{\alpha\cdot\sigma^{-1},\,\sigma^{-1}} \text{ for } H\right)^* &= m_{H^o}^{[k]} \circ \underbrace{\sigma^{-1} \circ P_{\alpha\cdot\sigma^{-1}}}_{\substack{=P_{\alpha\cdot\sigma^{-1}\cdot\left(\sigma^{-1}\right)^{-1}}\circ\sigma^{-1} \\ \text{(by (10))}}} \circ \Delta_{H^o}^{[k]} \\ &= m_{H^o}^{[k]} \circ \underbrace{P_{\alpha\cdot\sigma^{-1}\cdot\left(\sigma^{-1}\right)^{-1}}}_{=P_\alpha} \circ \sigma^{-1} \circ \Delta_{H^o}^{[k]} \\ &= m_{H^o}^{[k]} \circ P_\alpha \circ \sigma^{-1} \circ \Delta_{H^o}^{[k]} = \left(p_{\alpha,\sigma} \text{ for } H^o\right)\end{aligned}$$

(again by the definition of $p_{\alpha,\sigma}$). This proves Proposition 1.40. □

1.11. Have we found them all?

Now let H be a connected graded bialgebra. As we already mentioned, each $p_{\alpha,\sigma}$ belongs to the **k**-module $\mathbf{E}(H)$ of all graded **k**-module endomorphisms of H that annihilate all but finitely many degrees of H. Thus, the same holds for any **k**-linear combination of the $p_{\alpha,\sigma}$. The fact that each $p_{\alpha,\sigma}$ annihilates all but the $|\alpha|$-th graded component of H allows us to form infinite **k**-linear combinations $\sum\limits_{k\in\mathbb{N}} \sum\limits_{\alpha\in\{1,2,3,\ldots\}^k} \sum\limits_{\sigma\in\mathfrak{S}_k} \lambda_{\alpha,\sigma} p_{\alpha,\sigma}$ of these $p_{\alpha,\sigma}$ as well. Each such combination belongs to $\operatorname{End}_{\mathrm{gr}} H$ (although not to $\mathbf{E}(H)$ any more) and (since it is natural in H) is therefore a natural graded **k**-module endomorphism of H defined for any connected graded bialgebra H.

Question 1.41. Are these combinations the only natural graded **k**-module endomorphisms of H defined for any connected graded bialgebra H ?

In other words: Let g be a natural graded **k**-module endomorphism on the category of connected graded **k**-Hopf algebras. (That is, for each connected graded Hopf algebra H, we have a graded **k**-module endomorphism g_H, and each graded Hopf algebra morphism $\varphi : H \to H'$ gives a commutative diagram.) Is it true that g is an infinite **k**-linear combination of $p_{\alpha,\sigma}$'s?

I have so far been unable to answer this question, for lack of convenient free objects in the relevant category. Nevertheless, I suspect that the answer is positive (i.e., every natural endomorphism is an infinite **k**-linear combination of $p_{\alpha,\sigma}$'s), at least when **k** is a field of characteristic 0.

Question 1.41 can also be asked without the first "graded". That is, we can ask about natural **k**-module endomorphisms of H that are not necessarily graded.

A similar question can be asked for general (as opposed to connected) graded bialgebras. However, it requires some more care in its formulation, as not every infinite **k**-linear combination $\sum_{k\in\mathbb{N}} \sum_{\alpha\in\mathbb{N}^k} \sum_{\sigma\in\mathfrak{S}_k} \lambda_{\alpha,\sigma} p_{\alpha,\sigma}$ is well-defined (and we cannot restrict the second sum to the $\alpha \in \{1,2,3,\ldots\}^k$ only).

2. The combinatorial Hopf algebra PNSym

Let us now recall the connected graded Hopf algebra NSym of noncommutative symmetric functions (introduced in [GKLLRT94], recently exposed in [GriRei20, §5.4] and [Meliot17, Definition 6.1][13]). As a **k**-algebra, it is free with countably many generators $\mathbf{H}_1, \mathbf{H}_2, \mathbf{H}_3, \ldots$, and its comultiplication is given by $\Delta(\mathbf{H}_m) = \sum_{i=0}^{m} \mathbf{H}_i \otimes \mathbf{H}_{m-i}$, where $\mathbf{H}_0 := 1$. (We are using the notation $\mathbf{H}_k$ for the k-th complete homogeneous noncommutative symmetric function in NSym, which is denoted H_k in [GriRei20, Theorem 5.4.2], and denoted S_k in [GKLLRT94, (22)] and [Meliot17].)

For any weak composition $(\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{N}^n$, we set

$$\mathbf{H}_{(\alpha_1,\alpha_2,\ldots,\alpha_n)} := \mathbf{H}_{\alpha_1}\mathbf{H}_{\alpha_2}\cdots\mathbf{H}_{\alpha_n} \in \mathrm{NSym}.$$

Thus, the family $(\mathbf{H}_\alpha)_{\alpha \text{ is a composition}}$ is a basis of the **k**-module NSym. (Note that $\mathbf{H}_\alpha$ is called S^α in [GKLLRT94].)

An *internal product* $*$ is defined on NSym in [GKLLRT94, §5.1]. It is explicitly given by the formula

$$\mathbf{H}_\alpha * \mathbf{H}_\beta = \sum_{\substack{\gamma_{i,j}\in\mathbb{N} \text{ for all } i\in[k] \text{ and } j\in[\ell];\\ \gamma_{i,1}+\gamma_{i,2}+\cdots+\gamma_{i,\ell}=\alpha_i \text{ for all } i\in[k];\\ \gamma_{1,j}+\gamma_{2,j}+\cdots+\gamma_{k,j}=\beta_j \text{ for all } j\in[\ell]}} \mathbf{H}_{(\gamma_{1,1},\gamma_{1,2},\ldots,\gamma_{k,\ell})}$$

for all compositions α and β (see [GKLLRT94, Proposition 5.1]). As mentioned in the introduction, Patras's composition formula (1) is structurally identical to this expression. This shows that any cocommutative connected graded bialgebra H is a module over the non-unital algebra $\mathrm{NSym}^{(2)}$ of noncommutative symmetric functions equipped with its internal product. (Each complete noncommutative symmetric function $\mathbf{H}_\alpha$ acts as the operator $p_{\alpha,\mathrm{id}}$ on H.)

It is natural to ask whether a similar construction can be made for any connected graded bialgebra H using Theorem 1.19. Thus, we are looking for a non-unital algebra that contains elements $F_{\alpha,\sigma}$ for all compositions $\alpha \in \mathbb{N}^k$ and all permutations

[13] These three references give slightly different definitions of this Hopf algebra NSym, but all these definitions are easily seen to be equivalent (e.g., using [GKLLRT94, Note 3.5, Proposition 3.8, Proposition 3.9] and [Meliot17, Proposition 6.2, Proposition 6.3]). They also denote it by different symbols: It is called **Sym** in [GKLLRT94], called NSym in [GriRei20, §5.4], and called NCSym in [Meliot17, Definition 6.1] (a name that means a different Hopf algebra in most of the literature).

$\sigma \in \mathfrak{S}_k$, and which acts on any connected graded bialgebra H by having each $F_{\alpha,\sigma}$ act as the operator $p_{\alpha,\sigma}$.

In this section, we shall construct such an algebra – which I name $\operatorname{PNSym}^{(2)}$ for "permuted noncommutative symmetric functions". Besides having an "internal product" $*$, it has an "external product" $\cdot$ (corresponding to the convolution of operators on H) and a coproduct Δ (corresponding to acting on a tensor product of bialgebras).

2.1. Mopiscotions and weak mopiscotions

To construct this algebra, we need to make some implicit things explicit and introduce some more notation:

Definition 2.1. A *mopiscotion* (short for "permuted composition") is a pair (α, σ), where α is a composition of length k (for some $k \in \mathbb{N}$) and σ is a permutation in $\mathfrak{S}_k$.

Definition 2.2. A *weak mopiscotion* is a pair (α, σ), where α is a weak composition of length k (for some $k \in \mathbb{N}$) and σ is a permutation in $\mathfrak{S}_k$.

Clearly, any mopiscotion is a weak mopiscotion. In the reverse direction, we can transform any weak mopiscotion (α, σ) into a mopiscotion (β, τ) (non-injectively) by removing the zeroes from the composition α while appropriately reducing the permutation σ as well. To give a precise definition, we need the concept of *standardization* ([GriRei20, Definition 5.3.3]):

Definition 2.3. Let $w = (w_1, w_2, \ldots, w_h)$ be any finite list of integers (or of elements of any totally ordered set). Then, the *standardization* of w is defined as the unique permutation $\sigma \in \mathfrak{S}_h$ with the property that for every two elements a and b of $[h]$ satisfying $a < b$, we have the equivalence $(\sigma(a) < \sigma(b)) \iff (w_a \leq w_b)$.

Roughly speaking, the standardization of the list w is the permutation whose values are in the same relative order as the entries of w; when w has equal entries, we count entries lying further left as being smaller. For example, the two lists $(5, 7, 1, 8, 2)$ and $(2, 2, 0, 2, 1)$ have the same standardization, namely the permutation $\sigma \in \mathfrak{S}_5$ with one-line notation $[3, 4, 1, 5, 2]$. (In this paper, we will only use standardizations of lists that consist of distinct entries. Thus, any subtleties related to equal entries can be ignored, and we can simply compute the standardization of a list by replacing its smallest entry by 1, its second-smallest entry by 2, and so on, and finally read the resulting list as the one-line notation of a permutation.)

Definition 2.4. Let (α, σ) be a weak mopiscotion, with $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$ and $\sigma \in \mathfrak{S}_k$. Let $(j_1 < j_2 < \cdots < j_h)$ be the list of all elements i of $[k]$ satisfying $\alpha_i \neq 0$, in increasing order. Let $\tau \in \mathfrak{S}_h$ be the standardization of the list $(\sigma(j_1), \sigma(j_2), \ldots, \sigma(j_h))$. Let $\operatorname{red}\alpha$ denote the composition $(\alpha_{j_1}, \alpha_{j_2}, \ldots, \alpha_{j_h})$ (which consists of all nonzero entries of α). Then, we define $\operatorname{red}(\alpha, \sigma)$ to be the mopiscotion $(\operatorname{red}\alpha, \tau)$. We call $\operatorname{red}(\alpha, \sigma)$ the *reduction* of (α, σ).

For example,

$$\operatorname{red}((3,0,1,2,0), [4,5,1,3,2]) = ((3,1,2), [3,1,2]),$$

where the square brackets indicate a permutation written in one-line notation. For another example,

$$\operatorname{red}((3,0,1,2,0), [4,1,3,2,5]) = ((3,1,2), [3,2,1]).$$

Clearly, if (α, σ) is a mopiscotion (i.e., if all entries of α are nonzero), then $\operatorname{red}(\alpha, \sigma) = (\alpha, \sigma)$.

Mopiscotions have already appeared under the guise of "weighted permutations" in work by Foissy and Patras [FoiPat13]. Indeed, the set $\mathcal{S}$ defined in [FoiPat13, §3] can be identified with the set of all mopiscotions, as long as we translate each pair $(\sigma, d) \in \mathfrak{S}_k \times \operatorname{Hom}([k], \mathbb{N}^*) \subseteq \mathcal{S}$ (the notations used here are those of [FoiPat13]) into the mopiscotion (δ, σ), where $\delta := (d(1), d(2), \ldots, d(k))$. These weighted permutations $(\sigma, d) \in \mathcal{S}$ are used in [FoiPat13] to define certain linear endomorphisms of a shuffle Hopf algebra, and might be generalizable to arbitrary graded bialgebras with a dendriform structure, but do not agree with our maps $p_{\alpha,\sigma}$.

Let us now return to our maps $p_{\alpha,\sigma}$. If H is connected, we can dispense with the ones in which α contains zeroes, since there is a way to reduce all such $p_{\alpha,\sigma}$ to the ones where α is a proper composition (i.e., contains no zeroes):

Proposition 2.5. Let H be a connected graded bialgebra. Let (α, σ) be a weak mopiscotion, and let $(\beta, \tau) = \operatorname{red}(\alpha, \sigma)$. Then,

$$p_{\alpha,\sigma} = p_{\beta,\tau}.$$

Proof idea. This is easy if one does not try to be rigorous. Here is a handwavy proof using Sweedler notation:

The connectedness of H yields $p_0(h) = \epsilon(h) \cdot 1_H$ for each $h \in H$. Hence, in the formula

$$p_{\alpha,\sigma}(x) = \sum_{(x)} p_{\alpha_1}\left(x_{(\sigma(1))}\right) p_{\alpha_2}\left(x_{(\sigma(2))}\right) \cdots p_{\alpha_k}\left(x_{(\sigma(k))}\right)$$

(from Remark 1.9), all the factors $p_{\alpha_i}\left(x_{(\sigma(i))}\right)$ with $\alpha_i = 0$ can be rewritten as $\epsilon\left(x_{(\sigma(i))}\right)$ (the 1_H gets swallowed by the product) and thus can be removed completely (using the $\sum_{(h)} \epsilon\left(h_{(1)}\right) h_{(2)} = \sum_{(h)} h_{(1)} \epsilon\left(h_{(2)}\right) = h$ axiom of a coalgebra), as long as we remember to adjust the permutation σ accordingly (removing its value $\sigma(i)$ and decreasing all values larger than $\sigma(i)$ by 1). At the end of this process, we end up with $p_{\beta,\tau}(x)$. This shows that $p_{\alpha,\sigma}(x) = p_{\beta,\tau}(x)$ for each $x \in H$. Proposition 2.5 follows.

A more rigorous version of this proof can be found in the Appendix (Section A). □

2.2. PNSym

We can now define the combinatorial Hopf algebra that will occupy us for the rest of this work:

Definition 2.6. Let PNSym be the free **k**-module with basis $(F_{\alpha,\sigma})_{(\alpha,\sigma) \text{ is a mopiscotion}}$.

For any weak mopiscotion (α, σ), we set

$$F_{\alpha,\sigma} := F_{\beta,\tau},$$

where $(\beta, \tau) = \operatorname{red}(\alpha, \sigma)$.

Define two multiplications on PNSym: one "external multiplication" (which mirrors convolution of $p_{\alpha,\sigma}$'s as expressed in Proposition 1.15) given by

$$F_{\alpha,\sigma} \cdot F_{\beta,\tau} = F_{\alpha\beta,\sigma\oplus\tau};$$

and another "internal multiplication" (which mirrors composition of $p_{\alpha,\sigma}$'s as expressed in Theorem 1.19) given by

$$F_{\alpha,\sigma} * F_{\beta,\tau} = \sum_{\substack{\gamma_{i,j}\in\mathbb{N} \text{ for all } i\in[k] \text{ and } j\in[\ell];\\ \gamma_{i,1}+\gamma_{i,2}+\cdots+\gamma_{i,\ell}=\alpha_i \text{ for all } i\in[k];\\ \gamma_{1,j}+\gamma_{2,j}+\cdots+\gamma_{k,j}=\beta_j \text{ for all } j\in[\ell]}} F_{(\gamma_{1,1},\gamma_{1,2},\ldots,\gamma_{k,\ell}),\tau[\sigma]}$$

(where $\alpha \in \mathbb{N}^k$ and $\beta \in \mathbb{N}^\ell$). Also, we define a comultiplication $\Delta : \text{PNSym} \to \text{PNSym} \otimes \text{PNSym}$ on PNSym by

$$\Delta(F_{\alpha,\sigma}) = \sum_{\substack{\beta,\gamma \text{ weak compositions;}\\ \text{entrywise sum } \beta+\gamma=\alpha}} F_{\beta,\sigma} \otimes F_{\gamma,\sigma}$$

(mirroring the formula from Proposition 1.39).

We also equip the **k**-module PNSym with a grading by letting each $F_{\alpha,\sigma}$ be homogeneous of degree $|\alpha|$.

These operations on PNSym behave as nicely as the analogous operations on NSym:

Theorem 2.7. The $\mathbf{k}$-module PNSym becomes a connected graded cocommutative Hopf algebra when equipped with the external multiplication $\cdot$, and a (non-graded) non-unital bialgebra when equipped with the internal multiplication $*$. In particular, both multiplications are associative.

There are two ways to prove this. I shall very briefly outline both:

First proof idea for Theorem 2.7. Most claims can be derived from properties of the operators $p_{\alpha,\sigma}$, using the H from Theorem 1.36 **(a)** as a faithful representation.

For an example, let us prove that the internal multiplication $*$ on PNSym is associative.

Let H be any connected graded $\mathbf{k}$-bialgebra. Let $\text{ev}_H : \text{PNSym} \to \text{End}\, H$ be the $\mathbf{k}$-linear map that sends any $F_{\alpha,\sigma}$ to the operator $p_{\alpha,\sigma} \in \text{End}\, H$ for any mopiscotion (α, σ). Note that

$$\text{ev}_H (F_{\alpha,\sigma}) = p_{\alpha,\sigma} \tag{43}$$

is true not only for all mopiscotions (α, σ), but also for all weak mopiscotions (α, σ) (because if (α, σ) is any weak mopiscotion, and if $(\beta, \tau) = \text{red}(\alpha, \sigma)$, then $p_{\alpha,\sigma} = p_{\beta,\tau}$ and $F_{\alpha,\sigma} = F_{\beta,\tau}$).

Now, let H be the connected graded Hopf algebra H from Theorem 1.36 **(a)**. Then, Theorem 1.36 **(a)** says that the family $(p_{\alpha,\sigma})_{(\alpha,\sigma) \text{ is a mopiscotion}}$ is $\mathbf{k}$-linearly independent. Hence, the linear map ev_H is injective.

