Comments on arXiv:2105.00538v3

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This is a comment on the preprint arXiv:2105.00538v3 (Eoghan McDowell, Mark Wildon, *Modular plethystic isomorphisms for two-dimensional linear groups*, arXiv:2105.00538v3, to appear in Journal of Algebra), in which I (believe I) prove Theorem 1.4 and Theorem 1.2 of said preprint in more transparent (and certainly less combinatorial) ways. I try to imitate the notation of the preprint, but in some places I cannot help reverting to my own (in particular, my groups all act from the left). I let K be an arbitrary commutative ring (not necessarily a field) throughout this comment (however, I will only work with free K-modules).

0.1. The modular Wronskian isomorphism

Let K be an arbitrary commutative ring. Let E be the natural representation of the group $\operatorname{GL}_2(K)$ on the free K-module K^2. Let $m, \ell \in \mathbb{N}$. In Theorem 1.4 of arXiv:2105.00538v3, you show that

$$\operatorname{Sym}_m \operatorname{Sym}^\ell E \otimes (\det E)^{\otimes m(m-1)/2} \cong \Lambda^m \operatorname{Sym}^{\ell+m-1} E$$

as $\operatorname{GL}_2(K)$-representations via a certain isomorphism that you call ζ in Definition 4.2.

Let me prove this in a somewhat simpler way. Specifically, I will show that ζ is $\operatorname{GL}_2(K)$-equivariant. The bijectivity of ζ follows easily enough from a triangularity argument (which is what you do in your proof of Lemma 4.5).

Let $K[X,Y]$ be the polynomial ring in 2 variables X, Y. We will identify $\operatorname{Sym} E$ with $K[X,Y]$, thus writing elements of $\operatorname{Sym} E$ as $f(X,Y)$. The group $\operatorname{GL}_2(K)$ acts on $K[X,Y]$ by the rule

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot f(X,Y) = f(aX+cY, bX+dY)$$
$$\text{for all } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \operatorname{GL}_2(K) \text{ and } f(X,Y) \in K[X,Y].$$

Let $K[\mathbf{X},\mathbf{Y}]$ be the polynomial ring in $2m$ variables $X_1, X_2, \ldots, X_m, Y_1, Y_2, \ldots, Y_m$. The symmetric group S_m acts on this ring $K[\mathbf{X},\mathbf{Y}]$ by K-algebra automorphisms (with any $\sigma \in S_m$ sending each X_i to $X_{\sigma(i)}$ and sending each Y_i to $Y_{\sigma(i)}$). Let $K[\mathbf{X}, \mathbf{Y}]^{\mathrm{sym}}$ be the invariant ring of this action. The group $\mathrm{GL}_2(K)$ acts on $K[\mathbf{X}, \mathbf{Y}]$ by acting on each (X_i, Y_i)-pair separately – i.e., by the rule

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot f(X_i, Y_i) = f(aX_i + cY_i, bX_i + dY_i)$$
$$\text{for all } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(K) \text{ and } i \in [m] \text{ and } f(X_i, Y_i) \in K[X_i, Y_i].$$

This $\mathrm{GL}_2(K)$-action commutes with the action of S_m, and thus induces a $\mathrm{GL}_2(K)$-action on $K[\mathbf{X}, \mathbf{Y}]^{\mathrm{sym}}$.

We will build a commutative diagram

$$\begin{array}{ccccc}
\mathrm{Sym}_m\left(\mathrm{Sym}^\ell E\right) & \hookrightarrow & \left(\mathrm{Sym}^\ell E\right)^{\otimes m} & \xrightarrow{\omega} & \Lambda^m \mathrm{Sym}^{\ell+m-1} E \\
\downarrow \alpha & & \downarrow \iota & & \downarrow \gamma \\
\mathrm{Sym}_m(\mathrm{Sym}\, E) & \hookrightarrow & (\mathrm{Sym}\, E)^{\otimes m} & & \Lambda^m \mathrm{Sym}\, E \\
\cong \downarrow \beta & & \cong \downarrow \kappa & & \downarrow \delta \\
K[\mathbf{X}, \mathbf{Y}]^{\mathrm{sym}} & \hookrightarrow & K[\mathbf{X}, \mathbf{Y}] & \xrightarrow[\psi]{} & K[\mathbf{X}, \mathbf{Y}]
\end{array}$$

where the horizontal $\hookrightarrow$ maps are the obvious inclusions and where the other maps (all of them K-linear) are defined as follows:

- The injection $\alpha : \mathrm{Sym}_m\left(\mathrm{Sym}^\ell E\right) \hookrightarrow \mathrm{Sym}_m(\mathrm{Sym}\, E)$ is obtained by applying the Sym_m functor to the (split) injection $\mathrm{Sym}^\ell E \hookrightarrow \mathrm{Sym}\, E$. This is clearly $\mathrm{GL}_2(K)$-equivariant.

- The isomorphism $\kappa : (\mathrm{Sym}\, E)^{\otimes m} \to K[\mathbf{X}, \mathbf{Y}]$ is the well-known isomorphism that sends each tensor

$$f_1(X, Y) \otimes f_2(X, Y) \otimes \cdots \otimes f_m(X, Y)$$
$$\text{to } f_1(X_1, Y_1) f_2(X_2, Y_2) \cdots f_m(X_m, Y_m).$$

  This is clearly $\mathrm{GL}_2(K)$-equivariant and S_m-equivariant.

- The isomorphism $\beta : \mathrm{Sym}_m(\mathrm{Sym}\, E) \to K[\mathbf{X}, \mathbf{Y}]^{\mathrm{sym}}$ is the restriction of κ to the S_m-fixed spaces. It is clearly $\mathrm{GL}_2(K)$-equivariant (since κ is).

- The injection $\gamma : \Lambda^m \mathrm{Sym}^{\ell+m-1} E \hookrightarrow \Lambda^m \mathrm{Sym}\, E$ is obtained by applying the Λ^m functor to the (split) injection $\mathrm{Sym}^{\ell+m-1} E \hookrightarrow \mathrm{Sym}\, E$. This is clearly $\mathrm{GL}_2(K)$-equivariant.

- The map $\delta : \Lambda^m \operatorname{Sym} E \hookrightarrow K[\mathbf{X}, \mathbf{Y}]$ sends each

$$f_1(X,Y) \wedge f_2(X,Y) \wedge \cdots \wedge f_m(X,Y)$$
$$\text{to } \sum_{\sigma \in S_m} (\operatorname{sgn} \sigma) \cdot f_{\sigma(1)}(X_1, Y_1) f_{\sigma(2)}(X_2, Y_2) \cdots f_{\sigma(m)}(X_m, Y_m).$$

  This map δ is clearly $\mathrm{GL}_2(K)$-equivariant, and furthermore is easily seen to be injective (since it factors as $\delta = \kappa \circ \delta'$, where $\delta' : \Lambda^m \operatorname{Sym} E \hookrightarrow (\operatorname{Sym} E)^{\otimes m}$ is the canonical injection sending each $f_1(X,Y) \wedge f_2(X,Y) \wedge \cdots \wedge f_m(X,Y)$ to $\sum_{\sigma \in S_m} (\operatorname{sgn} \sigma) \cdot f_{\sigma(1)}(X,Y) \otimes f_{\sigma(2)}(X,Y) \otimes \cdots \otimes f_{\sigma(m)}(X,Y)$).

- The injection $\iota : \left(\operatorname{Sym}^\ell E\right)^{\otimes m} \hookrightarrow (\operatorname{Sym} E)^{\otimes m}$ is obtained by applying the ${}^{\otimes m}$ functor to the (split) injection $\operatorname{Sym}^\ell E \hookrightarrow \operatorname{Sym} E$. This is clearly $\mathrm{GL}_2(K)$-equivariant.

- The map $\omega : \left(\operatorname{Sym}^\ell E\right)^{\otimes m} \to \Lambda^m \operatorname{Sym}^{\ell+m-1} E$ sends each tensor

$$f_1(X,Y) \otimes f_2(X,Y) \otimes \cdots \otimes f_m(X,Y)$$
$$\text{to } f_1(X,Y) X^{m-1} Y^0 \wedge f_2(X,Y) X^{m-2} Y^1 \wedge \cdots \wedge f_m(X,Y) X^{m-m} Y^{m-1}.$$

  This is the map you describe in Theorem 1.4.

- We let $\operatorname{alt} : K[\mathbf{X}, \mathbf{Y}] \to K[\mathbf{X}, \mathbf{Y}]$ be the K-linear map that sends each polynomial $g \in K[\mathbf{X}, \mathbf{Y}]$ to

$$\operatorname{alt} g := \sum_{\sigma \in S_m} (\operatorname{sgn} \sigma) \cdot \underbrace{(\sigma \cdot g)}_{= g\left(X_{\sigma(1)}, X_{\sigma(2)}, \ldots, X_{\sigma(m)}, Y_{\sigma(1)}, Y_{\sigma(2)}, \ldots, Y_{\sigma(m)}\right)}.$$

  This map alt is called the *alternator*, and its image is the space of all alternating polynomials in $K[\mathbf{X}, \mathbf{Y}]$ (but we won't use this). The map alt is easily seen to be a $K[\mathbf{X}, \mathbf{Y}]^{\mathrm{sym}}$-module morphism: i.e., we have

$$\operatorname{alt}(fg) = f \cdot \operatorname{alt} g \tag{1}$$

  for any $f \in K[\mathbf{X}, \mathbf{Y}]^{\mathrm{sym}}$ and any $g \in K[\mathbf{X}, \mathbf{Y}]$.

- The map $\psi : K[\mathbf{X}, \mathbf{Y}] \to K[\mathbf{X}, \mathbf{Y}]$ sends each polynomial $f \in K[\mathbf{X}, \mathbf{Y}]$ to $\operatorname{alt}\left(f \mathbf{X}^d \mathbf{Y}^e\right)$, where

$$\mathbf{X}^d := X_1^{m-1} X_2^{m-2} \cdots X_m^{m-m} = \prod_{i=1}^m X_i^{m-i}$$
$$\text{and} \qquad \mathbf{Y}^e := Y_1^0 Y_2^1 \cdots Y_m^{m-1} = \prod_{i=1}^m Y_i^{i-1}.$$

  This is not per se $\mathrm{GL}_2(K)$-equivariant (for $m > 1$ and $K \neq 0$).

The commutativity of the above diagram is easy to see: The top-left and the bottom-left square obviously commute, while the commutativity of the right rectangle boils down to the equality

$$\sum_{\sigma\in S_m} (\operatorname{sgn}\sigma)\cdot\prod_{i=1}^{m}\left(f_i\left(X_{\sigma(i)},Y_{\sigma(i)}\right)X_{\sigma(i)}^{m-i}Y_{\sigma(i)}^{i-1}\right)$$
$$=\sum_{\sigma\in S_m} (\operatorname{sgn}\sigma)\cdot\prod_{i=1}^{m}\left(f_{\sigma(i)}\left(X_i,Y_i\right)X_i^{m-\sigma(i)}Y_i^{\sigma(i)-1}\right)$$

(for all $f_1, f_2, \ldots, f_m \in K[X,Y]$), which is easily checked (just substitute σ^{-1} for σ in the sum).

Now, your ζ is the restriction $\omega\mid_{\operatorname{Sym}_m \operatorname{Sym}^\ell E}$. Thus, we need to prove that ω restricts to an $\mathrm{GL}_2(K)$-equivariant map

$$\text{from } \operatorname{Sym}_m \operatorname{Sym}^\ell E \otimes (\det E)^{\otimes m(m-1)/2} \text{ to } \Lambda^m \operatorname{Sym}^{\ell+m-1} E$$

(where "$\otimes (\det E)^{\otimes m(m-1)/2}$" is seen as a twist of the $\mathrm{GL}_2(K)$-action – i.e., we identify $\operatorname{Sym}_m \operatorname{Sym}^\ell E \otimes (\det E)^{\otimes m(m-1)/2}$ with $\operatorname{Sym}_m \operatorname{Sym}^\ell E$ as a K-module[1]).