The formula for $F_{\alpha,\sigma} * F_{\beta,\tau}$ that we used to define the internal multiplication $*$ is very similar to the formula for $p_{\alpha,\sigma} \circ p_{\beta,\tau}$ in Theorem 1.19. In view of (43), this entails that

$$\text{ev}_H \left(F_{\alpha,\sigma} * F_{\beta,\tau}\right) = p_{\alpha,\sigma} \circ p_{\beta,\tau} = \text{ev}_H (F_{\alpha,\sigma}) \circ \text{ev}_H \left(F_{\beta,\tau}\right)$$

for any two mopiscotions (α, σ) and (β, τ). By bilinearity, this entails that

$$\text{ev}_H (f * g) = (\text{ev}_H f) \circ (\text{ev}_H g)$$

for any $f, g \in \text{PNSym}$. Thus, the injective $\mathbf{k}$-linear map $\text{ev}_H : \text{PNSym} \to \text{End}\, H$ embeds the $\mathbf{k}$-module PNSym with its binary operation $*$ into the algebra $\text{End}\, H$ with its binary operation $\circ$. Since the latter operation $\circ$ is associative, it thus follows that the former operation $*$ is associative as well.

Similarly, we can show that the operation $\cdot$ on PNSym is associative and unital with the unity $1 = F_{\varnothing,\varnothing}$ (although this is pretty obvious).

It is very easy to see that the cooperation Δ is coassociative, counital and cocommutative. It is also clear that both the multiplication $\cdot$ and the comultiplication Δ on PNSym are graded.

The next difficulty is to prove that Δ is a $\mathbf{k}$-algebra homomorphism, i.e., that $\Delta(fg) = \Delta(f) \cdot \Delta(g)$ for all $f, g \in \text{PNSym}$ (where the "$\cdot$" on the right hand side is the extension of the external multiplication $\cdot$ to $\text{PNSym} \otimes \text{PNSym}$). Here, we can argue as above, using the fact (a consequence of Theorem 1.39) that

$$\text{ev}_{H \otimes H} (f) = (\text{ev}_H \otimes \text{ev}_H) (\Delta (f)) \in \text{End} (H \otimes H) \quad \text{for every } f \in \text{PNSym}$$

to make sense of Δ), and using the fact that the map

$$\mathrm{ev}_H \otimes \mathrm{ev}_H : \mathrm{PNSym} \otimes \mathrm{PNSym} \to \mathrm{End}\, H \otimes \mathrm{End}\, H \to \mathrm{End}\,(H \otimes H)$$

is injective (this is not hard to show using the argument used in the proof of Theorem 1.36).

What we have shown so far yields that PNSym (equipped with $\cdot$ and Δ) is a connected graded $\mathbf{k}$-bialgebra. Thus, PNSym is a Hopf algebra (since any connected graded $\mathbf{k}$-bialgebra is a Hopf algebra).

It remains to show that PNSym (equipped with $*$ and Δ) is a non-unital $\mathbf{k}$-bialgebra. Having already verified that $*$ is associative, we only need to show that $\Delta(f * g) = \Delta(f) * \Delta(g)$ for all $f, g \in \mathrm{PNSym}$. But this is similar to the proof of $\Delta(fg) = \Delta(f) \cdot \Delta(g)$ above. Thus, the proof of Theorem 2.7 is complete. $\square$

Second proof idea for Theorem 2.7. There is also a more direct combinatorial approach to this theorem. First, we shall define two simpler bialgebras WNSym and Perm, and then present PNSym as a quotient of their tensor product WNSym $\otimes$ Perm.

Here are some details:

We define WNSym to be the free $\mathbf{k}$-module with basis $(C_\alpha)_{\alpha \text{ is a weak composition}}$. We equip this $\mathbf{k}$-module WNSym with an "external multiplication" defined by

$$C_\alpha \cdot C_\beta = C_{\alpha\beta}$$

(where $\alpha\beta$ is the concatenation of α and β), and an "internal multiplication" defined by

$$C_\alpha * C_\beta = \sum_{\substack{\gamma_{i,j} \in \mathbb{N} \text{ for all } i \in [k] \text{ and } j \in [\ell]; \\ \gamma_{i,1} + \gamma_{i,2} + \cdots + \gamma_{i,\ell} = \alpha_i \text{ for all } i \in [k]; \\ \gamma_{1,j} + \gamma_{2,j} + \cdots + \gamma_{k,j} = \beta_j \text{ for all } j \in [\ell]}} C_{(\gamma_{1,1}, \gamma_{1,2}, \ldots, \gamma_{k,\ell})}$$

(where $\alpha \in \mathbb{N}^k$ and $\beta \in \mathbb{N}^\ell$), and a comultiplication $\Delta : \mathrm{WNSym} \to \mathrm{WNSym} \otimes \mathrm{WNSym}$ defined by

$$\Delta(C_\alpha) = \sum_{\substack{\beta, \gamma \in \mathbb{N}^k; \\ \text{entrywise sum } \beta + \gamma = \alpha}} C_\beta \otimes C_\gamma \qquad \text{for any } \alpha \in \mathbb{N}^k.$$

It is not too hard to show that WNSym thus becomes a graded (but not connected!) cocommutative $\mathbf{k}$-bialgebra when equipped with the external multiplication $\cdot$, and a (non-graded) non-unital bialgebra when equipped with the internal multiplication $*$. (Indeed, this WNSym is a mild variation on the Hopf algebra NSym of noncommutative symmetric functions, which is studied (e.g.) in [GKLLRT94] or [GriRei20, §5.4]; the only difference is that compositions have been replaced by weak compositions. The first letter "W" in WNSym refers to this weakness.)

We define $\mathfrak{S}$ to be the disjoint union $\bigsqcup_{k \in \mathbb{N}} \mathfrak{S}_k$ of all symmetric groups $\mathfrak{S}_k$ for all $k \in \mathbb{N}$. We define Perm to be the free $\mathbf{k}$-module with basis $(P_\sigma)_{\sigma \in \mathfrak{S}}$. We equip this $\mathbf{k}$-module Perm with an "external multiplication" $\cdot$ defined by

$$P_\sigma \cdot P_\tau = P_{\sigma \oplus \tau},$$

and an "internal multiplication" $*$ defined by

$$P_\sigma * P_\tau = P_{\tau[\sigma]},$$

and a comultiplication $\Delta : \operatorname{Perm} \to \operatorname{Perm} \otimes \operatorname{Perm}$ defined by

$$\Delta\left(P_\sigma\right) = P_\sigma \otimes P_\sigma.$$

It is not too hard to show that Perm becomes a (non-graded) cocommutative $\mathbf{k}$-bialgebra when equipped with either of the two multiplications. Indeed, in both cases, it becomes the monoid algebra of an appropriate monoid on the set $\mathfrak{S}$. The hardest part of the proof is to check the associativity of the internal multiplication; this can be done as follows:

> *Claim 1:* We have $\tau\left[\sigma\left[\rho\right]\right] = \left(\tau\left[\sigma\right]\right)\left[\rho\right]$ for any three permutations $\tau, \sigma, \rho \in \mathfrak{S}$.

Proof of Claim 1. We can regard the set $\mathfrak{S}$ as a skeletal groupoid with objects $0, 1, 2, \ldots$ and morphism sets $\mathfrak{S}\left(k,k\right) = \mathfrak{S}_k$ and $\mathfrak{S}\left(k,\ell\right) = \varnothing$ for all $k \neq \ell$. However, the definition of $\tau\left[\sigma\right]$ becomes cleaner if we "de-skeletize" $\mathfrak{S}$ to a larger category. Namely, we define a *tormutation* to be a bijection (not necessarily order-preserving) between two finite totally ordered sets. Clearly, each permutation $\sigma \in \mathfrak{S}_k$ is a tormutation $[k] \to [k]$. Conversely, any tormutation $\phi : A \to B$ induces a canonical permutation $\overline{\phi} \in \mathfrak{S}_{|A|}$ by the rule

$$\overline{\phi} := \operatorname{inc}_{B \to [|B|]} \circ \phi \circ \operatorname{inc}_{[|A|] \to A} : [|A|] \to [|B|],$$

where $\operatorname{inc}_{X \to Y}$ denotes the unique order isomorphism between two given finite totally ordered sets X and Y. We may call $\overline{\phi}$ the *standardization* of ϕ. Thus, the skeletal groupoid $\mathfrak{S}$ is a skeleton of the groupoid $\widetilde{\mathfrak{S}}$ whose objects are the finite totally ordered sets and whose morphisms are the tormutations.

Given two tormutations $\phi : A \to B$ and $\phi' : A' \to B'$, we now define a tormutation $\phi'\left\langle\phi\right\rangle : A \times A' \to B' \times B$ by

$$\left(\phi'\left\langle\phi\right\rangle\right)\left(a,a'\right) = \left(\phi'\left(a'\right), \phi\left(a\right)\right) \qquad \text{for all } \left(a,a'\right) \in A \times A'.$$

In other words, $\phi'\left\langle\phi\right\rangle$ applies ϕ and ϕ' to the respective entries of the input, then swaps the outputs.

It is now easy to see that $\overline{\phi'\left\langle\phi\right\rangle} = \overline{\phi'}\left[\overline{\phi}\right]$ for any two tormutations ϕ and ϕ'. Thus, in order to prove that $\tau\left[\sigma\left[\rho\right]\right] = \left(\tau\left[\sigma\right]\right)\left[\rho\right]$ for any three permutations $\tau, \sigma, \rho \in \mathfrak{S}$, it suffices to show that $\overline{\phi''\left\langle\phi'\left\langle\phi\right\rangle\right\rangle} = \overline{\left(\phi''\left\langle\phi'\right\rangle\right)\left\langle\phi\right\rangle}$ for any three tormutations ϕ, ϕ', ϕ''. But the latter is easy: The map $\phi''\left\langle\phi'\left\langle\phi\right\rangle\right\rangle$ sends each $\left(\left(a,a'\right),a''\right)$ to $\left(\phi''\left(a''\right), \left(\phi'\left(a'\right), \phi\left(a\right)\right)\right)$, whereas the map $\left(\phi''\left\langle\phi'\right\rangle\right)\left\langle\phi\right\rangle$ sends each $\left(a, \left(a', a''\right)\right)$ to $\left(\left(\phi''\left(a''\right), \phi'\left(a'\right)\right), \phi\left(a\right)\right)$. Because of the canonical order isomorphism $A \times \left(B \times C\right) \cong \left(A \times B\right) \times C$ for any three totally ordered sets A, B, C, these two maps are therefore equivalent, i.e., have the same canonical permutation. This proves Claim 1. $\square$

We now know that Perm becomes a (non-graded) cocommutative $\mathbf{k}$-bialgebra when equipped with either of the two multiplications.

Now, let WPNSym be the tensor product $\text{WNSym} \otimes \text{Perm}$. We equip this tensor product WPNSym with an "external multiplication" (obtained by tensoring the external multiplications of WNSym and of Perm), an "internal multiplication" (obtained similarly) and a comultiplication (also obtained similarly). We furthermore set

$$\widehat{F}_{\alpha,\sigma} := C_\alpha \otimes P_\sigma \in \text{WPNSym} \qquad \text{for any weak mopiscotion } (\alpha, \sigma).$$

Then, $\left(\widehat{F}_{\alpha,\sigma}\right)_{(\alpha,\sigma) \text{ is a weak mopiscotion}}$ is a basis of the $\mathbf{k}$-module WPNSym, and our operations $\cdot$, $*$ and Δ on WPNSym satisfy the same relations for this basis as the analogous operations on PNSym do for the basis $(F_{\alpha,\sigma})_{(\alpha,\sigma) \text{ is a mopiscotion}}$ (we just have to replace each "F" by "$\widehat{F}$"). It is thus easy to show that WPNSym is a graded (but not connected) cocommutative $\mathbf{k}$-bialgebra when equipped with $\cdot$, and a (non-graded) non-unital $\mathbf{k}$-bialgebra when equipped with $*$. (Here we use the facts that the tensor product of two cocommutative bialgebras is a cocommutative bialgebra, and that the tensor product of two non-unital bialgebras is a non-unital bialgebra.)

As we said, WPNSym is almost the PNSym that we care about. But WPNSym does not satisfy the rule

$$F_{\alpha,\sigma} = F_{\beta,\tau} \qquad \text{for } (\beta, \tau) = \text{red}(\alpha, \sigma)$$

that is fundamental to the definition of PNSym. Hence, PNSym is not quite WPNSym but rather a quotient of WPNSym. To be specific, we define a $\mathbf{k}$-submodule I_{red} of WPNSym by

$$\begin{aligned} I_{\text{red}} &:= \text{span}_{\mathbf{k}} \left(\widehat{F}_{\alpha,\sigma} - \widehat{F}_{\beta,\tau} \mid (\beta, \tau) = \text{red}(\alpha, \sigma) \right) \\ &= \text{span}_{\mathbf{k}} \left(\widehat{F}_{\alpha,\sigma} - \widehat{F}_{\beta,\tau} \mid \text{red}(\beta, \tau) = \text{red}(\alpha, \sigma) \right). \end{aligned}$$

It is not too hard to show that this I_{red} is an ideal of WPNSym with respect to both $\cdot$ and $*$ and a coideal with respect to Δ. Hence, the quotient $\text{WPNSym} / I_{\text{red}}$ inherits all operations of WPNSym, thus becoming a graded bialgebra under $\cdot$ and Δ and a non-unital bialgebra under $*$ and Δ. Moreover, the graded bialgebra $\text{WPNSym} / I_{\text{red}}$ is connected (since $\text{red}(\alpha, \sigma) = (\varnothing, \varnothing)$ whenever $|\alpha| = 0$), and thus is a Hopf algebra. As we recall, this means that PNSym is a well-defined connected graded Hopf algebra (since $\text{PNSym} \cong \text{WPNSym} / I_{\text{red}}$). This completes the proof of Theorem 2.7 again. □

Proposition 2.8. Let $n \in \mathbb{N}$. Then, the n-th graded component PNSym_n of PNSym is a free $\mathbf{k}$-module of rank

$$\sum_{k=0}^{n} \binom{n-1}{n-k} k!.$$

Proof idea. Clearly, PNSym_n is a free $\mathbf{k}$-module with a basis consisting of all $F_{\alpha,\sigma}$ where (α, σ) ranges over all mopiscotions satisfying $|\alpha| = n$. It remains to show that the number of such mopiscotions is $\sum\limits_{k=0}^{n} \dbinom{n-1}{n-k} k!$. But this is easy: For any k, the number of such mopiscotions in which α has length k is $\dbinom{n-1}{n-k} k!$ (since there are $\dbinom{n-1}{n-k}$ compositions of n into k parts, and $k!$ permutations $\sigma \in \mathfrak{S}_k$). □

We note that the addend $\dbinom{n-1}{n-k} k!$ in Proposition 2.8 can also be rewritten as $k \cdot (n-1)(n-2) \cdots (n-k+1)$ when n is positive (but not when $n = 0$).

Here are the ranks of the free $\mathbf{k}$-modules PNSym_n for the first few values of n:

n	0	1	2	3	4	5	6	7
$\operatorname{rank}(\operatorname{PNSym}_n)$	1	1	3	11	49	261	1631	11743

.

Starting at $n = 1$, this sequence of ranks is known to the OEIS as Sequence A001339, and has appeared in the theory of combinatorial Hopf algebras before ([HiNoTh06, §3.8.3]), although the Hopf algebra considered there appears to be different (perhaps dual to ours?).[14]

Theorem 2.9. Let $\operatorname{PNSym}^{(2)}$ be the non-unital algebra PNSym with multiplication $*$. Then, every connected graded bialgebra H becomes a $\operatorname{PNSym}^{(2)}$-module, with $F_{\alpha,\sigma}$ acting as $p_{\alpha,\sigma}$. Moreover, the action of an external product uv of two elements $u, v \in \operatorname{PNSym}$ on H is the convolution of the action of u with the action of v.

Proof idea. The first claim follows from Theorem 1.19, the second from Proposition 1.15. □

There is also an analogue of the "splitting formula" ([GKLLRT94, Proposition 5.2]) for PNSym, connecting the two products (internal and external) with the comultiplication:

[14] Some words about the connection to [HiNoTh06, §3.8.3] are in order. One of the many concepts studied in [HiNoTh06] is the *stalactic equivalence*, an equivalence relation on the words over a given alphabet A. It is the equivalence relation on A^* defined by the single axiom $uawav \equiv uaawv$ for any words $u, v, w \in A^*$ and any letter $a \in A$. It is easy to see that two words $u, v \in A^*$ are equivalent if and only if they contain each letter the same number of times, and the order in which the letters **first** appear in each word is the same for both words. Thus, each stalactic equivalence class is uniquely determined by the multiplicities of its letters and by the order in which they first appear. For *initial* words (i.e., for words over the alphabet $\{1, 2, 3, \ldots\}$ whose set of letters is $[k]$ for some $k \in \mathbb{N}$), this data is equivalent to a mopiscotion (α, σ) (where α determines the multiplicities of letters, and σ determines their order of first appearance). The number of stalactic equivalence classes of initial words of length n thus equals the number of mopiscotions (α, σ) with $|\alpha| = n$.

Theorem 2.10. Let $f, g, h \in \operatorname{PNSym}$. Write $\Delta(h)$ as $\Delta(h) = \sum_{(h)} h_{(1)} \otimes h_{(2)}$ (using Sweedler notation). Then,

$$(fg) * h = \sum_{(h)} \left(f * h_{(1)}\right)\left(g * h_{(2)}\right).$$

(To be more precise, the analogue of [GKLLRT94, Proposition 5.2] would be the generalization of this formula to iterated coproducts $\Delta^{[r]}(h)$, but this generalization follows from Theorem 2.10 by a straightforward induction on r.)

Proof idea for Theorem 2.10. This can be easily shown using Definition 2.6. (An important first step is to show that the formulas for $F_{\alpha,\sigma} \cdot F_{\beta,\tau}$ and $F_{\alpha,\sigma} * F_{\beta,\tau}$ hold not only for mopiscotions (α, σ) and (β, τ) but also for weak mopiscotions (α, σ) and (β, τ).)

Alternatively, we can derive Theorem 2.10 from the analogous claim about WPNSym, which in turn follows from the analogous claims about WNSym and Perm. (For example, the analogous claim for Perm holds because $\omega[\sigma \oplus \tau] = (\omega[\sigma]) \oplus (\omega[\tau])$ for any three permutations σ, τ, ω.) □

Remark 2.11. The $\mathbf{k}$-linear map

$$\begin{aligned} \mathfrak{p} : \operatorname{PNSym} &\to \operatorname{NSym}, \\ F_{\alpha,\sigma} &\mapsto \mathbf{H}_\alpha \qquad \text{for any mopiscotion } (\alpha, \sigma) \end{aligned}$$

is a surjection that respects all structures (external and internal multiplication, comultiplication and grading).

This surjection is furthermore split: The $\mathbf{k}$-linear map

$$\begin{aligned} \mathfrak{i} : \operatorname{NSym} &\to \operatorname{PNSym}, \\ \mathbf{H}_\alpha &\mapsto F_{\alpha,\mathrm{id}} \qquad \text{for any composition } \alpha \end{aligned}$$

(where id denotes the identity permutation in $\mathfrak{S}_k$ where α has length k) is an injection that is right-inverse to $\mathfrak{p}$ and respects external multiplication, comultiplication and grading (but does not respect internal multiplication).

Proposition 2.12. The $\mathbf{k}$-algebra PNSym (equipped with the external multiplication) is free.

Proof idea. This algebra is just the monoid algebra of the monoid of mopiscotions under the operation $(\alpha, \sigma) \cdot (\beta, \tau) = (\alpha\beta, \sigma \oplus \tau)$. But this monoid is free, with the generators being the mopiscotions (α, σ) for which the permutation σ is connected[15] (as can be easily verified). □

[15] See [GriRei20, Exercise 8.1.10] for the definition of a connected permutation.

2.3. The Ω involution

If A is any $\mathbf{k}$-algebra, then its *opposite algebra* A^{op} is defined as the $\mathbf{k}$-algebra obtained from A by "reversing the order of factors in a product": i.e., as a $\mathbf{k}$-module, $A^{\text{op}} = A$, but the product ab of two elements of A^{op} equals their product ba when viewed as elements of A. This definition makes sense for both unital and non-unital $\mathbf{k}$-algebras.

A $\mathbf{k}$-algebra A (unital or not) is said to be *self-opposite* if it is isomorphic to its opposite algebra A^{op}. The non-unital $\mathbf{k}$-algebra $\text{NSym}^{(2)}$ – that is, NSym equipped with the internal multiplication $*$ – is not self-opposite (at least not in a graded way) in general[16]. However, the non-unital $\mathbf{k}$-algebra $\text{PNSym}^{(2)}$ (as defined in Theorem 2.9) is self-opposite. Even better, the following holds:

Proposition 2.13. The $\mathbf{k}$-linear map

$$\Omega : \text{PNSym} \to \text{PNSym},$$
$$F_{\alpha,\sigma} \mapsto F_{\alpha\cdot\sigma^{-1},\ \sigma^{-1}}$$

is a graded $\mathbf{k}$-bialgebra morphism from PNSym to PNSym (in particular, it respects the external multiplication $\cdot$ and the comultiplication Δ) and is furthermore a non-unital $\mathbf{k}$-algebra morphism from $\text{PNSym}^{(2)}$ to $\left(\text{PNSym}^{(2)}\right)^{\text{op}}$ (that is, it satisfies $\Omega(a * b) = \Omega(b) * \Omega(a)$ for all $a, b \in \text{PNSym}$). Moreover, it is an involution (i.e., satisfies $\Omega \circ \Omega = \text{id}$).

Proof. This can be checked by hand. Alternatively, use Proposition 1.40 in a faithful representation argument along the lines of the First proof idea for Theorem 2.7. □

2.4. The Janus monoid and the structure of internal multiplication

We shall now take a closer look at the internal multiplication $*$ on PNSym.

For this entire section, we fix an $n \in \mathbb{N}$. Consider the n-th graded component PNSym_n of PNSym. As we know, this component PNSym_n is closed under the

[16]Namely, its 5-th graded component $\text{NSym}_5^{(2)}$ is not self-opposite when $\mathbf{k}$ is a field of characteristic 0. Indeed, this is implicit in [Saliol08]: By [Saliol08, Theorem 2.1], it suffices to prove that the S_5-invariant subalgebra $(\mathbf{k}\mathcal{F})^{S_5}$ is not self-opposite. By [Saliol08, Proposition 4.1], this subalgebra is a split basic algebra. If it was self-opposite, its quiver (see [Saliol08, §3]) would thus be isomorphic to the quiver obtained from it by reversing all arrows. But the quiver of $(\mathbf{k}\mathcal{F})^{S_5}$ is described explicitly in [Saliol08, Theorem 8.1], and has three sources (i.e., vertices with no incoming arcs) but only two sinks (i.e., vertices with no outgoing arcs). Thus, this quiver cannot be isomorphic to the quiver obtained from it by reversing all arrows. This shows that the $\mathbf{k}$-algebra $\text{NSym}_5^{(2)}$ is not self-opposite when $\mathbf{k}$ is a field of characteristic 0.

Hence, $\text{NSym}^{(2)}$ is not self-opposite as a graded non-unital $\mathbf{k}$-algebra under this condition. Theoretically, this does not preclude it being self-opposite without the grading, but I don't consider this to be very likely.

internal multiplication $*$ (indeed, this is clear from the definition of $*$). Moreover, the element $F_{((n),\mathrm{id})}$ of PNSym_n is a neutral element for $*$. Thus, PNSym_n becomes a (unital) **k**-algebra under the internal comultiplication $*$. Let us denote this **k**-algebra by $\mathrm{PNSym}_n^{(2)}$.

As we saw in Remark 2.11, the surjection $\mathfrak{p} : \mathrm{PNSym} \to \mathrm{NSym}$ respects $*$. Thus, $\mathrm{PNSym}_n^{(2)}$ surjects onto the **k**-algebra $\mathrm{NSym}_n^{(2)}$, which is known to be isomorphic to the descent algebra of the symmetric group $\mathfrak{S}_n$ (see [GKLLRT94, §5.1]). Hence, $\mathrm{PNSym}_n^{(2)}$ can be viewed as a "twisted" version of the descent algebra (though not in the same sense as in [PatSch06]). Therefore, we can try to extend to $\mathrm{PNSym}_n^{(2)}$ the rich theory that has been developed for the descent algebra ([GarReu89], [Bidiga97], [Schock04], etc.).

An important tool for understanding the descent algebra is the *face monoid* (aka *Tits monoid*) of the braid arrangement. Its use has been pioneered by Bidigare in [Bidiga97]; a more recent exposition is found in [Saliol06]. As we will soon see, the analogous role for $\mathrm{PNSym}_n^{(2)}$ is played by the *Janus algebra* of the braid arrangement, defined by Aguiar and Mahajan in [AguMah20, §1.9.3]. We will introduce this algebra in a combinatorial language, avoiding any mention of hyperplane arrangements, as the combinatorial arguments stand well on their own legs here and the geometric viewpoint would be a distraction.

2.4.1. Set compositions and their friends

We begin by defining a number of combinatorial structures related to *set compositions* (also known as *ordered set partitions*):

- A *weak set composition* of $[n]$ means a (finite) tuple $(A_1, A_2, \ldots, A_k)$ of disjoint subsets of $[n]$ such that $A_1 \cup A_2 \cup \cdots \cup A_k = [n]$. The entries A_i of such a tuple are called its *blocks*. Note that some of these blocks A_i can be empty, so k can be arbitrarily large. We let WSC be the set of all weak set compositions of $[n]$. This is an infinite set.

- A *set composition* of $[n]$ means a weak set composition of $[n]$ whose blocks are all nonempty. For instance, $(\{2,5\}, \{1\}, \{3,4\})$ is a set composition of $[5]$, but $(\{2,5\}, \varnothing, \{1\}, \{3,4\})$ is only a weak set composition of $[5]$. We let SC be the set of all set compositions of $[n]$. This is a finite set, whose size is known as the n-th *ordered Bell number*.

- A *weak set mopiscotion* of $[n]$ means a pair $(\mathbf{A}, \sigma)$ consisting of a weak set composition $\mathbf{A}$ of $[n]$ that has k blocks and a permutation $\sigma \in \mathfrak{S}_k$. Likewise we define *set mopiscotions*. We let WSM be the set of all weak set mopiscotions of $[n]$, and we let SM be the set of all set mopiscotions of $[n]$.

- If $\mathbf{A} = (A_1, A_2, \ldots, A_k)$ is a weak set composition of $[n]$, then $\operatorname{comp} \mathbf{A}$ is the weak composition of n defined by

$$\operatorname{comp} \mathbf{A} := (|A_1|, |A_2|, \ldots, |A_k|).$$

If $\mathbf{A}$ is a set composition, then $\operatorname{comp} \mathbf{A}$ is a composition.

- If $\mathbf{A} = (A_1, A_2, \ldots, A_k) \in \mathrm{WSC}$ is a weak set composition with k blocks, and if $\sigma \in \mathfrak{S}_k$, then $\mathbf{A} \cdot \sigma$ is the weak set composition $\left(A_{\sigma(1)}, A_{\sigma(2)}, \ldots, A_{\sigma(k)}\right)$. This gives a right action of $\mathfrak{S}_k$ on the set of all weak set compositions of $[n]$ with k blocks. Two weak set compositions $\mathbf{A}$ and $\mathbf{B}$ are said to be *support-equivalent* if $\mathbf{A} = \mathbf{B} \cdot \sigma$ for some $\sigma \in \mathfrak{S}_k$ (that is, if $\mathbf{A}$ can be obtained from $\mathbf{B}$ by reordering the blocks). This is an equivalence relation; its equivalence classes are called the *weak set partitions* of $[n]$. (They can be viewed as set partitions that allow empty blocks.) Likewise, we define *set partitions* of $[n]$ as support-equivalence classes of set compositions. (These can be identified with the usual set partitions from enumerative combinatorics.)

  If $\mathbf{A}$ is a weak set composition, then its support-equivalence class is called its *support* and is denoted $\operatorname{supp} \mathbf{A}$.

  For instance, the two set compositions $(\{2,5\}, \{1\}, \{3,4\})$ and $(\{1\}, \{3,4\}, \{2,5\})$ are equivalent.

- A *weak set bicomposition* of $[n]$ means a pair $(\mathbf{A}, \mathbf{B})$ of two weak set compositions $\mathbf{A}$ and $\mathbf{B}$ that have the same support (i.e., satisfy $\mathbf{A} = \mathbf{B} \cdot \sigma$ for some $\sigma \in \mathfrak{S}_k$, where k is the number of blocks of $\mathbf{B}$). We let WJ be the set of all weak set bicompositions of $[n]$. Likewise, we define *set bicompositions*, and we call their set J. The letter J is for "Janus", as we regard $\mathbf{A}$ and $\mathbf{B}$ as the two "faces" of a weak set bicomposition $(\mathbf{A}, \mathbf{B})$; this terminology goes back to [AguMah20, §1.9.3].

Between all these sets we have a number of maps:

- We have the so-called *reduction maps*, whose purpose is to remove empty blocks from weak set compositions. Specifically, these are the maps

$$\begin{aligned} \operatorname{red} &: \mathrm{WSC} \to \mathrm{SC}, \\ \operatorname{red} &: \mathrm{WSM} \to \mathrm{SM}, \\ \operatorname{red} &: \mathrm{WJ} \to \mathrm{J} \end{aligned}$$

  defined as follows:

  - If $\mathbf{A} = (A_1, A_2, \ldots, A_k) \in \mathrm{WSC}$ is a weak set composition, then $\operatorname{red} \mathbf{A} := \left(A_{j_1}, A_{j_2}, \ldots, A_{j_h}\right) \in \mathrm{SC}$, where $(j_1 < j_2 < \cdots < j_h)$ is the list of all elements i of $[k]$ satisfying $A_i \neq \varnothing$.
  - If $(\mathbf{A}, \sigma) \in \mathrm{WSM}$ is a weak set mopiscotion with $\mathbf{A} = (A_1, A_2, \ldots, A_k)$ and $\sigma \in \mathfrak{S}_k$, then

$$\operatorname{red}(\mathbf{A}, \sigma) := (\operatorname{red} \mathbf{A}, \tau) \in \mathrm{SM},$$

    where $(j_1 < j_2 < \cdots < j_h)$ is the list of all elements i of $[k]$ satisfying $A_i \neq \varnothing$, and where $\tau \in \mathfrak{S}_h$ is the standardization of the list $(\sigma(j_1), \sigma(j_2), \ldots, \sigma(j_h))$ (as in Definition 2.4).

  - If $(\mathbf{A}, \mathbf{B}) \in \mathrm{WJ}$ is a weak set bicomposition, then $\operatorname{red}(\mathbf{A}, \mathbf{B}) := (\operatorname{red}\mathbf{A}, \operatorname{red}\mathbf{B}) \in \mathrm{J}$.

- We have a map

$$\begin{aligned} g_{\mathrm{W}} : \mathrm{WSM} &\to \mathrm{WJ}, \\ (\mathbf{A}, \sigma) &\mapsto \left(\mathbf{A},\ \mathbf{A} \cdot \sigma^{-1}\right) \end{aligned}$$

  and an analogously defined map $g : \mathrm{SM} \to \mathrm{J}$ (which is simply the restriction of g_{W} to the subset SM).

Most importantly, we have monoid structures on some of the above sets:

- The *lexicographic meet* $\mathbf{A} \wedge \mathbf{B}$ of two weak set compositions $\mathbf{A} = (A_1, A_2, \ldots, A_k)$ and $\mathbf{B} = (B_1, B_2, \ldots, B_\ell)$ of $[n]$ is defined to be the weak set composition

$$\begin{aligned} \mathbf{A} \wedge \mathbf{B} := (&A_1 \cap B_1,\ A_1 \cap B_2,\ \ldots,\ A_1 \cap B_\ell, \\ &A_2 \cap B_1,\ A_2 \cap B_2,\ \ldots,\ A_2 \cap B_\ell, \\ &\ldots, \\ &A_k \cap B_1,\ A_k \cap B_2,\ \ldots,\ A_k \cap B_\ell) \end{aligned}$$

  of $[n]$. (This is just the $k\ell$-tuple consisting of all the intersections $A_i \cap B_j$, listed in the order of lexicographically increasing (i, j).) This operation $(\mathbf{A}, \mathbf{B}) \mapsto \mathbf{A} \wedge \mathbf{B}$ makes WSC into a monoid (the associativity is easy to check); its neutral element is the 1-block weak set composition $([n])$.

- The *reduced lexicographic meet* $\mathbf{AB}$ of two set compositions $\mathbf{A}$ and $\mathbf{B}$ of $[n]$ is defined to be the set composition $\operatorname{red}(\mathbf{A} \wedge \mathbf{B})$. This operation $(\mathbf{A}, \mathbf{B}) \mapsto \mathbf{AB}$ makes SC into a monoid (but not into a submonoid of WSC). This monoid is a *band* – i.e., a monoid in which every element is idempotent. It is known as the *face monoid* (or *Tits monoid*) of the type-A braid arrangement (due to a geometric interpretation of set compositions as faces of the arrangement – see, e.g., [Saliol06, §1.2.1]). The map $\operatorname{red} : \mathrm{WSC} \to \mathrm{SC}$ is a surjective monoid morphism.

- The *lexicographic meet* $(\mathbf{A}, \sigma) \wedge (\mathbf{B}, \tau)$ of two weak set mopiscotions $(\mathbf{A}, \sigma)$ and $(\mathbf{B}, \tau)$ is defined to be the weak set mopiscotion $(\mathbf{A} \wedge \mathbf{B},\ \tau[\sigma])$. This makes WSM into a monoid (actually a submonoid of $\mathrm{WSC} \times \mathfrak{S}$, where $\mathfrak{S}$ is as defined in the Second proof idea for Theorem 2.7). Its neutral element is $\left(([n])\,,\ \mathrm{id}_{[1]}\right)$.

- The *reduced lexicographic meet* $(\mathbf{A}, \sigma)(\mathbf{B}, \tau)$ of two set mopiscotions $(\mathbf{A}, \sigma)$ and $(\mathbf{B}, \tau)$ is defined to be the set mopiscotion $\operatorname{red}((\mathbf{A}, \sigma) \wedge (\mathbf{B}, \tau))$. This makes SM into a monoid. The map $\operatorname{red} : \mathrm{WSM} \to \mathrm{SM}$ is a surjective monoid morphism.

- The *lexicographic meet* $(\mathbf{A}, \mathbf{A}') \wedge (\mathbf{B}, \mathbf{B}')$ of two weak set bicompositions $(\mathbf{A}, \mathbf{A}')$ and $(\mathbf{B}, \mathbf{B}')$ is defined to be the weak set bicomposition $(\mathbf{A} \wedge \mathbf{B},\ \mathbf{B}' \wedge \mathbf{A}')$ (note the reverse order of factors in the second argument!). This makes WJ into a monoid, known as the (type-A) *weak Janus monoid*.