In other words, we need to prove that

$$A\cdot\omega(v) = (\det A)^{m(m-1)/2}\cdot\omega(Av) \tag{2}$$

for any $A \in \mathrm{GL}_2(K)$ and any $v \in \operatorname{Sym}_m \operatorname{Sym}^\ell E$. In view of the above commutative diagram (and in view of the injectivity of γ and δ), it suffices to show that

$$A\cdot\psi(w) = (\det A)^{m(m-1)/2}\cdot\psi(Aw) \tag{3}$$

for any $A \in \mathrm{GL}_2(K)$ and any $w \in K[\mathbf{X},\mathbf{Y}]^{\mathrm{sym}}$ (because if we have proved (3), then we can apply (3) to $w = \beta(\alpha(v))$ and then "unapply" the $\mathrm{GL}_2(K)$-equivariant map $\delta\circ\gamma$ to obtain (2)).

So let us prove (3). We fix $A \in \mathrm{GL}_2(K)$ and $w \in K[\mathbf{X},\mathbf{Y}]^{\mathrm{sym}}$. Then, $Aw \in K[\mathbf{X},\mathbf{Y}]^{\mathrm{sym}}$ (since the actions of $\mathrm{GL}_2(K)$ and of S_m on $K[\mathbf{X},\mathbf{Y}]$ commute). Now, the definition of ψ yields

$$\psi(Aw) = \operatorname{alt}\left((Aw)\,\mathbf{X}^d\mathbf{Y}^e\right) = (Aw)\cdot\operatorname{alt}\left(\mathbf{X}^d\mathbf{Y}^e\right) \tag{4}$$

(by (1), since $Aw \in K[\mathbf{X},\mathbf{Y}]^{\mathrm{sym}}$). On the other hand,

$$\psi(w) = \operatorname{alt}\left(w\mathbf{X}^d\mathbf{Y}^e\right) = w\cdot\operatorname{alt}\left(\mathbf{X}^d\mathbf{Y}^e\right) \qquad \left(\text{by (1), since } w\in K[\mathbf{X},\mathbf{Y}]^{\mathrm{sym}}\right),$$

so that

$$A\cdot\psi(w) = A\cdot\left(w\cdot\operatorname{alt}\left(\mathbf{X}^d\mathbf{Y}^e\right)\right) = (Aw)\cdot A\left(\operatorname{alt}\left(\mathbf{X}^d\mathbf{Y}^e\right)\right). \tag{5}$$

[1]because $(\det E)^{\otimes m(m-1)/2}$ is canonically isomorphic to K as a K-module, and of course we have $W\otimes K \cong W$ for any K-module W

However, I claim that $A\left(\operatorname{alt}\left(\mathbf{X}^d\mathbf{Y}^e\right)\right) = (\det A)^{m(m-1)/2} \cdot \operatorname{alt}\left(\mathbf{X}^d\mathbf{Y}^e\right)$. Indeed, this is probably easiest to see by factoring $\operatorname{alt}\left(\mathbf{X}^d\mathbf{Y}^e\right)$ explicitly: We have

$$\begin{aligned}
\operatorname{alt}\left(\mathbf{X}^d\mathbf{Y}^e\right) &= \sum_{\sigma\in S_m} (\operatorname{sgn}\sigma) \cdot \underbrace{\left(\sigma \cdot \left(\mathbf{X}^d\mathbf{Y}^e\right)\right)}_{=\prod_{i=1}^{m}\left(X_{\sigma(i)}^{m-i}Y_{\sigma(i)}^{i-1}\right)} = \sum_{\sigma\in S_m} (\operatorname{sgn}\sigma) \cdot \prod_{i=1}^{m}\left(X_{\sigma(i)}^{m-i}Y_{\sigma(i)}^{i-1}\right) \\
&= \det\left(\left(X_j^{m-i}Y_j^{i-1}\right)_{1\le i\le m,\ 1\le j\le m}\right) \\
&= (Y_1Y_2\cdots Y_m)^{m-1} \underbrace{\det\left(\left(\left(\frac{X_j}{Y_j}\right)^{m-i}\right)_{1\le i\le m,\ 1\le j\le m}\right)}_{\substack{=\prod_{1\le i<j\le m}\left(\frac{X_i}{Y_i}-\frac{X_j}{Y_j}\right) \\ \text{(by the Vandermonde determinant)}}} \\
&= (Y_1Y_2\cdots Y_m)^{m-1} \prod_{1\le i<j\le m}\left(\frac{X_i}{Y_i}-\frac{X_j}{Y_j}\right) \\
&= \prod_{1\le i<j\le m}\left(X_iY_j - X_jY_i\right),
\end{aligned}$$