- The *reduced lexicographic meet* $(\mathbf{A}, \mathbf{A}')\,(\mathbf{B}, \mathbf{B}')$ of two set bicompositions $(\mathbf{A}, \mathbf{A}')$ and $(\mathbf{B}, \mathbf{B}')$ is defined to be the set bicomposition $\operatorname{red}((\mathbf{A}, \mathbf{A}') \wedge (\mathbf{B}, \mathbf{B}')) = (\operatorname{red}(\mathbf{A} \wedge \mathbf{B}),\ \operatorname{red}(\mathbf{B}' \wedge \mathbf{A}'))$. This makes J into a monoid, known as the (type-A) *Janus monoid*. The map $\operatorname{red} : \mathrm{WJ} \to \mathrm{J}$ is a surjective monoid morphism.

Proposition 2.14. The diagram

$$\begin{array}{ccc} \mathrm{WSM} & \xrightarrow{g_{\mathrm{W}}} & \mathrm{WJ} \\ {\scriptstyle \operatorname{red}}\downarrow & & \downarrow{\scriptstyle \operatorname{red}} \\ \mathrm{SM} & \xrightarrow[g]{} & \mathrm{J} \end{array}$$

is commutative.

Proof idea. A refreshing exercise in the definition of standardization.

Let $(\mathbf{A}, \sigma) \in \mathrm{WSM}$. We must show that $\operatorname{red}(g_{\mathrm{W}}(\mathbf{A}, \sigma)) = g(\operatorname{red}(\mathbf{A}, \sigma))$.

Write the weak set composition $\mathbf{A}$ as $\mathbf{A} = (A_1, A_2, \ldots, A_k)$. Then, the definition of $\operatorname{red} : \mathrm{WSM} \to \mathrm{SM}$ yields

$$\operatorname{red}(\mathbf{A}, \sigma) = (\operatorname{red}\mathbf{A}, \tau), \tag{44}$$

where $(j_1 < j_2 < \cdots < j_h)$ is the list of all elements i of $[k]$ satisfying $A_i \neq \varnothing$, and where $\tau \in \mathfrak{S}_h$ is the standardization of the list $(\sigma(j_1), \sigma(j_2), \ldots, \sigma(j_h))$ (as in Definition 2.4). Thus,

$$g(\operatorname{red}(\mathbf{A}, \sigma)) = g(\operatorname{red}\mathbf{A}, \tau) = \left(\operatorname{red}\mathbf{A},\ (\operatorname{red}\mathbf{A}) \cdot \tau^{-1}\right) \tag{45}$$

by the definition of g.

By the definition of g_{W}, we have $g_{\mathrm{W}}(\mathbf{A}, \sigma) = \left(\mathbf{A},\ \mathbf{A} \cdot \sigma^{-1}\right)$, so that

$$\operatorname{red}(g_{\mathrm{W}}(\mathbf{A}, \sigma)) = \operatorname{red}\left(\mathbf{A},\ \mathbf{A} \cdot \sigma^{-1}\right) = \left(\operatorname{red}\mathbf{A},\ \operatorname{red}\left(\mathbf{A} \cdot \sigma^{-1}\right)\right). \tag{46}$$

We must prove that the left hand sides of the equalities (46) and (45) are equal. Clearly, it suffices to show that the right hand sides are equal. In other words, it suffices to show that $\operatorname{red}\left(\mathbf{A} \cdot \sigma^{-1}\right) = (\operatorname{red}\mathbf{A}) \cdot \tau^{-1}$.

Both set compositions $\operatorname{red}\left(\mathbf{A} \cdot \sigma^{-1}\right)$ and $(\operatorname{red}\mathbf{A}) \cdot \tau^{-1}$ are obtained from $\mathbf{A}$ by removing all empty blocks and permuting the blocks (either before or after the removal of empty blocks). Thus, they both consist of the nonempty blocks of $\mathbf{A}$ in some order. In order to show that $\operatorname{red}\left(\mathbf{A} \cdot \sigma^{-1}\right) = (\operatorname{red}\mathbf{A}) \cdot \tau^{-1}$, we only need to prove that the orders in which they contain these nonempty blocks are the same. In other words, we must prove that if p and q are two distinct elements of $[k]$ such that $A_p \neq \varnothing$ and $A_q \neq \varnothing$, then the equivalence

$$\begin{aligned} &\left(A_p \text{ occurs before } A_q \text{ in } \operatorname{red}\left(\mathbf{A} \cdot \sigma^{-1}\right)\right) \\ \Longleftrightarrow\ &\left(A_p \text{ occurs before } A_q \text{ in } (\operatorname{red}\mathbf{A}) \cdot \tau^{-1}\right) \end{aligned} \tag{47}$$

holds. So let us prove this. Let p and q be two distinct elements of $[k]$ such that $A_p \neq \varnothing$ and $A_q \neq \varnothing$. Then, we have $p = j_x$ and $q = j_y$ for some $x, y \in [h]$. But we have the following chain of equivalences:

$$\begin{aligned}
&\left(A_p \text{ occurs before } A_q \text{ in } \operatorname{red}\left(\mathbf{A} \cdot \sigma^{-1}\right)\right) \\
\Longleftrightarrow\ & (\sigma(p) < \sigma(q)) \\
\Longleftrightarrow\ & (\sigma(j_x) < \sigma(j_y)) \qquad (\text{since } p = j_x \text{ and } q = j_y) \\
\Longleftrightarrow\ & (\tau(x) < \tau(y))
\end{aligned}$$

(since τ is the standardization of the list $(\sigma(j_1), \sigma(j_2), \ldots, \sigma(j_h))$, so that the relative order of the values of τ agrees with the relative order of the entries of $(\sigma(j_1), \sigma(j_2), \ldots, \sigma(j_h))$). We also have the following chain of equivalences:

$$\begin{aligned}
&\left(A_p \text{ occurs before } A_q \text{ in } (\operatorname{red} \mathbf{A}) \cdot \tau^{-1}\right) \\
\Longleftrightarrow\ & \left(A_{j_x} \text{ occurs before } A_{j_y} \text{ in } (\operatorname{red} \mathbf{A}) \cdot \tau^{-1}\right) \qquad (\text{since } p = j_x \text{ and } q = j_y) \\
\Longleftrightarrow\ & \left(A_{j_x} \text{ occurs before } A_{j_y} \text{ in } \left(A_{j_{\tau^{-1}(1)}}, A_{j_{\tau^{-1}(2)}}, \ldots, A_{j_{\tau^{-1}(h)}}\right)\right) \\
& \left(\begin{array}{c} \text{since the definition of } \operatorname{red} \mathbf{A} \text{ shows} \\ \text{that } \operatorname{red} \mathbf{A} = \left(A_{j_1}, A_{j_2}, \ldots, A_{j_h}\right) \text{ and} \\ \text{thus } (\operatorname{red} \mathbf{A}) \cdot \tau^{-1} = \left(A_{j_{\tau^{-1}(1)}}, A_{j_{\tau^{-1}(2)}}, \ldots, A_{j_{\tau^{-1}(h)}}\right) \end{array}\right) \\
\Longleftrightarrow\ & \left(j_x \text{ occurs before } j_y \text{ in } \left(\tau^{-1}(1),\ \tau^{-1}(2),\ \ldots,\ \tau^{-1}(h)\right)\right) \\
\Longleftrightarrow\ & (\tau(x) < \tau(y)).
\end{aligned}$$

Comparing these two chains of equivalences, we obtain the desired equivalence (47), which is all that stood between us and the proof of Proposition 2.14. ☐

Proposition 2.15. The maps $g_W : \mathrm{WSM} \to \mathrm{WJ}$ and $g : \mathrm{SM} \to \mathrm{J}$ are monoid morphisms.

Proof idea. An easy exercise about $\tau[\sigma]$.

In more detail: It suffices to show that $g_W : \mathrm{WSM} \to \mathrm{WJ}$ is a monoid morphism, since then the commutative diagram in Proposition 2.14 (and the surjectivity of $\operatorname{red} : \mathrm{WSM} \to \mathrm{SM}$) will yield the same claim about $g : \mathrm{SM} \to \mathrm{J}$.

So we must prove the equality $g_W((\mathbf{A}, \sigma) \wedge (\mathbf{B}, \tau)) = g_W(\mathbf{A}, \sigma) \wedge g_W(\mathbf{B}, \tau)$ for any two weak set mopiscotions $(\mathbf{A}, \sigma)$ and $(\mathbf{B}, \tau)$. Fix two weak set mopiscotions $(\mathbf{A}, \sigma)$ and $(\mathbf{B}, \tau)$. The definition of g_W yields $g_W(\mathbf{A}, \sigma) = \left(\mathbf{A},\ \mathbf{A} \cdot \sigma^{-1}\right)$ and $g_W(\mathbf{B}, \tau) = \left(\mathbf{B},\ \mathbf{B} \cdot \tau^{-1}\right)$. Thus,

$$\begin{aligned}
g_W(\mathbf{A}, \sigma) \wedge g_W(\mathbf{B}, \tau) &= \left(\mathbf{A},\ \mathbf{A} \cdot \sigma^{-1}\right) \wedge \left(\mathbf{B},\ \mathbf{B} \cdot \tau^{-1}\right) \\
&= \left(\mathbf{A} \wedge \mathbf{B},\ \left(\mathbf{B} \cdot \tau^{-1}\right) \wedge \left(\mathbf{A} \cdot \sigma^{-1}\right)\right). \qquad (48)
\end{aligned}$$

On the other hand, we have $(\mathbf{A}, \sigma) \wedge (\mathbf{B}, \tau) = (\mathbf{A} \wedge \mathbf{B},\ \tau[\sigma])$, so that

$$\begin{aligned}
g_W((\mathbf{A}, \sigma) \wedge (\mathbf{B}, \tau)) &= g_W((\mathbf{A} \wedge \mathbf{B},\ \tau[\sigma])) \\
&= \left(\mathbf{A} \wedge \mathbf{B},\ (\mathbf{A} \wedge \mathbf{B}) \cdot (\tau[\sigma])^{-1}\right). \qquad (49)
\end{aligned}$$

We must prove that $g_{\mathrm{W}}\left((\mathbf{A},\sigma)\wedge(\mathbf{B},\tau)\right) = g_{\mathrm{W}}(\mathbf{A},\sigma)\wedge g_{\mathrm{W}}(\mathbf{B},\tau)$. In other words, we must prove that the left hand sides of the equalities (48) and (49) are equal. Clearly, it suffices to show that the right hand sides are equal. This boils down to showing that

$$\left(\mathbf{B}\cdot\tau^{-1}\right)\wedge\left(\mathbf{A}\cdot\sigma^{-1}\right) = (\mathbf{A}\wedge\mathbf{B})\cdot(\tau[\sigma])^{-1}. \qquad (50)$$

But this is all about permutations: Both weak set compositions $(\mathbf{A}\wedge\mathbf{B})\cdot(\tau[\sigma])^{-1}$ and $\left(\mathbf{B}\cdot\tau^{-1}\right)\wedge\left(\mathbf{A}\cdot\sigma^{-1}\right)$ contain the same blocks (namely, the pairwise intersections $A_i\cap B_j$), but we must show that these blocks also appear in the same order. Both of our weak set compositions have $k\ell$ entries each, but it is best to index them not by the numbers $i\in[k\ell]$ but rather by the pairs $(j,i)\in[\ell]\times[k]$ (in lexicographically increasing order). Upon this reindexing, the (j,i)-th entry of $\left(\mathbf{B}\cdot\tau^{-1}\right)\wedge\left(\mathbf{A}\cdot\sigma^{-1}\right)$ becomes $B_{\tau^{-1}(j)}\cap A_{\sigma^{-1}(i)} = A_{\sigma^{-1}(i)}\cap B_{\tau^{-1}(j)}$. What about the (j,i)-th entry of $(\mathbf{A}\wedge\mathbf{B})\cdot(\tau[\sigma])^{-1}$? Recall the tormutation $\phi'\langle\phi\rangle$ defined in the second proof idea for Theorem 2.7. As we know, $\tau[\sigma]$ is really just the standardization of the tormutation $\tau\langle\sigma\rangle : [k]\times[\ell]\to[\ell]\times[k]$ that sends each (u,v) to $(\tau(v),\sigma(u))$. Our reindexing has turned $(\mathbf{A}\wedge\mathbf{B})\cdot(\tau[\sigma])^{-1}$ into $(\mathbf{A}\wedge\mathbf{B})\cdot(\tau\langle\sigma\rangle)^{-1}$. But $(\tau\langle\sigma\rangle)^{-1}(j,i) = \left(\sigma^{-1}(i),\tau^{-1}(j)\right)$. Hence, the (j,i)-th entry of $(\mathbf{A}\wedge\mathbf{B})\cdot(\tau[\sigma])^{-1}$ is the $\left(\sigma^{-1}(i),\tau^{-1}(j)\right)$-th entry of $\mathbf{A}\wedge\mathbf{B}$; but the latter entry is $A_{\sigma^{-1}(i)}\cap B_{\tau^{-1}(j)}$. This is the same answer that we obtained for the (j,i)-th entry of $\left(\mathbf{B}\cdot\tau^{-1}\right)\wedge\left(\mathbf{A}\cdot\sigma^{-1}\right)$. Hence, we have shown that respective entries of $(\mathbf{A}\wedge\mathbf{B})\cdot(\tau[\sigma])^{-1}$ and $\left(\mathbf{B}\cdot\tau^{-1}\right)\wedge\left(\mathbf{A}\cdot\sigma^{-1}\right)$ are equal. Thus, (50) follows, and we are done. □

Proposition 2.16. The map $g : \mathrm{SM}\to\mathrm{J}$ is a monoid isomorphism.

Proof idea. Also easy (but note that this does not apply to g_{W}, since g_{W} is not injective!).

In more detail: We already know from Proposition 2.15 that g is a monoid morphism. It remains to prove that g is bijective. Surjectivity is clear, since each $(\mathbf{A},\mathbf{B})\in\mathrm{J}$ satisfies $\mathbf{A}=\mathbf{B}\cdot\sigma$ for some $\sigma\in\mathfrak{S}_k$. To prove injectivity, we must show that this σ is unique. But this is clear, because the blocks of $\mathbf{B}$ are distinct (being nonempty disjoint sets) and thus cannot be nontrivially permuted without changing $\mathbf{B}$. □

We could also define a monoid structure on the sets of (weak) set partitions, i.e., on the images of the support maps. Then, the support maps would be monoid morphisms.

All the above-defined monoids come with a left action of the symmetric group $\mathfrak{S}_n$ (unrelated to the right action of $\mathfrak{S}_k$ that was used in defining supports). Namely:

- The symmetric group $\mathfrak{S}_n$ acts on WSC and on SC by the rule

  $$\sigma\cdot(A_1,A_2,\ldots,A_k) := (\sigma(A_1),\ \sigma(A_2),\ \ldots,\ \sigma(A_k)).$$

  (Here, $\sigma(A_i)$ is understood as usual – i.e., as the image of the subset A_i of $[n]$ under σ.)

- The symmetric group $\mathfrak{S}_n$ acts on WSM and on SM by the rule
$$\sigma \cdot (\mathbf{A}, \tau) := (\sigma \cdot \mathbf{A},\ \tau).$$
(Thus, σ only affects the first argument of a weak set mopiscotion.)

- The symmetric group $\mathfrak{S}_n$ acts on WJ and on J by the rule
$$\sigma \cdot (\mathbf{A}, \mathbf{B}) := (\sigma \cdot \mathbf{A},\ \sigma \cdot \mathbf{B}).$$
(That is, σ acts diagonally on the two arguments.)

All these actions are actions by monoid automorphisms.

Remark 2.17. Let us briefly discuss the connection of our Janus monoid J to the Janus monoid of Aguiar and Mahajan.

In [AguMah20, §1.9.3], Aguiar and Mahajan define the *Janus monoid* $\mathrm{J}[\mathcal{A}]$ of a hyperplane arrangement $\mathcal{A}$. It consists of the so-called *bifaces*, which are the pairs (U, V) of two faces of $\mathcal{A}$ such that U and V have the same support (i.e., supporting hyperplane). When $\mathcal{A}$ is the braid arrangement in $\mathbb{R}^n$, the faces of $\mathcal{A}$ are in canonical bijection with the set compositions of $[n]$, and the support of a face corresponds to the (unordered) set partition obtained by forgetting the order of the blocks in the corresponding set composition. Thus, the bifaces of the braid arrangement are what we call set bicompositions. The multiplication of bifaces that defines the product in $\mathrm{J}[\mathcal{A}]$ turns out to be precisely our reduced lexicographic meet operation on set bicompositions.

2.4.2. Algebras and the master diagram

Each monoid M induces a monoid algebra $\mathbf{k}[M]$, which is a $\mathbf{k}$-algebra that is spanned freely (as a $\mathbf{k}$-module) by the elements of M and whose multiplication is just the multiplication of M extended by linearity. In particular, all the above-defined monoids WSC, SC, WSM, SM, WJ and J have monoid algebras $\mathbf{k}[\mathrm{WSC}]$, $\mathbf{k}[\mathrm{SC}]$, $\mathbf{k}[\mathrm{WSM}]$, $\mathbf{k}[\mathrm{SM}]$, $\mathbf{k}[\mathrm{WJ}]$ and $\mathbf{k}[\mathrm{J}]$. Some of these algebras have names: $\mathbf{k}[\mathrm{J}]$ is known as the *Janus algebra*, while $\mathbf{k}[\mathrm{SC}]$ is known as the *face algebra* or the *Tits algebra*. The symmetric group $\mathfrak{S}_n$ acts on each of the above-listed monoids by monoid automorphisms, and thus acts on all their monoid algebras by $\mathbf{k}$-algebra automorphisms.

Any monoid morphism $f : M \to N$ induces a $\mathbf{k}$-algebra morphism $\mathbf{k}[M] \to \mathbf{k}[N]$, which we shall again denote by f by abuse of notation. In particular, the commutative diagram in Proposition 2.14 thus induces a commutative diagram of $\mathbf{k}$-algebras and $\mathbf{k}$-algebra morphisms

$$\begin{array}{ccc} \mathbf{k}[\mathrm{WSM}] & \xrightarrow{g_{\mathrm{W}}} & \mathbf{k}[\mathrm{WJ}] \\ {\scriptstyle \mathrm{red}}\downarrow & & \downarrow{\scriptstyle \mathrm{red}} \\ \mathbf{k}[\mathrm{SM}] & \xrightarrow[g]{} & \mathbf{k}[\mathrm{J}] \end{array}$$

(by Proposition 2.15).

However, we can do more with the algebras than we could with the monoids. For this purpose, we recall the $\mathbf{k}$-algebra $\operatorname{PNSym}_n^{(2)}$, which has basis

$$\left(F_{\alpha,\sigma}\right)_{(\alpha,\sigma) \text{ is a mopiscotion with } |\alpha|=n}.$$

We also consider the analogously defined $\mathbf{k}$-algebra $\operatorname{WPNSym}_n^{(2)}$, which has basis

$$\left(\widehat{F}_{\alpha,\sigma}\right)_{(\alpha,\sigma) \text{ is a weak mopiscotion with } |\alpha|=n}$$

(see the Second proof idea for Theorem 2.7 for its precise definition). There is a surjective $\mathbf{k}$-algebra morphism

$$\begin{aligned} \operatorname{red} : \operatorname{WPNSym}_n^{(2)} &\to \operatorname{PNSym}_n^{(2)}, \\ \widehat{F}_{\alpha,\sigma} &\mapsto F_{\alpha,\sigma} \end{aligned}$$

(since the rule for multiplying $F_{\alpha,\sigma}$'s in $\operatorname{PNSym}_n^{(2)}$ is exactly the rule for multiplying $\widehat{F}_{\alpha,\sigma}$'s in $\operatorname{WPNSym}_n^{(2)}$). Now we claim the following.

Theorem 2.18. Consider the two $\mathbf{k}$-linear maps

$$\begin{aligned} f_{\mathrm{W}} : \operatorname{WPNSym}_n^{(2)} &\to \mathbf{k}\left[\mathrm{WSM}\right], \\ \widehat{F}_{\alpha,\sigma} &\mapsto \sum_{\substack{\mathbf{A} \in \mathrm{WSC}; \\ \operatorname{comp} \mathbf{A} = \alpha}} \left(\mathbf{A}, \sigma\right) \end{aligned}$$

and

$$\begin{aligned} f : \operatorname{PNSym}_n^{(2)} &\to \mathbf{k}\left[\mathrm{SM}\right], \\ F_{\alpha,\sigma} &\mapsto \sum_{\substack{\mathbf{A} \in \mathrm{SC}; \\ \operatorname{comp} \mathbf{A} = \alpha}} \left(\mathbf{A}, \sigma\right). \end{aligned}$$

Then:

(a) Both maps f_{W} and f are injective $\mathbf{k}$-algebra morphisms.

(b) The diagram

$$\begin{array}{ccccc} \operatorname{WPNSym}_n^{(2)} & \xrightarrow{f_{\mathrm{W}}} & \mathbf{k}\left[\mathrm{WSM}\right] & \xrightarrow{g_{\mathrm{W}}} & \mathbf{k}\left[\mathrm{WJ}\right] \\ \downarrow{\scriptstyle \operatorname{red}} & & \downarrow{\scriptstyle \operatorname{red}} & & \downarrow{\scriptstyle \operatorname{red}} \\ \operatorname{PNSym}_n^{(2)} & \xrightarrow[f]{} & \mathbf{k}\left[\mathrm{SM}\right] & \xrightarrow[g]{} & \mathbf{k}\left[\mathrm{J}\right] \end{array}$$

is commutative.

(c) Recall the $\mathfrak{S}_n$-actions on $\mathbf{k}\left[\mathrm{WSM}\right]$ and $\mathbf{k}\left[\mathrm{SM}\right]$. Then, the image of f_{W} is the $\mathfrak{S}_n$-invariant part of $\mathbf{k}\left[\mathrm{WSM}\right]$ (that is, the $\mathbf{k}$-subalgebra of $\mathbf{k}\left[\mathrm{WSM}\right]$ consisting of all vectors $v \in \mathbf{k}\left[\mathrm{WSM}\right]$ that satisfy $\sigma \cdot v = v$ for all $\sigma \in \mathfrak{S}_n$). Likewise, the image of f is the $\mathfrak{S}_n$-invariant part of $\mathbf{k}\left[\mathrm{SM}\right]$.

Proof idea. **(b)** Thanks to Proposition 2.14, we only need to show the commutativity of the left square. But this is fairly easy.

(a) Injectivity is easy for both f_W and f. Next, it is easy to prove that f_W is a $\mathbf{k}$-algebra morphism. Since $\operatorname{red} : \mathrm{WPNSym}_n^{(2)} \to \mathrm{PNSym}_n^{(2)}$ is a surjective $\mathbf{k}$-algebra morphism, we conclude that f is a $\mathbf{k}$-algebra morphism as well.

(c) This is pretty obvious, since the images of $\widehat{F}_{\alpha,\sigma}$ under f_W (resp., the images of $F_{\alpha,\sigma}$ under f) are precisely the orbit sums of the $\mathfrak{S}_n$-action on WSM (resp. SM). □

Corollary 2.19. The $\mathbf{k}$-algebra $\mathrm{PNSym}_n^{(2)}$ is isomorphic to the $\mathfrak{S}_n$-invariant part of $\mathbf{k}[\mathrm{J}]$.

Proof idea. Consider the following chain of $\mathbf{k}$-algebra isomorphisms:

$$\begin{aligned}\mathrm{PNSym}_n^{(2)} &\cong f\left(\mathrm{PNSym}_n^{(2)}\right) && \text{(by Theorem 2.18 **(a)**)} \\ &= \left(\mathfrak{S}_n\text{-invariant part of } \mathbf{k}[\mathrm{SM}]\right) && \text{(by Theorem 2.18 **(c)**)} \\ &\cong \left(\mathfrak{S}_n\text{-invariant part of } \mathbf{k}[\mathrm{J}]\right)\end{aligned}$$

(since $g : \mathrm{SM} \to \mathrm{J}$ is an $\mathfrak{S}_n$-equivariant monoid isomorphism by Proposition 2.16, and thus gives rise to an $\mathfrak{S}_n$-equivariant $\mathbf{k}$-algebra isomorphism $\mathbf{k}[\mathrm{SM}] \to \mathbf{k}[\mathrm{J}]$). □

Note that we have not used the associativity of the operation $*$ in our proof of Corollary 2.19 (except in order to call $\mathrm{PNSym}_n^{(2)}$ a $\mathbf{k}$-algebra; but we could have just as well called it a nonassociative $\mathbf{k}$-algebra until we knew that it is associative). Thus, Corollary 2.19 provides a new proof of the associativity of the operation $*$ on PNSym_n (and thus on all of PNSym).

2.4.3. The structure of $\mathrm{PNSym}_n^{(2)}$

We shall next combine Theorem 2.18 with some known results about bands ([Brown04, Appendices A and B]) to uncover the structure of the $\mathbf{k}$-algebra $\mathrm{PNSym}_n^{(2)}$.

We recall that a *band* means a semigroup in which every element is idempotent. It is easy to see that SC and J are bands.

Let us say some general words about finite bands first.

Let S be any finite band. Then, a partial preorder (i.e., reflexive and transitive relation) $\succcurlyeq$ is defined on S by setting

$$(x \succcurlyeq y) \iff (x = xyx).$$

Like any partial preorder, this preorder $\succcurlyeq$ induces an equivalence relation $\sim$ defined by

$$(x \sim y) \iff (x \succcurlyeq y \text{ and } y \succcurlyeq x).$$

The equivalence class of an element $x \in S$ is called the *support* of x, denoted $\operatorname{supp} x$. The set of all $\operatorname{supp} x$ with $x \in S$ is called the *semilattice of supports* of S, and is denoted L. It is a partially ordered set (since the preorder $\succcurlyeq$ descends to a partial order on L), and in fact a semilattice (by [Brown04, Theorem A.11]). The join operation of this semilattice does in fact descend from the multiplication on S: that is, we have $(\operatorname{supp} x) \vee (\operatorname{supp} y) = \operatorname{supp}(xy)$ for all $x, y \in S$ (by [Brown04, (A.3)]). Hence, L (equipped with the join operation) is a semigroup, and the map $\operatorname{supp} : S \to L$ is a surjective semigroup morphism. By linearizing supp, we obtain a surjective non-unital $\mathbf{k}$-algebra morphism $\operatorname{supp} : \mathbf{k}[S] \to \mathbf{k}[L]$. (In all cases we are interested in, S is actually a monoid, so all these $\mathbf{k}$-algebras and morphisms are unital, but the theory works just as well for proper semigroups.)

The monoid algebra $\mathbf{k}[L]$ is isomorphic to the direct-power algebra $\mathbf{k}^L$ via a $\mathbf{k}$-algebra isomorphism $\phi : \mathbf{k}[L] \to \mathbf{k}^L$ (see [Brown04, §B.1]). Composing this isomorphism ϕ with $\operatorname{supp} : \mathbf{k}[S] \to \mathbf{k}[L]$, we obtain a surjective non-unital $\mathbf{k}$-algebra morphism $\psi = \phi \circ \operatorname{supp} : \mathbf{k}[S] \to \mathbf{k}^L$. It is shown in [Brown04, Theorem B.1] that the kernel $\operatorname{Ker}\psi$ is a nilpotent ideal of $\mathbf{k}[S]$. If $\mathbf{k}$ is a field, this entails that the $\mathbf{k}$-algebra $\mathbf{k}[S]$ has Jacobson radical (and nilradical) equal to $\operatorname{Ker}\psi$ and semisimple quotient isomorphic to $\mathbf{k}^L$ (since $\mathbf{k}[S] / \operatorname{Ker}\psi \cong \operatorname{Im}\psi = \mathbf{k}^L$ is clearly semisimple). When S is a monoid, all these algebras and morphisms are unital.

Applying all this to $S = \mathrm{SC}$, we recover classical properties of the face algebra $\mathbf{k}[\mathrm{SC}]$ ([Bidiga97, §2.3.3]), once we understand what the semilattice of supports L is. Namely, if $S = \mathrm{SC}$, then the equivalence relation $\sim$ on S we have defined above turns out to be precisely the support-equivalence relation for set compositions. Thus, the semilattice L in this case is naturally isomorphic to the (reversed)[17] lattice of partitions of the set $[n]$; the support map $\operatorname{supp} : \mathrm{SC} \to L$ essentially turns a set composition into a set partition by forgetting the order of the blocks. Using the identification of the descent algebra with the $\mathfrak{S}_n$-invariant part of $\mathbf{k}[\mathrm{SC}]$, we can furthermore derive structural properties of the descent algebra from this ([Solomo76, Theorem 3 in the type-A case], see also [Bidiga97, §3.8.5] and [GarReu89, Theorem 1.1]).

It is not much harder to analyze the case of $S = \mathrm{J}$ in the same way. The relevant lemma is the following (easy proof left to the reader):

Lemma 2.20. Let $(\mathbf{A}, \mathbf{A}')$ and $(\mathbf{B}, \mathbf{B}')$ be two set bicompositions. Then, $\operatorname{supp}(\mathbf{A}, \mathbf{A}') = \operatorname{supp}(\mathbf{B}, \mathbf{B}')$ if and only if $\operatorname{supp}\mathbf{A} = \operatorname{supp}\mathbf{B} = \operatorname{supp}\mathbf{A}' = \operatorname{supp}\mathbf{B}'$.

Thus, the semilattice of supports L for $S = \mathrm{J}$ is the same as for $S = \mathrm{SC}$, namely the (reversed) lattice of partitions of $[n]$. The above-mentioned general results about bands then apply again. We obtain the following:

Theorem 2.21. **(a)** There is a surjective $\mathbf{k}$-algebra morphism $\psi : \mathbf{k}[\mathrm{J}] \to \mathbf{k}^L$, where L is the set of all set partitions of $[n]$. Its kernel $\operatorname{Ker}\psi$ is nilpotent. Furthermore,

[17] The reversal is annoying but necessary: The join operation $\vee$ on the semilattice S is the meet operation $\wedge$ in our above terminology.

this morphism ψ is $\mathfrak{S}_n$-equivariant (i.e., respects the $\mathfrak{S}_n$-module structures on $\mathbf{k}[J]$ and $\mathbf{k}^L$).

(b) Now assume that $n!$ is invertible in $\mathbf{k}$. Then, there is a surjective $\mathbf{k}$-algebra morphism $\psi' : \operatorname{PNSym}_n^{(2)} \to \mathbf{k}^P$, where P is the set of all partitions of n. Its kernel $\operatorname{Ker}(\psi')$ is nilpotent.

Proof idea. **(a)** All of this follows from the discussion above the theorem, except for the last claim about $\mathfrak{S}_n$-equivariance, which is obvious.

(b) We note the following general fact:

> *Claim 1:* Let $f : V \to W$ be a surjective morphism of representations of $\mathfrak{S}_n$. Then, the restriction of f to the $\mathfrak{S}_n$-invariant part of V is a surjective map onto the $\mathfrak{S}_n$-invariant part of W.

Proof of Claim 1. Let w belong to the $\mathfrak{S}_n$-invariant part of W. Then, $w = f(v)$ for some $v \in V$ (since f is surjective). Consider this v, and set $v' := \dfrac{1}{n!} \sum_{\sigma \in \mathfrak{S}_n} \sigma v$ (which is well-defined, since $n!$ is invertible in $\mathbf{k}$). Then, v' lies in the $\mathfrak{S}_n$-invariant part of V, and satisfies

$$\begin{aligned} f(v') &= \frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_n} \underbrace{f(\sigma v)}_{= \sigma f(v)} = \frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_n} \sigma \underbrace{f(v)}_{= w} \\ &= \frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_n} \underbrace{\sigma w}_{\substack{= w \\ \text{(since } w \text{ is } \mathfrak{S}_n\text{-invariant)}}} = \frac{1}{n!} \underbrace{\sum_{\sigma \in \mathfrak{S}_n} w}_{\substack{= |\mathfrak{S}_n| w \\ = n! w}} = \frac{1}{n!} n! w = w. \end{aligned}$$

This shows that w is an image under the restriction of f to the $\mathfrak{S}_n$-invariant part of V. This proves the desired surjectivity, i.e., Claim 1. □

Now, the morphism ψ from part **(a)** is $\mathfrak{S}_n$-equivariant, hence is a morphism of representations of $\mathfrak{S}_n$. Thus, it restricts to a morphism ψ' from the $\mathfrak{S}_n$-invariant part of $\mathbf{k}[J]$ to the $\mathfrak{S}_n$-invariant part of $\mathbf{k}^L$. This restriction ψ' must again be surjective (by Claim 1, since ψ is surjective). Thus, ψ' is a surjective $\mathbf{k}$-algebra morphism from the $\mathfrak{S}_n$-invariant part of $\mathbf{k}[J]$ to the $\mathfrak{S}_n$-invariant part of $\mathbf{k}^L$.

But the former part is isomorphic to $\operatorname{PNSym}_n^{(2)}$ (by Corollary 2.19), whereas the latter part is isomorphic to $\mathbf{k}^P$ (since it is spanned by the orbit sums in $\mathbf{k}^L$, but there are clearly $|P|$ such orbit sums – one for each partition of n – and these orbit sums are orthogonal idempotents). Thus, Theorem 2.21 **(b)** follows. □

When $\mathbf{k}$ is a field of characteristic 0, Theorem 2.21 **(b)** shows that the kernel $\operatorname{Ker}(\pi')$ is the Jacobson radical (and the nilradical) of $\operatorname{PNSym}_n^{(2)}$, while $\mathbf{k}^P$ is the semisimple quotient of $\operatorname{PNSym}_n^{(2)}$. Note that $\mathbf{k}^P$ can also be interpreted as the center of the symmetric group algebra $\mathbf{k}[\mathfrak{S}_n]$ when $n!$ is invertible in $\mathbf{k}$.

Remark 2.22. As we pointed out in Remark 2.17, our Janus monoid J is isomorphic to the Janus monoid $\mathrm{J}[\mathcal{A}]$ of the type-A braid arrangement, as defined in [AguMah20, §1.9.3]. Thus, our Janus algebra $\mathbf{k}[\mathcal{A}]$ is isomorphic to the Janus algebra $\mathrm{J}[\mathcal{A}]$ defined loc. cit.. Hence, our **k**-algebra $\mathrm{PNSym}_n^{(2)}$ is isomorphic to the $\mathfrak{S}_n$-invariant part of the latter. This invariant part has also been studied by Aguiar and Mahajan [AguMah22, §1.12.3] (who have even introduced a q-deformation thereof). While their focus is different from ours, there are some overlaps in the results obtained. In particular, the $\mathrm{PNSym}_n^{(2)} \cong \left(\mathrm{PNSym}_n^{(2)}\right)^{\mathrm{op}}$ part of Proposition 2.13 appears in [AguMah22], being entirely obvious from the Janus viewpoint (the isomorphism corresponds to sending $(\mathbf{A}, \mathbf{B}) \mapsto (\mathbf{B}, \mathbf{A})$ in the Janus monoid), and parts of Theorem 2.21 **(a)** appear in [AguMah22, §1.12.4]. Finally, [AguMah20, Proposition 11.6] is an analogue of our Theorem 2.9 for $\mathcal{A}$-bimonoids instead of connected graded bialgebras.

Aguiar and Mahajan mostly study a single hyperplane arrangement $\mathcal{A}$ in isolation, so they do not appear to get any results about the external multiplication and the coproduct on PNSym. However, it might be possible to extend their theory to towers of arrangements, and interpret these operations in geometric terms as well.