so that

$$\begin{aligned}
A\left(\operatorname{alt}\left(\mathbf{X}^d\mathbf{Y}^e\right)\right) &= A\left(\prod_{1\le i<j\le m}\left(X_iY_j - X_jY_i\right)\right) = \prod_{1\le i<j\le m} \underbrace{A\left(X_iY_j - X_jY_i\right)}_{\substack{=\det A\cdot\left(X_iY_j-X_jY_i\right) \\ \text{(straightforward to verify)}}} \\
&= (\det A)^{m(m-1)/2} \cdot \underbrace{\prod_{1\le i<j\le m}\left(X_iY_j - X_jY_i\right)}_{=\operatorname{alt}\left(\mathbf{X}^d\mathbf{Y}^e\right)} \\
&= (\det A)^{m(m-1)/2} \cdot \operatorname{alt}\left(\mathbf{X}^d\mathbf{Y}^e\right).
\end{aligned}$$

Thus, (5) becomes

$$\begin{aligned}
A\cdot\psi(w) &= (Aw)\cdot \underbrace{A\left(\operatorname{alt}\left(\mathbf{X}^d\mathbf{Y}^e\right)\right)}_{=(\det A)^{m(m-1)/2}\cdot\operatorname{alt}\left(\mathbf{X}^d\mathbf{Y}^e\right)} \\
&= (\det A)^{m(m-1)/2} \cdot \underbrace{(Aw)\cdot\operatorname{alt}\left(\mathbf{X}^d\mathbf{Y}^e\right)}_{\substack{=\psi(Aw) \\ \text{(by (4))}}} = (\det A)^{m(m-1)/2} \cdot \psi(Aw).
\end{aligned}$$

This proves (3), and thus (2), and with it the Wronskian isomorphism.

0.2. The complementary partition isomorphism

Let K be an arbitrary commutative ring. Let $d \in \mathbb{N}$. Let G be a group, and let V be the K-module K^d with some action of G. Let $s \in \mathbb{N}$, and let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_d)$ be a partition with length $\leq d$ and largest part $\leq s$. Let $\lambda^\circ = (s - \lambda_d, s - \lambda_{d-1}, \ldots, s - \lambda_1)$ denote the complement of λ in the $d \times s$-rectangle. In Theorem 1.2 of arXiv:2105.00538v3, you claim that

$$\nabla^\lambda V \cong \nabla^{\lambda^\circ} V^\star \otimes (\det V)^{\otimes s}$$

as G-modules.

Again, let me prove this in a way I believe to be simpler. I will use Section 8.1 of [Ful97] (Fulton's book *Young tableaux*). In this section, Fulton defines and analyzes the Schur module E^λ, which (as we will soon see) is isomorphic to $\nabla^\lambda V$. (Note that, just as we do, Fulton works in full generality, not just over the field $\mathbb{C}$.)