2.5. Questions

Much remains to be understood about PNSym. Many combinatorial Hopf algebras can be embedded into algebras of noncommutative formal power series (i.e., completions of free algebras) such that the comultiplication Δ is an "alphabet doubling" operation. For instance, NSym has such an embedding[18].

Question 2.23. Does PNSym have such an embedding as well?

We can ask about some other features that certain combinatorial Hopf algebras have:

Question 2.24. Does PNSym have a categorification (e.g., a presentation as K_0 of some category)?

Question 2.25. Is there a cancellation-free formula for the antipode of PNSym ?

Question 2.26. Is PNSym isomorphic to the dual of the commutative combinatorial Hopf algebra arising from [HiNoTh06, §3.8.3]?

[18] This is explained, e.g., in [GriRei20, §8.1]: Namely, [GriRei20, Corollary 8.1.14(b)] embeds NSym in the Hopf algebra FQSym, whereas [GriRei20, (8.1.3)] embeds FQSym in the algebra $\mathbf{k}\langle\langle X_1, X_2, X_3, \ldots \rangle\rangle$ of noncommutative formal power series.

Question 2.27. What are the primitive elements of PNSym ? We recall that the primitive elements of NSym correspond to the *Lie quasi-idempotents* in the descent algebra ([GKLLRT94, Corollary 5.17]), and include several renowned families, such as the noncommutative power sums of the first two kinds ([GKLLRT94, §5.2], [GriRei20, Exercises 5.4.5 and 5.4.12]) and the third kind ([KrLeTh97, Definition 5.26]) and the Klyachko elements ([KrLeTh97, §6.2]). What is the meaning of primitive elements of PNSym ? Do some of them act as quasi-idempotents on every connected graded bialgebra H ?

Question 2.28. Is the dual of PNSym a polynomial ring? (For comparison, the dual of NSym is QSym, which is a polynomial ring by a result of Hazewinkel [GriRei20, Theorem 6.4.3]. Note that the dual of PNSym is a commutative connected graded $\mathbf{k}$-bialgebra, and thus a polynomial ring whenever $\mathbf{k}$ is a field of characteristic 0 by Leray's theorem [GriRei20, Remark 1.7.30 (a)]. The interesting question is what happens for arbitrary $\mathbf{k}$.)

Question 2.29. Let us rename the Janus monoid J as J_n to stress its dependence on n. Can the Janus algebras $\mathbf{k}[\mathrm{J}_n]$ for varying $n \in \mathbb{N}$ be combined into a combinatorial Hopf algebra on the $\mathbf{k}$-module $\bigoplus_{n\in\mathbb{N}} \mathbf{k}[\mathrm{J}_n]$ equipped with a second multiplication, extending PNSym? (The second multiplication would be coming from the $\mathbf{k}[\mathrm{J}_n]$'s, while the first multiplication and the comultiplication would extend those on PNSym.) Such a Hopf algebra would probably be a Janus analogue of the *NCQSym** from [BerZab09, §5.2].

Sarah Brauner suggests another question: It is known that for any $n \in \mathbb{N}$, the span of the elements $\sum_{\alpha \text{ is a composition of } n;\ \ell(\alpha)=k} \mathbf{H}_\alpha$ over all $k \in \{0,1,\ldots,n\}$ is a commutative subalgebra of the $\mathbf{k}$-algebra $\mathrm{NSym}_n^{(2)}$; this subalgebra is known as the *Eulerian subalgebra* (this appears, e.g., in [Schock04, §4.2] in the language of the descent algebra). Is there an analogous "Eulerian" subalgebra of $\mathrm{PNSym}_n^{(2)}$? The following appears to work for small n:

Question 2.30. Fix $n \in \mathbb{N}$, and set $\mathfrak{e}_{k,\sigma} := \sum_{\alpha \text{ is a composition of } n;\ \ell(\alpha)=k} F_{\alpha,\sigma} \in \mathrm{PNSym}_n$ for each $k \in \mathbb{N}$ and each $\sigma \in \mathfrak{S}_k$. Is the span of these $\mathfrak{e}_{k,\sigma}$ a $\mathbf{k}$-subalgebra of $\mathrm{PNSym}_n^{(2)}$ (that is, closed under $*$)?

The answer is positive for all $n \leq 4$, but this subalgebra is not commutative for $n = 3$ already.

3. Application to identity checking

In this last section, we shall outline a way in which the above results can be used. For an example, let us prove that every connected graded bialgebra H satisfies

$$(p_1 \star p_2 - p_2 \star p_1)^5 = 0 \qquad (51)$$

(where the 5-th power is taken with respect to the composition $\circ$). To prove this equality, we can rewrite the left hand side as $\left(p_{(1,2),\mathrm{id}} - p_{(2,1),\mathrm{id}}\right)^5$ (where id is the identity permutation in $\mathfrak{S}_2$), and expand this (using Theorem 1.19 and Proposition 1.15) as a $\mathbb{Z}$-linear combination of $p_{\alpha,\sigma}$'s. According to Theorem 1.36, for the above equality (51) to hold, this $\mathbb{Z}$-linear combination has to be trivial (i.e., all coefficients must be zeroes), which we can directly check. Alternatively, using Theorem 2.9, we can reduce the equality (51) to the equality

$$\left(F_{(1,2),\mathrm{id}} - F_{(2,1),\mathrm{id}}\right)^{*5} = 0 \qquad \text{in PNSym}$$

(where the "$*5$" exponent means a 5-th power with respect to the internal product $*$). This equality is straightforward to check using the definition of $*$.

Likewise, any identity such as (51) (that is, any identity whose both sides are formed from the maps $p_{\alpha,\sigma}$ by addition, convolution and composition) can be proved mechanically using computations inside PNSym. This approach can be useful even if we don't end up making the computations. For example, it shows that when proving an identity such as (51) (with integer coefficients), it suffices to prove it for $\mathbf{k} = \mathbb{Q}$ (since the Hopf algebra PNSym with all its structures is defined over $\mathbb{Z}$ and is free as a $\mathbf{k}$-module, so that its $\mathbb{Z}$-version embeds naturally into its $\mathbb{Q}$-version). Moreover, it suffices to prove it in the case when all graded components H_n of H are finite-dimensional $\mathbf{k}$-vector spaces (thanks to Theorem 1.36 **(b)**). This makes a number of methods available (graded duals, coradical filtration[19]) that could not be used in the a-priori generality of a connected graded bialgebra over an arbitrary commutative ring. While this is perhaps not very unexpected, it is reassuring and helpful.

This all can be applied to a slightly wider class of identities. Indeed, while in (51) we don't allow the use of the identity map $\mathrm{id}_H : H \to H$, we can actually express id_H as the infinite sum $p_0 + p_1 + p_2 + \cdots$. Infinite sums are not allowed in (51), but we can replace them by finite partial sums if we restrict ourselves to the submodule $H_0 + H_1 + \cdots + H_k$ of H. For example, if we want to prove the equality

$$(p_1 \star \mathrm{id}_H - 2\,\mathrm{id}_H) \circ (p_1 \star \mathrm{id}_H)^2 = 0 \qquad \text{on } H_2$$

for any connected graded bialgebra H (this identity is part of [Grinbe22, Theorem 1.4 **(b)**]), then we can replace each id_H by $p_0 + p_1 + p_2$, which renders the equality

[19] Both of these methods have been used in proving such identities (see, e.g., [AguLau14], [Pang18, §3]).

amenable to our PNSym-method. Likewise, the antipode S of a connected graded Hopf algebra H is not directly of a form supported by our method, but can be written as an infinite sum

$$\sum_{k\in\mathbb{N}} (-1)^k (p_1+p_2+p_3+\cdots)^{\star k} = \sum_{k\in\mathbb{N}} (-1)^k \sum_{\alpha\in\mathbb{N}^k \text{ is a composition}} p_{\alpha,\mathrm{id}},$$

which – when restricted to a submodule $H_0+H_1+\cdots+H_k$ – can be replaced by a finite partial sum. Thus, claims such as Aguiar's and Lauve's result "$\left(S^2-\mathrm{id}\right)^k=0$ on H_k for any connected graded Hopf algebra H" (see [AguLau14, Proposition 7]) can (for a fixed $k\in\mathbb{N}$) be checked using the PNSym-method (although this particular claim has been extended and proved in a much more elementary way in [Grinbe21]). I believe that the field of Hopf algebra identities is vast and mostly unexplored so far.

Using Theorem 1.37, we can obtain a similar method for checking identities for general (not just connected) graded bialgebras. Such identities can be viewed as equalities in the bialgebra WPNSym (constructed in our second proof of Theorem 2.7) instead of PNSym. Incidentally, this shows that any such identity that holds for all graded **k**-Hopf algebras will necessarily hold for all graded **k**-bialgebras as well.

Question 3.1. Is there a similar method for checking identities between maps $H^{\otimes k}\to H^{\otimes \ell}$ as opposed to only maps $H\to H$?

The same problem without the grading has been solved by categorical methods in Pirashvili's 2002 paper [Pirash02].

We end this section with an open question that generalizes (51):

Question 3.2. Let $i,j\in\mathbb{N}$. What is the smallest integer $k(i,j)\in\mathbb{N}$ for which every connected graded bialgebra H satisfies

$$\left(p_i\star p_j - p_j\star p_i\right)^{k(i,j)} = 0$$

(where the power is with respect to composition)? In other words, what is the smallest integer $k(i,j)\in\mathbb{N}$ for which

$$\left(F_{(i,j),\mathrm{id}} - F_{(j,i),\mathrm{id}}\right)^{*k(i,j)} = 0 \qquad \text{in } \mathrm{PNSym}^{(2)} \ ?$$

Some known values (computed using SageMath) are

$$k(0,j)=1, \quad k(i,i)=1, \quad k(1,2)=5, \quad k(1,3)=7,$$
$$k(1,4)=9, \quad k(1,5)=11, \quad k(1,6)=13, \quad k(2,3)=9,$$
$$k(2,4)=9.$$

(Of course, $k(i,j)=k(j,i)$ for all i and j.) It is striking that all these known values are odd.

A. Appendix: Rigorous proof of Proposition 2.5

We proved Proposition 2.5 using combinatorial handwaving; let us now give a rigorous (if rather cumbersome and tiring) proof. Arguably, this is just the above informal proof, rewritten without Sweedler notation and handwaving, but the rewrite has turned out to be surprisingly nontrivial.

Rigorous proof of Proposition 2.5. Forget that we fixed (α, σ) and (β, τ).

In the following, the "hat" symbol $\widehat{\ }$ over an entry of a list will signify that this entry must be omitted from the list. For instance, "$(\alpha_1, \alpha_2, \ldots, \widehat{\alpha_q}, \ldots, \alpha_k)$" denotes the list $(\alpha_1, \alpha_2, \ldots, \alpha_k)$ without its q-th entry.

For each weak mopiscotion (α, σ) with $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k) \in \mathbb{N}^k$ and $\sigma \in \mathfrak{S}_k$, and for each index $q \in [k]$ that satisfies $\alpha_q = 0$, we define a weak mopiscotion $\operatorname{del}(\alpha, \sigma, q)$ as follows:

- Let α' denote the weak composition $(\alpha_1, \alpha_2, \ldots, \widehat{\alpha_q}, \ldots, \alpha_k)$. This is the $(k-1)$-tuple obtained from α by removing the q-th entry, which is $\alpha_q = 0$. Note that any 0's that appear in α in positions other than the q-th are retained in α'.
- Let $\sigma' \in \mathfrak{S}_{k-1}$ be the standardization of the list $\left(\sigma(1), \sigma(2), \ldots, \widehat{\sigma(q)}, \ldots, \sigma(k)\right)$.
- Then, we define $\operatorname{del}(\alpha, \sigma, q)$ to be the weak mopiscotion (α', σ').

We call $\operatorname{del}(\alpha, \sigma, q)$ the *deletion* of q from (α, σ).

We can view this deletion as a single step from the weak mopiscotion (α, σ) to its reduction $\operatorname{red}(\alpha, \sigma)$. The formal reason for this is the following lemma:

> *Claim 1:* Let (α, σ) be a weak mopiscotion with $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k) \in \mathbb{N}^k$. Let $q \in [k]$ be such that $\alpha_q = 0$. Let $(\alpha', \sigma') := \operatorname{del}(\alpha, \sigma, q)$. Then,
>
> $$\operatorname{red}(\alpha, \sigma) = \operatorname{red}(\alpha', \sigma').$$
>
> In other words, deleting q from (α, σ) and then reducing the resulting weak mopiscotion produces the same result as reducing (α, σ) right away.

Proof of Claim 1. This is one of the kind of proofs that give combinatorics a bad name; but we have little choice. Set

$$(\beta, \tau) := \operatorname{red}(\alpha, \sigma) \qquad \text{and} \qquad (\gamma, \omega) := \operatorname{red}(\alpha', \sigma').$$

Thus, we must show that $(\beta, \tau) = (\gamma, \omega)$.

Both compositions β and γ are obtained from α by removing all zeroes (either all at once in the case of β, or starting with α_q and then the rest in the case of γ). Hence, $\beta = \gamma$. It remains to show that $\tau = \omega$.

Let $(r_1 < r_2 < \cdots < r_{k-1})$ be the list of all elements of $[k] \setminus \{q\}$, in increasing order. Thus,

$$[k] \setminus \{q\} = \{r_1 < r_2 < \cdots < r_{k-1}\}$$

and

$$(1 < 2 < \cdots < \widehat{q} < \cdots < k) = (r_1 < r_2 < \cdots < r_{k-1}) \tag{52}$$

(an equality of strictly increasing $(k-1)$-tuples). Explicitly, we have $r_i = i + [i \geq q]$ for each $i \in [k-1]$, where we use the Iverson bracket notation ($[\mathcal{A}]$ denotes the truth value of a statement $\mathcal{A}$).

By the definition of $\operatorname{del}(\alpha, \sigma, q)$, we have $\alpha' = \left(\alpha_1, \alpha_2, \ldots, \widehat{\alpha_q}, \ldots, \alpha_k\right)$. Let us also denote this $(k-1)$-tuple α' as $\alpha' = \left(\alpha'_1, \alpha'_2, \ldots, \alpha'_{k-1}\right)$. Thus,

$$\begin{aligned} \left(\alpha'_1, \alpha'_2, \ldots, \alpha'_{k-1}\right) &= \alpha' = \left(\alpha_1, \alpha_2, \ldots, \widehat{\alpha_q}, \ldots, \alpha_k\right) \\ &= \left(\alpha_{r_1}, \alpha_{r_2}, \ldots, \alpha_{r_{k-1}}\right) \qquad \text{(by (52))}. \end{aligned}$$

In other words,

$$\alpha'_i = \alpha_{r_i} \qquad \text{for each } i \in [k-1]. \tag{53}$$

Let $(j_1 < j_2 < \cdots < j_h)$ be the list of all elements i of $[k-1]$ satisfying $\alpha'_i \neq 0$, in increasing order. Then, by the definition of a reduction, ω is the standardization of $(\sigma'(j_1), \sigma'(j_2), \ldots, \sigma'(j_h))$ (since $(\gamma, \omega) = \operatorname{red}(\alpha', \sigma')$). Hence, by the definition of standardization, we know that for every two elements a and b of $[h]$ satisfying $a < b$, we have the equivalence

$$(\omega(a) < \omega(b)) \iff \left(\sigma'(j_a) \leq \sigma'(j_b)\right). \tag{54}$$

Recall that $(j_1 < j_2 < \cdots < j_h)$ is the list of all elements i of $[k-1]$ satisfying $\alpha'_i \neq 0$. Thus, the number of these elements is h. In other words, the $(k-1)$-tuple α' has exactly h nonzero entries. Hence, the k-tuple α has exactly h nonzero entries, too (since the $(k-1)$-tuple α' is obtained from α by removing the zero entry $\alpha_q = 0$, and thus has the same nonzero entries as α).

Next, we observe that the chain of inequalities $r_{j_1} < r_{j_2} < \cdots < r_{j_h}$ holds, since it is a sub-chain of $r_1 < r_2 < \cdots < r_{k-1}$ (because the subscripts satisfy $j_1 < j_2 < \cdots < j_h$).

Recall that $(j_1 < j_2 < \cdots < j_h)$ is the list of all elements i of $[k-1]$ satisfying $\alpha'_i \neq 0$. Thus, for each $i \in \{j_1 < j_2 < \cdots < j_h\}$, we have $\alpha'_i \neq 0$ and therefore $\alpha_{r_i} \neq 0$ (since (53) yields $\alpha_{r_i} = \alpha'_i \neq 0$). In other words, the numbers $\alpha_{r_{j_1}}, \alpha_{r_{j_2}}, \ldots, \alpha_{r_{j_h}}$ are all nonzero. Thus, $\alpha_{r_{j_1}}, \alpha_{r_{j_2}}, \ldots, \alpha_{r_{j_h}}$ are h nonzero entries of α (all occupying different positions in α, because $r_{j_1} < r_{j_2} < \cdots < r_{j_h}$). There cannot be any further nonzero entries of α besides these (since α has exactly h nonzero entries). Thus, $\alpha_{r_{j_1}}, \alpha_{r_{j_2}}, \ldots, \alpha_{r_{j_h}}$ are **all** the nonzero entries of α, listed from left to right (since $r_{j_1} < r_{j_2} < \cdots < r_{j_h}$).

In other words, $\left(r_{j_1} < r_{j_2} < \cdots < r_{j_h}\right)$ is the list of all elements i of $[k]$ satisfying $\alpha_i \neq 0$, in increasing order.

Therefore, by the definition of reduction, τ is the standardization of $\left(\sigma\left(r_{j_1}\right), \sigma\left(r_{j_2}\right), \ldots, \sigma\left(r_{j_h}\right)\right)$ (since $(\beta, \tau) = \operatorname{red}(\alpha, \sigma)$). Hence, by the definition of standardization, we know that for every two elements a and b of $[h]$ satisfying $a < b$, we have the equivalence

$$(\tau(a) < \tau(b)) \iff \left(\sigma\left(r_{j_a}\right) \leq \sigma\left(r_{j_b}\right)\right). \tag{55}$$

Finally, recall that $(\alpha', \sigma') = \operatorname{del}(\alpha, \sigma, q)$. Hence, the definition of deletion yields that σ' is the standardization of $\left(\sigma(1), \sigma(2), \ldots, \widehat{\sigma(q)}, \ldots, \sigma(k)\right)$. In other words, σ' is the standardization of $(\sigma(r_1), \sigma(r_2), \ldots, \sigma(r_{k-1}))$ (since (52) shows that $\left(\sigma(1), \sigma(2), \ldots, \widehat{\sigma(q)}, \ldots, \sigma(k)\right) = (\sigma(r_1), \sigma(r_2), \ldots, \sigma(r_{k-1}))$). Hence, for every two elements a and b of $[k-1]$ satisfying $a < b$, we have the equivalence

$$\left(\sigma'(a) < \sigma'(b)\right) \iff (\sigma(r_a) \leq \sigma(r_b)) \tag{56}$$

(by the definition of standardization).

Now, for any two elements a and b of $[h]$ satisfying $a < b$, we have the following chain of equivalences:

$$\begin{aligned}
(\omega(a) < \omega(b)) &\iff (\sigma'(j_a) \leq \sigma'(j_b)) \qquad \text{(by (54))} \\
&\iff (\sigma'(j_a) < \sigma'(j_b)) \\
&\qquad \left(\begin{array}{c}\text{since } a < b \text{ entails } j_a < j_b \\ \text{(because } j_1 < j_2 < \cdots < j_h\text{) and} \\ \text{thus } j_a \neq j_b \text{ and thus } \sigma'(j_a) \neq \sigma'(j_b) \\ \text{(since } \sigma' \text{ is a permutation)}\end{array}\right) \\
&\iff (\sigma(r_{j_a}) \leq \sigma(r_{j_b})) \qquad \left(\begin{array}{c}\text{by (56)} \\ \text{(applied to } j_a \text{ and } j_b \text{ instead of } a \text{ and } b\text{),} \\ \text{since } a < b \text{ entails } j_a < j_b \\ \text{(because } j_1 < j_2 < \cdots < j_h\text{)}\end{array}\right) \\
&\iff (\tau(a) < \tau(b)) \qquad \text{(by (55))} \\
&\iff (\tau(a) \leq \tau(b)) \qquad \left(\begin{array}{c}\text{since } a < b \text{ entails } a \neq b \text{ and} \\ \text{thus } \tau(a) \neq \tau(b) \text{ (since } \tau \text{ is a permutation)}\end{array}\right).
\end{aligned}$$

This entails that ω is the standardization of the list $(\tau(1), \tau(2), \ldots, \tau(h))$. But the latter standardization is just τ itself (since τ is a permutation of $[h]$). Hence, we have proved that ω is just τ. In other words, $\tau = \omega$. Combining this with $\beta = \gamma$, we obtain $(\beta, \tau) = (\gamma, \omega)$. As explained, this proves Claim 1. □

We will need one further combinatorial lemma about deletion:

> *Claim 2:* Let (α, σ) be a weak mopiscotion with $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k) \in \mathbb{N}^k$. Let $q \in [k]$ be such that $\alpha_q = 0$. Let $(\beta, \tau) := \operatorname{del}(\alpha, \sigma, q)$ and $r := \sigma(q)$. Write the $(k-1)$-tuple β as $\beta = (\beta_1, \beta_2, \ldots, \beta_{k-1})$.
>
> Let $(x_1, x_2, \ldots, x_k)$ be a k-tuple of arbitrary objects. Let $(y_1, y_2, \ldots, y_{k-1}) := (x_1, x_2, \ldots, \widehat{x_r}, \ldots, x_k)$. Then,
>
> $$\left(x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(q-1)}\right) = \left(y_{\tau(1)}, y_{\tau(2)}, \ldots, y_{\tau(q-1)}\right) \tag{57}$$
>
> and
>
> $$\left(x_{\sigma(q+1)}, x_{\sigma(q+2)}, \ldots, x_{\sigma(k)}\right) = \left(y_{\tau(q)}, y_{\tau(q+1)}, \ldots, y_{\tau(k-1)}\right) \tag{58}$$
>
> and
>
> $$\left(\alpha_1, \alpha_2, \ldots, \alpha_{q-1}\right) = \left(\beta_1, \beta_2, \ldots, \beta_{q-1}\right) \tag{59}$$
>
> and
>
> $$\left(\alpha_{q+1}, \alpha_{q+2}, \ldots, \alpha_k\right) = \left(\beta_q, \beta_{q+1}, \ldots, \beta_{k-1}\right). \tag{60}$$

Proof of Claim 2. By assumption, we have $(\beta, \tau) = \operatorname{del}(\alpha, \sigma, q)$. By the definition of deletion, this means that $\beta = \left(\alpha_1, \alpha_2, \ldots, \widehat{\alpha_q}, \ldots, \alpha_k\right)$, whereas τ is the standardization of the list $\left(\sigma(1), \sigma(2), \ldots, \widehat{\sigma(q)}, \ldots, \sigma(k)\right)$.

We have

$$\begin{aligned}\left(\alpha_1, \alpha_2, \ldots, \alpha_{q-1}, \alpha_{q+1}, \alpha_{q+2}, \ldots, \alpha_k\right) &= \left(\alpha_1, \alpha_2, \ldots, \widehat{\alpha_q}, \ldots, \alpha_k\right) \\ &= \beta = \left(\beta_1, \beta_2, \ldots, \beta_{k-1}\right).\end{aligned}$$

This is an equality between $(k-1)$-tuples. Splitting it into two parts (between the $(q-1)$-th and q-th entries), we obtain

$$\left(\alpha_1, \alpha_2, \ldots, \alpha_{q-1}\right) = \left(\beta_1, \beta_2, \ldots, \beta_{q-1}\right)$$

and

$$\left(\alpha_{q+1}, \alpha_{q+2}, \ldots, \alpha_k\right) = \left(\beta_q, \beta_{q+1}, \ldots, \beta_{k-1}\right).$$

Thus, we have proved (59) and (60). It remains to prove (57) and (58).

Let ∂_r be the unique strictly increasing bijection from $[k-1]$ to $[k] \setminus \{r\}$. Explicitly, the map ∂_r sends the numbers $1, 2, \ldots, k-1$ to $1, 2, \ldots, \widehat{r}, \ldots, k$, respectively. Similarly, let ∂_q be the unique strictly increasing bijection from $[k-1]$ to $[k] \setminus \{q\}$. We have

$$\left(y_1, y_2, \ldots, y_{k-1}\right) = \left(x_1, x_2, \ldots, \widehat{x_r}, \ldots, x_k\right) = \left(x_{\partial_r(1)}, x_{\partial_r(2)}, \ldots, x_{\partial_r(k-1)}\right)$$

(since the definition of ∂_r yields $(1, 2, \ldots, \widehat{r}, \ldots, k) = (\partial_r(1), \partial_r(2), \ldots, \partial_r(k-1))$). In other words,

$$y_i = x_{\partial_r(i)} \qquad \text{for each } i \in [k-1]. \tag{61}$$

Recall that (α, σ) is a weak mopiscotion and $\alpha \in \mathbb{N}^k$. Hence, $\sigma \in \mathfrak{S}_k$.

The map σ is a bijection from $[k]$ to $[k]$ (since $\sigma \in \mathfrak{S}_k$) and sends q to r (since $r = \sigma(q)$). Hence, it restricts to a bijection $\overline{\sigma}$ from $[k] \setminus \{q\}$ to $[k] \setminus \{r\}$. Consider this bijection $\overline{\sigma}$. The composition $\partial_r^{-1} \circ \overline{\sigma} \circ \partial_q$ is thus a bijection from $[k-1]$ to $[k-1]$, that is, a permutation of $[k-1]$. It is easy to see that this composition $\partial_r^{-1} \circ \overline{\sigma} \circ \partial_q$ is precisely the permutation τ [20]. In other words, $\partial_r^{-1} \circ \overline{\sigma} \circ \partial_q = \tau$. Hence, $\overline{\sigma} \circ \partial_q = \partial_r \circ \tau$. Therefore, each $i \in [k-1]$ satisfies

$$\begin{aligned}\sigma\left(\partial_q(i)\right) &= \overline{\sigma}\left(\partial_q(i)\right) && (\text{since } \overline{\sigma} \text{ is a restriction of } \sigma) \\ &= \left(\overline{\sigma} \circ \partial_q\right)(i) = \left(\partial_r \circ \tau\right)(i) && \left(\text{since } \overline{\sigma} \circ \partial_q = \partial_r \circ \tau\right) \\ &= \partial_r(\tau(i)).\end{aligned} \tag{62}$$

Now, for each $i \in [k-1]$, we have

$$\begin{aligned}y_{\tau(i)} &= x_{\partial_r(\tau(i))} && (\text{by (61), applied to } \tau(i) \text{ instead of } i) \\ &= x_{\sigma\left(\partial_q(i)\right)} && \left(\text{since (62) yields } \partial_r(\tau(i)) = \sigma\left(\partial_q(i)\right)\right).\end{aligned}$$

[20] *Proof.* Let $\psi := \partial_r^{-1} \circ \overline{\sigma} \circ \partial_q$. As we know, this ψ is a permutation of $[k-1]$. Hence, the standardization of the list $(\psi(1), \psi(2), \ldots, \psi(k-1))$ is ψ itself.

Recall that τ is the standardization of the list

$$\left(\sigma(1), \sigma(2), \ldots, \widehat{\sigma(q)}, \ldots, \sigma(k)\right) = \left(\sigma\left(\partial_q(1)\right), \sigma\left(\partial_q(2)\right), \ldots, \sigma\left(\partial_q(k-1)\right)\right)$$

(since the definition of ∂_q yields $(1, 2, \ldots, \widehat{q}, \ldots, k) = \left(\partial_q(1), \partial_q(2), \ldots, \partial_q(k-1)\right)$). By the definition of standardization, this shows that for every two elements a and b of $[k-1]$ satisfying $a < b$, we have the equivalence $(\tau(a) < \tau(b)) \iff \left(\sigma\left(\partial_q(a)\right) \leq \sigma\left(\partial_q(b)\right)\right)$. Thus, for every

In other words,

$$\begin{aligned}\left(y_{\tau(1)}, y_{\tau(2)}, \ldots, y_{\tau(k-1)}\right) &= \left(x_{\sigma\left(\partial_q(1)\right)}, x_{\sigma\left(\partial_q(2)\right)}, \ldots, x_{\sigma\left(\partial_q(k-1)\right)}\right) \\ &= \left(x_{\sigma(1)}, x_{\sigma(2)}, \ldots, \widehat{x_{\sigma(q)}}, \ldots, x_{\sigma(k)}\right)\end{aligned}$$

(since the definition of ∂_q yields $\left(\partial_q(1), \partial_q(2), \ldots, \partial_q(k-1)\right) = (1, 2, \ldots, \widehat{q}, \ldots, k)$). In other words,

$$\left(x_{\sigma(1)}, x_{\sigma(2)}, \ldots, \widehat{x_{\sigma(q)}}, \ldots, x_{\sigma(k)}\right) = \left(y_{\tau(1)}, y_{\tau(2)}, \ldots, y_{\tau(k-1)}\right).$$

This is an equality between two $(k-1)$-tuples. Splitting it into two parts (between the $(q-1)$-th and q-th entries), we obtain

$$\left(x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(q-1)}\right) = \left(y_{\tau(1)}, y_{\tau(2)}, \ldots, y_{\tau(q-1)}\right)$$

and

$$\left(x_{\sigma(q+1)}, x_{\sigma(q+2)}, \ldots, x_{\sigma(k)}\right) = \left(y_{\tau(q)}, y_{\tau(q+1)}, \ldots, y_{\tau(k-1)}\right).$$

Thus, (57) and (58) are proved. This completes the proof of Claim 2. □

Thanks to Claim 1, we can easily reduce Proposition 2.5 to the following claim about deletion:

> *Claim 3:* Let (α, σ) be a weak mopiscotion with $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k) \in \mathbb{N}^k$. Let $q \in [k]$ be such that $\alpha_q = 0$. Let $(\beta, \tau) := \operatorname{del}(\alpha, \sigma, q)$. Then, $p_{\alpha,\sigma} = p_{\beta,\tau}$.

---

two elements a and b of $[k-1]$ satisfying $a < b$, we have the chain of equivalences

$$\begin{aligned}(\tau(a) < \tau(b)) &\iff \left(\sigma\left(\partial_q(a)\right) \leq \sigma\left(\partial_q(b)\right)\right) \\ &\iff \left(\overline{\sigma}\left(\partial_q(a)\right) \leq \overline{\sigma}\left(\partial_q(b)\right)\right) \\ &\qquad \left(\begin{array}{c}\text{since } \overline{\sigma} \text{ is a restriction of the map } \sigma, \\ \text{and thus we have } \sigma\left(\partial_q(a)\right) = \overline{\sigma}\left(\partial_q(a)\right) \\ \text{and } \sigma\left(\partial_q(b)\right) = \overline{\sigma}\left(\partial_q(b)\right)\end{array}\right) \\ &\iff \left(\partial_r^{-1}\left(\overline{\sigma}\left(\partial_q(a)\right)\right) \leq \partial_r^{-1}\left(\overline{\sigma}\left(\partial_q(b)\right)\right)\right) \\ &\qquad \left(\begin{array}{c}\text{since the map } \partial_r^{-1} \text{ is strictly increasing} \\ \text{(being the inverse of the strictly increasing} \\ \text{bijection } \partial_r\text{), and thus preserves inequalities}\end{array}\right) \\ &\iff \left(\underbrace{\left(\partial_r^{-1} \circ \overline{\sigma} \circ \partial_q\right)}_{=\psi}(a) \leq \underbrace{\left(\partial_r^{-1} \circ \overline{\sigma} \circ \partial_q\right)}_{=\psi}(b)\right) \\ &\iff (\psi(a) \leq \psi(b)).\end{aligned}$$

Hence, τ is the standardization of the list $(\psi(1), \psi(2), \ldots, \psi(k-1))$. In other words, τ is ψ (since the standardization of the list $(\psi(1), \psi(2), \ldots, \psi(k-1))$ is ψ itself). Thus, $\tau = \psi = \partial_r^{-1} \circ \overline{\sigma} \circ \partial_q$. That is, the permutation $\partial_r^{-1} \circ \overline{\sigma} \circ \partial_q$ is just τ.

Proof of Claim 3. Let $r := \sigma(q)$. Write the $(k-1)$-tuple β as $\beta = (\beta_1, \beta_2, \ldots, \beta_{k-1})$.

Define the $\mathbf{k}$-linear maps

$$f : H^{\otimes k} \to H^{\otimes (k-1)},$$
$$x_1 \otimes x_2 \otimes \cdots \otimes x_k \mapsto \epsilon(x_r) \cdot x_1 \otimes x_2 \otimes \cdots \otimes \widehat{x_r} \otimes \cdots \otimes x_k$$

and

$$g : H^{\otimes (k-1)} \to H^{\otimes k},$$
$$x_1 \otimes x_2 \otimes \cdots \otimes x_{k-1} \mapsto x_1 \otimes x_2 \otimes \cdots \otimes x_{q-1} \otimes 1_H \otimes x_q \otimes x_{q+1} \otimes \cdots \otimes x_{k-1}.$$

It is well-known and easy to prove that that $f \circ \Delta^{[k]} = \Delta^{[k-1]}$ and $m^{[k]} \circ g = m^{[k-1]}$. (Indeed, the latter equality says that a product of the form $x_1 x_2 \cdots x_{q-1} 1_H x_q x_{q+1} \cdots x_{k-1}$ can be simplified by removing the factor 1_H; the former equality is just the dual result with q replaced by r.) Now, if we can show that

$$P_\alpha \circ \sigma^{-1} = g \circ P_\beta \circ \tau^{-1} \circ f \tag{63}$$

as maps from $H^{\otimes k}$ to $H^{\otimes k}$, then we will be able to conclude that

$$\begin{aligned} p_{\alpha,\sigma} &= m^{[k]} \circ \underbrace{P_\alpha \circ \sigma^{-1}}_{= g \circ P_\beta \circ \tau^{-1} \circ f} \circ \Delta^{[k]} \qquad \text{(by the definition of } p_{\alpha,\sigma}\text{)} \\ &= \underbrace{m^{[k]} \circ g}_{= m^{[k-1]}} \circ P_\beta \circ \tau^{-1} \circ \underbrace{f \circ \Delta^{[k]}}_{= \Delta^{[k-1]}} \\ &= m^{[k-1]} \circ P_\beta \circ \tau^{-1} \circ \Delta^{[k-1]} = p_{\beta,\tau} \qquad \left(\text{by the definition of } p_{\beta,\tau}\right), \end{aligned}$$

and so Claim 3 will be proved.