We WLOG assume that $G = \operatorname{GL}_d(K)$, since any group action on K^d factors through $\operatorname{GL}_d(K)$. Consider the polynomial ring $K[Z]$ in d^2 indeterminates

$$\begin{aligned} &z_{1,1}, \quad z_{1,2}, \quad \ldots, \quad z_{1,d}, \\ &z_{2,1}, \quad z_{2,2}, \quad \ldots, \quad z_{2,d}, \\ &\vdots \\ &z_{d,1}, \quad z_{d,2}, \quad \ldots, \quad z_{d,d} \end{aligned}$$

over K. Let Z be the $d \times d$-matrix $(z_{i,j})_{1 \leq i \leq d,\ 1 \leq j \leq d}$ over $K[Z]$. The determinant $\det Z$ of this matrix is a regular element of $K[Z]$. Hence, the ring $K[Z]$ embeds as a subring into its localization $K[Z]_{\det Z}$ at the multiplicative set of all powers of $\det Z$. The latter localization $K[Z]_{\det Z}$ is, of course, the coordinate ring of the affine group scheme GL_d. We will work in the K-algebra $K[Z]_{\det Z}$, so that Z^{-1} and $(\det Z)^{-1}$ are well-defined.

We let $[p] := \{1, 2, \ldots, p\}$ for any $p \in \mathbb{N}$. For any $U \subseteq [d]$, we let $\widetilde{U}$ denote the set $[d] \setminus U$ (that is, the complement of U in $[d]$). For any $p \times q$-matrix A and any two subsets $U \subseteq [p]$ and $V \subseteq [q]$, we let $A_{U,V}$ denote the $|U| \times |V|$-submatrix of A obtained by removing all rows other than the U-rows and removing all columns other than the V-columns. The *Jacobi complementary minor theorem* says that if A is an invertible $d \times d$-matrix over some commutative ring, and if $U \subseteq [d]$ and $V \subseteq [d]$, then

$$\det(A_{U,V}) = \pm \det A \cdot \det\left(\left(A^{-1}\right)_{\widetilde{V},\widetilde{U}}\right). \tag{6}$$

(The $\pm$ sign is actually $(-1)^{\operatorname{sum} U + \operatorname{sum} V}$, where $\operatorname{sum} U$ denotes the sum of all elements of U. But we don't care what it is.) Applying (6) to $A = Z$, we find

$$\det(Z_{U,V}) = \pm \det Z \cdot \det\left(\left(Z^{-1}\right)_{\widetilde{V},\widetilde{U}}\right),$$

so that

$$\det\left(\left(Z^{-1}\right)_{\widetilde{V},\widetilde{U}}\right) = \pm\frac{\det\left(Z_{U,V}\right)}{\det Z}. \tag{7}$$

We WLOG assume that $\mathcal{B} = \{1, 2, \ldots, d\}$, and we let $(v_1, v_2, \ldots, v_d)$ be the standard basis of $V = K^d$. We identify $\operatorname{Sym} V$ with the polynomial ring $K[v_1, v_2, \ldots, v_d]$. We consider the K-algebra isomorphism

$$\mathbf{i} : (\operatorname{Sym} V)^{\otimes d} \to K[Z],$$
$$f_1 \otimes \cdots \otimes f_d \mapsto \prod_{i=1}^{d} f_i\left(z_{i,1}, z_{i,2}, \ldots, z_{i,d}\right).$$

We will use this isomorphism $\mathbf{i}$ to identify $(\operatorname{Sym} V)^{\otimes d}$ with $K[Z]$.

We will use the notation $A_{u,v}$ for the (u,v)-th entry of a matrix A. Thus, in particular, $Z_{u,v} = z_{u,v}$ for all $u, v \in [d]$.

The polynomial ring $K[Z]$ is a G-module[2], with G acting by K-algebra automorphisms according to the rule

$$A \cdot z_{i,j} = (ZA)_{i,j} = \sum_{k=1}^{d} \underbrace{Z_{i,k}}_{=z_{i,k}} A_{k,j} = \sum_{k=1}^{d} z_{i,k} A_{k,j}$$
$$\text{for all } A \in G \text{ and } i, j \in [d].$$

In other words, a matrix $A \in G$ sends any polynomial $p(Z) \in K[Z]$ to $p(ZA) \in K[Z]$ (which is the result of substituting each $z_{i,j}$ by $(ZA)_{i,j}$ in the polynomial $p(Z)$). This is probably the more illuminating way of thinking about our action of G on $K[Z]$. In particular, it shows that each $A \in G$ satisfies

$$A \cdot \det Z = \det(ZA) = \det Z \cdot \det A. \tag{8}$$

Since $\det Z$ is invertible in $K[Z]_{\det Z}$, this shows that the action of G on $K[Z]$ can be extended to an action of G on the localization $K[Z]_{\det Z}$ (still by K-algebra automorphisms). Thus, $K[Z]$ is a G-submodule of $K[Z]_{\det Z}$.