Hence, it remains to prove (63). Let us compare what the maps $P_\alpha \circ \sigma^{-1}$ and $g \circ P_\beta \circ \tau^{-1} \circ f$ do to a pure tensor $x_1 \otimes x_2 \otimes \cdots \otimes x_k \in H^{\otimes k}$. For this purpose, we fix $x_1, x_2, \ldots, x_k \in H$, and we define the $(k-1)$-tuple

$$(y_1, y_2, \ldots, y_{k-1}) := (x_1, x_2, \ldots, \widehat{x_r}, \ldots, x_k) \in H^{k-1}.$$

Then,

$$y_1 \otimes y_2 \otimes \cdots \otimes y_{k-1} = x_1 \otimes x_2 \otimes \cdots \otimes \widehat{x_r} \otimes \cdots \otimes x_k.$$

We note that $p_0(h) = \epsilon(h) 1_H$ for each $h \in H$ (since H is connected). Thus, in particular,

$p_0(x_r) = \epsilon(x_r) 1_H$. Now

$$\begin{aligned}
&\left(P_\alpha \circ \sigma^{-1}\right)(x_1 \otimes x_2 \otimes \cdots \otimes x_k) \\
&= P_\alpha\left(\sigma^{-1}(x_1 \otimes x_2 \otimes \cdots \otimes x_k)\right) \\
&= P_\alpha\left(x_{\sigma(1)} \otimes x_{\sigma(2)} \otimes \cdots \otimes x_{\sigma(k)}\right) \qquad \left(\begin{array}{c}\text{by the definition of the} \\ \text{action of } \sigma^{-1} \text{ on } H^{\otimes k}\end{array}\right) \\
&= p_{\alpha_1}\left(x_{\sigma(1)}\right) \otimes p_{\alpha_2}\left(x_{\sigma(2)}\right) \otimes \cdots \otimes p_{\alpha_k}\left(x_{\sigma(k)}\right) \qquad (\text{by the definition of } P_\alpha) \\
&= \underbrace{p_{\alpha_1}\left(x_{\sigma(1)}\right) \otimes p_{\alpha_2}\left(x_{\sigma(2)}\right) \otimes \cdots \otimes p_{\alpha_{q-1}}\left(x_{\sigma(q-1)}\right)}_{\substack{=p_{\beta_1}\left(y_{\tau(1)}\right) \otimes p_{\beta_2}\left(y_{\tau(2)}\right) \otimes \cdots \otimes p_{\beta_{q-1}}\left(y_{\tau(q-1)}\right) \\ \text{(by (59) and (57))}}} \\
&\qquad \otimes \underbrace{p_{\alpha_q}\left(x_{\sigma(q)}\right)}_{\substack{=p_0(x_r) \\ \text{(since } \alpha_q = 0 \\ \text{and } \sigma(q) = r)}} \otimes \underbrace{p_{\alpha_{q+1}}\left(x_{\sigma(q+1)}\right) \otimes p_{\alpha_{q+2}}\left(x_{\sigma(q+2)}\right) \otimes \cdots \otimes p_{\alpha_k}\left(x_{\sigma(k)}\right)}_{\substack{=p_{\beta_q}\left(y_{\tau(q)}\right) \otimes p_{\beta_{q+1}}\left(y_{\tau(q+1)}\right) \otimes \cdots \otimes p_{\beta_{k-1}}\left(y_{\tau(k-1)}\right) \\ \text{(by (60) and (58))}}} \\
&= p_{\beta_1}\left(y_{\tau(1)}\right) \otimes p_{\beta_2}\left(y_{\tau(2)}\right) \otimes \cdots \otimes p_{\beta_{q-1}}\left(y_{\tau(q-1)}\right) \\
&\qquad \otimes \underbrace{p_0(x_r)}_{=\epsilon(x_r) 1_H} \otimes p_{\beta_q}\left(y_{\tau(q)}\right) \otimes p_{\beta_{q+1}}\left(y_{\tau(q+1)}\right) \otimes \cdots \otimes p_{\beta_{k-1}}\left(y_{\tau(k-1)}\right) \\
&= \epsilon(x_r) \cdot p_{\beta_1}\left(y_{\tau(1)}\right) \otimes p_{\beta_2}\left(y_{\tau(2)}\right) \otimes \cdots \otimes p_{\beta_{q-1}}\left(y_{\tau(q-1)}\right) \\
&\qquad \otimes 1_H \otimes p_{\beta_q}\left(y_{\tau(q)}\right) \otimes p_{\beta_{q+1}}\left(y_{\tau(q+1)}\right) \otimes \cdots \otimes p_{\beta_{k-1}}\left(y_{\tau(k-1)}\right).
\end{aligned}$$

Comparing this with

$$\begin{aligned}
&\left(g \circ P_\beta \circ \tau^{-1} \circ f\right)(x_1 \otimes x_2 \otimes \cdots \otimes x_k) \\
&= \left(g \circ P_\beta \circ \tau^{-1}\right)(f(x_1 \otimes x_2 \otimes \cdots \otimes x_k)) \\
&= \left(g \circ P_\beta \circ \tau^{-1}\right)\left(\epsilon(x_r) \cdot \underbrace{x_1 \otimes x_2 \otimes \cdots \otimes \widehat{x_r} \otimes \cdots \otimes x_k}_{= y_1 \otimes y_2 \otimes \cdots \otimes y_{k-1}}\right) \\
&\qquad \text{(by the definition of } f\text{)} \\
&= \left(g \circ P_\beta \circ \tau^{-1}\right)(\epsilon(x_r) \cdot y_1 \otimes y_2 \otimes \cdots \otimes y_{k-1}) \\
&= \epsilon(x_r) \cdot \left(g \circ P_\beta \circ \tau^{-1}\right)(y_1 \otimes y_2 \otimes \cdots \otimes y_{k-1}) \qquad \left(\text{by } \mathbf{k}\text{-linearity of } g \circ P_\beta \circ \tau^{-1}\right) \\
&= \epsilon(x_r) \cdot g\left(P_\beta\left(\tau^{-1}(y_1 \otimes y_2 \otimes \cdots \otimes y_{k-1})\right)\right) \\
&= \epsilon(x_r) \cdot g\left(P_\beta\left(y_{\tau(1)} \otimes y_{\tau(2)} \otimes \cdots \otimes y_{\tau(k-1)}\right)\right) \\
&\qquad \left(\text{by the definition of the action of } \tau^{-1} \text{ on } H^{\otimes(k-1)}\right) \\
&= \epsilon(x_r) \cdot g\left(p_{\beta_1}\left(y_{\tau(1)}\right) \otimes p_{\beta_2}\left(y_{\tau(2)}\right) \otimes \cdots \otimes p_{\beta_{k-1}}\left(y_{\tau(k-1)}\right)\right) \\
&\qquad \text{(by the definition of } P_\beta\text{)} \\
&= \epsilon(x_r) \cdot p_{\beta_1}\left(y_{\tau(1)}\right) \otimes p_{\beta_2}\left(y_{\tau(2)}\right) \otimes \cdots \otimes p_{\beta_{q-1}}\left(y_{\tau(q-1)}\right) \\
&\qquad \otimes 1_H \otimes p_{\beta_q}\left(y_{\tau(q)}\right) \otimes p_{\beta_{q+1}}\left(y_{\tau(q+1)}\right) \otimes \cdots \otimes p_{\beta_{k-1}}\left(y_{\tau(k-1)}\right) \\
&\qquad \text{(by the definition of } g\text{)},
\end{aligned}$$

we obtain

$$\left(P_\alpha \circ \sigma^{-1}\right)(x_1 \otimes x_2 \otimes \cdots \otimes x_k) = \left(g \circ P_\beta \circ \tau^{-1} \circ f\right)(x_1 \otimes x_2 \otimes \cdots \otimes x_k).$$

Since we have proved this for all $x_1, x_2, \ldots, x_k \in H$, we thus conclude that the two $\mathbf{k}$-linear maps $P_\alpha \circ \sigma^{-1}$ and $g \circ P_\beta \circ \tau^{-1} \circ f$ agree on all pure tensors. Thus, these two maps are identical. Hence, (63) is proved, and as we saw above, Claim 3 follows. □

Now, let us derive Proposition 2.5 from Claim 3.

Indeed, we define the *length* $\ell(\alpha)$ of a weak composition α to be the unique integer $k \in \mathbb{N}$ such that $\alpha \in \mathbb{N}^k$.

We shall now prove Proposition 2.5 by induction on the length $\ell(\alpha)$.

The *base case* ($\ell(\alpha) = 0$) is obvious (since in this case, α is the empty tuple $()$, and thus we have $(\beta, \tau) = (\alpha, \sigma)$).

For the *induction step*, we fix a positive integer k, and assume (as the induction hypothesis) that Proposition 2.5 holds whenever $\ell(\alpha) = k - 1$. We must now prove that Proposition 2.5 also holds whenever $\ell(\alpha) = k$. For this purpose, we fix a weak mopiscotion (α, σ) with $\ell(\alpha) = k$, and we set $(\beta, \tau) = \operatorname{red}(\alpha, \sigma)$. We must prove that $p_{\alpha,\sigma} = p_{\beta,\tau}$.

If all entries of α are nonzero, then this is obvious (since in this case we have $(\beta, \tau) = \operatorname{red}(\alpha, \sigma) = (\alpha, \sigma)$). Thus, we WLOG assume that not all entries of α are nonzero. In other words, some entry of α is 0.

Write α as $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$ (since $\alpha \in \mathbb{N}^k$). Then, there exists some $q \in [k]$ such that $\alpha_q = 0$ (since some entry of α is 0). Consider this q. Let $(\alpha', \sigma') := \operatorname{del}(\alpha, \sigma, q)$. Claim 3 (applied to (α', σ') instead of (β, τ)) then yields $p_{\alpha,\sigma} = p_{\alpha',\sigma'}$. But Claim 1 yields $\operatorname{red}(\alpha, \sigma) = \operatorname{red}(\alpha', \sigma')$, so that $(\beta, \tau) = \operatorname{red}(\alpha, \sigma) = \operatorname{red}(\alpha', \sigma')$. However, $\ell(\alpha') = k - 1$ (by the definition of deletion), and thus our induction hypothesis shows that Proposition 2.5 holds for the weak mopiscotion (α', σ') instead of (α, σ). Thus, we have $p_{\alpha',\sigma'} = p_{\beta,\tau}$ (since $(\beta, \tau) = \operatorname{red}(\alpha', \sigma')$). Thus, $p_{\alpha,\sigma} = p_{\alpha',\sigma'} = p_{\beta,\tau}$. Hence, Proposition 2.5 is proved for our (α, σ). This completes the induction step, and thus Proposition 2.5 is proved. □

References

[Aguiar17] Marcelo Aguiar, Swapneel Mahajan, *Topics in Hyperplane Arrangements*, Mathematical Surveys and Monographs **226**, AMS 2017.
See `http://pi.math.cornell.edu/~maguiar/surv-226.pdf` for a preprint.

[AguLau14] Marcelo Aguiar, Aaron Lauve, *The characteristic polynomial of the Adams operators on graded connected Hopf algebras*, arXiv:1403.7584v2. Published in: Algebra Number Theory **9** (3): 547–583 (2015). DOI: 10.2140/ant.2015.9.547.

[AguMah20] Marcelo Aguiar, Swapneel Mahajan, *Bimonoids for Hyperplane Arrangements*, Encyclopedia of Mathematics and its Applications **173**, Cambridge University Press 2020.

[AguMah22] Marcelo Aguiar, Swapneel Mahajan, *Coxeter Bialgebras*, Encyclopedia of Mathematics and its Applications **186**, Cambridge University Press 2020.

[BerZab09] Nantel Bergeron, Mike Zabrocki, *The Hopf algebras of symmetric functions and quasisymmetric functions in non-commutative variables are free and cofree*, J. of Algebra and its Applications **8** (2009), No 4, pp. 581–600.
See arXiv:0509265v3 for a preprint.

[Bidiga97] T. Patrick Bidigare, *Hyperplane Arrangement Face Algebras and their Associated Markov Chains*, PhD thesis at the University of Michigan, 1997.

[Brown04] Kenneth S. Brown, *Semigroup and Ring Theoretical Methods in Probability*, Fields Institute Communications, 2004.
`https://pi.math.cornell.edu/~kbrown/papers/toronto.pdf`

[CarPat21] Pierre Cartier, Frédéric Patras, *Classical Hopf Algebras and their Applications*, Algebra and Applications **29**, Springer 2021.

[FoiPat13] Loïc Foissy, Frédéric Patras, *Natural endomorphisms of shuffle algebras*, International Journal of Algebra and Computation **23** (2013), No. 04, pp.

989–1009.
A preprint is available at `http://loic.foissy.free.fr/pageperso/p21.pdf` .

[GarReu89] A. M. Garsia, C. Reutenauer, *A decomposition of Solomon's descent algebra*, Advances in Mathematics **77** (1989), Issue 2, pp. 189–262.

[GKLLRT94] Israel Gelfand, D. Krob, Alain Lascoux, B. Leclerc, V. S. Retakh, J.-Y. Thibon, *Noncommutative symmetric functions*, arXiv:hep-th/9407124v1, Advances in Mathematics **112** (1995), pp. 218–348.

[Grinbe17] Darij Grinberg, *On the logarithm of the identity on connected filtered bialgebras*, Version 0.19, 18 June 2021.
`https://www.cip.ifi.lmu.de/~grinberg/algebra/logid.pdf`

[Grinbe21] Darij Grinberg, *On the square of the antipode in a connected filtered Hopf algebra*, Communications in Mathematics **31** (2023), Issue 1, cm:10431, arXiv:2109.02101v3.
See also arXiv:2109.02101v2 for a more detailed version.

[Grinbe22] Darij Grinberg, *Project: Annihilating polynomials of Hopf algebra endomorphisms*, outline.
`https://www.cip.ifi.lmu.de/~grinberg/algebra/aphae-proj.pdf`

[GriRei20] Darij Grinberg, Victor Reiner, *Hopf algebras in Combinatorics*, version of 27 July 2020, `arXiv:1409.8356v7`.
See also `http://www.cip.ifi.lmu.de/~grinberg/algebra/HopfComb-sols.pdf` for a version that gets updated.

[Hazewi04] Michiel Hazewinkel, *Hopf algebras of endomorphisms of Hopf algebras*, 20 July 2004, arXiv:math/0410364v1.

[HiNoTh06] F. Hivert, J.-C. Novelli, J.-Y. Thibon, *Commutative combinatorial Hopf algebras*, J. Algebr. Comb. **28** (2008), pp. 65–95.

[JamKer81] Gordon James, Adalbert Kerber, *The representation theory of the symmetric group*, Encyclopedia of Mathematics and its Applications **16**, Addison-Wesley 1981.

[Kashin19] Yevgenia Kashina, *A generalized power map for Hopf algebras*, in: Stefaan Caenepeel, Freddy Van Oystaeyen (eds.), *Hopf algebras and quantum groups*, CRC Press 2000, pp. 159–176.

[KaSoZh06] Yevgenia Kashina, Yorck Sommerhäuser, Yongchang Zhu, *On higher Frobenius-Schur indicators*, Mem. Amer. Math. Soc. **181** (2006), number 855. A preprint is available at arXiv:math/0311199v2.

[KrLeTh97] D. Krob, B. Leclerc, J.-Y. Thibon, *Noncommutative symmetric functions. II. Transformations of alphabets*, Internat. J. Algebra Comput. **7** (1997), no. 2, pp. 181–264,
https://hal.science/hal-00018537/document

[Loday98] Jean-Louis Loday, *Cyclic Homology*, 2nd Edition, Springer 1998.

[LyuPak20] Hanbaek Lyu, Igor Pak, *On the number of contingency tables and the independence heuristic*, arXiv:2009.10810v1.

[Meliot17] Pierre-Loïc Méliot, *Representation Theory of Symmetric Groups*, Discrete Mathematics and its Applications, CRC Press 2017.

[Pang18] C.Y. Amy Pang, *Markov Chains from Descent Operators on Combinatorial Hopf Algebras*, arXiv:1609.04312v2.

[Pang21] C.Y. Amy Pang, *The Eigenvalues of Hyperoctahedral Descent Operators and Applications to Card-Shuffling*, arXiv:2108.09097v1.

[Patras92] Frédéric Patras, *Homotheties simpliciales*, PhD thesis at Université Paris VII.
https://math.univ-cotedazur.fr/~patras/articles/THESE_Patras_1992.pdf

[Patras94] F. Patras, *L'algèbre des descentes d'une bigèbre graduée*, Journal of Algebra **170** (1994), pp. 547–566.

[PatReu98] Frédéric Patras, Christophe Reutenauer, *Higher Lie idempotents*, 18 November 1998.

[PatSch06] Frédéric Patras, Manfred Schocker, *Twisted descent algebras and the Solomon–Tits algebra*, Advances in Mathematics **199** (2006), pp. 151–184.

[Pirash02] Teimuraz Pirashvili, *On the PROP corresponding to bialgebras*, Cahiers de topologie et géométrie différentielle catégoriques **43**, no. 3 (2002), pp. 221–239.

[Reuten93] Christophe Reutenauer, *Free Lie Algebras*, Oxford University Press 1993.

[Rousse94] G. Rousseau, *On the Jacobi Symbol*, Journal of Number Theory **48** (1994), pp. 109–111.

[Saliol06] Franco V. Saliola, *Hyperplane arrangements and descent algebras*, 1 October 2006.
https://saliola.github.io/maths/publications/LectureNotes/DesAlgLectureNotes.pdf
See https://www.cip.ifi.lmu.de/~grinberg/algebra/saliola-errata.pdf for inofficial errata.

[Saliol08] Franco V. Saliola, *On the quiver of the descent algebra*, Journal of Algebra **320** (2008), pp. 3866–3894.

[Schock04] Manfred Schocker, *The descent algebra of the symmetric group*, published in: Vlastimil Dlab and Claus Michael Ringel, Representations of Finite Dimensional Algebras and Related Topics in Lie Theory and Geometry, AMS 2004.

[Solomo76] Louis Solomon, *A Mackey Formula in the Group Ring of a Coxeter Group*, Journal of Algebra **41** (1976), pp. 255–268.