Our isomorphism $\mathbf{i} : \operatorname{Sym} V \to K[Z]$ is G-equivariant. (This is easiest to show by checking the commutativity on each v_j in each of the d factors of $(\operatorname{Sym} V)^{\otimes d}$, and then arguing that everything is a K-algebra homomorphism.)

The definition of $\operatorname{Sym}^\lambda V$ yields $\operatorname{Sym}^\lambda V \subseteq (\operatorname{Sym} V)^{\otimes d}$ (or at least that there is a canonical G-equivariant injection $\operatorname{Sym}^\lambda V \to (\operatorname{Sym} V)^{\otimes d}$, but we consider this injection as an inclusion). If t is a λ-tableau with entries in $\mathcal{B}$, then the element

[2] Recall that $G = \mathrm{GL}_d(K)$.

$\mathbf{e}(t) \in \operatorname{Sym}^\lambda V \subseteq (\operatorname{Sym} V)^{\otimes d}$ you define in §2.1 can now be rewritten as follows:[3]

$$\begin{aligned}
\mathbf{e}(t) &= \sum_{\sigma \in \operatorname{CPP}(\lambda)} \operatorname{sgn}(\sigma) \cdot \underbrace{\mathbf{s}(t\cdot\sigma)}_{\substack{= \bigotimes_{i=1}^{\ell(\lambda)} \prod_{j=1}^{\lambda_i} v_{(t\cdot\sigma)(i,j)} \\ \text{(by the definition of } \mathbf{s}(t\cdot\sigma))}} = \sum_{\sigma \in \operatorname{CPP}(\lambda)} \operatorname{sgn}(\sigma) \cdot \bigotimes_{i=1}^{\ell(\lambda)} \prod_{j=1}^{\lambda_i} v_{(t\cdot\sigma)(i,j)} \\
&= \sum_{\sigma \in \operatorname{CPP}(\lambda)} \operatorname{sgn}(\sigma) \cdot \underbrace{\prod_{i=1}^{\ell(\lambda)} \prod_{j=1}^{\lambda_i}}_{= \prod_{j=1}^{s} \prod_{i=1}^{\lambda'_j}} z_{i,(t\cdot\sigma)(i,j)} \qquad \left(\begin{array}{c}\text{since we are using } \mathbf{i} \text{ to} \\ \text{identify } (\operatorname{Sym} V)^{\otimes d} \text{ with } K[Z]\end{array}\right) \\
&= \sum_{\sigma \in \operatorname{CPP}(\lambda)} \operatorname{sgn}(\sigma) \cdot \prod_{j=1}^{s} \prod_{i=1}^{\lambda'_j} z_{i,(t\cdot\sigma)(i,j)} \\
&= \sum_{\sigma_1 \in S'_{\lambda_1}} \sum_{\sigma_2 \in S'_{\lambda_2}} \cdots \sum_{\sigma_s \in S'_{\lambda_s}} \operatorname{sgn}(\sigma_1) \operatorname{sgn}(\sigma_2) \cdots \operatorname{sgn}(\sigma_s) \cdot \prod_{j=1}^{s} \prod_{i=1}^{\lambda'_j} z_{i,t(\sigma_j(i),j)} \\
&\qquad \left(\begin{array}{c}\text{since the permutations } \sigma \in \operatorname{CPP}(\lambda) \text{ are in bijection with} \\ \text{the } s\text{-tuples of permutations } \sigma_1 \in S_{\lambda'_1}, \sigma_2 \in S_{\lambda'_2}, \ldots, \sigma_s \in S_{\lambda'_s}\end{array}\right) \\
&= \prod_{j=1}^{s} \underbrace{\left(\sum_{\sigma \in S_{\lambda'_j}} \operatorname{sgn}(\sigma) \cdot \prod_{i=1}^{\lambda'_j} z_{i,t(\sigma(i),j)} \right)}_{= \det\left(\left(z_{u,t(v,j)}\right)_{1\le u \le \lambda'_j,\ 1 \le v \le \lambda'_j} \right)} \qquad \text{(by the product rule)} \\
&= \prod_{j=1}^{s} \det \underbrace{\left(\left(z_{u,t(v,j)}\right)_{1\le u \le \lambda'_j,\ 1 \le v \le \lambda'_j} \right)}_{\substack{= Z_{[\lambda'_j], \operatorname{col}_j t} \\ \text{up to permuting the rows}}} = \pm \prod_{j=1}^{s} \det\left(Z_{[\lambda'_j], \operatorname{col}_j t} \right).
\end{aligned}$$

Thus,

$$\nabla^\lambda V = \operatorname{span}\left(\prod_{j=1}^{s} \det\left(Z_{[\lambda'_j], \operatorname{col}_j t} \right) \ \middle|\ t \text{ is a } \lambda\text{-tableau} \right) \tag{9}$$

(since $\nabla^\lambda V$ was defined as the span of all $\mathbf{e}(t)$ for λ-tableaux t). The right hand side of this equality is the G-module D^λ defined at the end of [Ful97, §8.1] (although

[3] I let $\operatorname{col}_j t$ denote the set of all entries of the j-th column of t.

My symmetric group actions might be in conflict with yours, since I'm used to everything acting from the left. I believe this shouldn't be a problem, since $\operatorname{sgn}(\sigma^{-1}) = \operatorname{sgn} \sigma$ for any permutation σ.

[Ful97] works with the monoid $M_d(K)$ instead of $\operatorname{GL}_d(K)$, and uses the notations R and m instead of K and d). Thus, $\nabla^\lambda V \cong D^\lambda \cong E^\lambda$ (by [Ful97, §8.1, Exercise 5]), where E^λ is the Schur module defined in [Ful97, §8.1]. This shows that $\nabla^\lambda V$ is always isomorphic to Fulton's E^λ, whatever K is.

We can replace "λ-tableau" by "column-standard λ-tableau" on the right hand side of (9), since any λ-tableau either yields a 0 determinant or can be made column-standard by a CPP-permutation (which can at most flip the sign of the respective product). Thus, (9) becomes

$$\begin{aligned}\nabla^\lambda V &= \operatorname{span}\left(\prod_{j=1}^{s} \det\left(Z_{\left[\lambda'_j\right],\operatorname{col}_j t}\right) \mid t \text{ is a column-standard } \lambda\text{-tableau}\right) \\ &= \operatorname{span}\left(\prod_{j=1}^{s} \det\left(Z_{\left[\lambda'_j\right],U_j}\right) \mid U_j \text{ is a } \lambda'_j\text{-element subset of } [d]\right). \end{aligned} \tag{10}$$

Applying this to λ° instead of λ, we obtain

$$\begin{aligned}\nabla^{\lambda^\circ} V &= \operatorname{span}\left(\prod_{j=1}^{s} \det\left(Z_{\left[d-\lambda'_j\right],U_j}\right) \mid U_j \text{ is a } \left(d-\lambda'_j\right)\text{-element subset of } [d]\right) \\ &\qquad \left(\text{since } (\lambda^\circ)'_j = d - \lambda'_j \text{ for each } j \in [s]\right) \\ &= \operatorname{span}\left(\prod_{j=1}^{s} \det\left(Z_{\left[d-\lambda'_j\right],\widetilde{U_j}}\right) \mid U_j \text{ is a } \lambda'_j\text{-element subset of } [d]\right) \end{aligned} \tag{11}$$

(here, we have substituted $\widetilde{U_j}$ for U_j).

Next, we need one more piece of notation. If γ is an automorphism of the group G, and if W is a G-module, then $W \circ \gamma$ shall mean the G-module W twisted by γ (that is, it is the same K-module as W, but the action of G on it is $\rho \circ \gamma : G \to \operatorname{GL}(W)$, where $\rho : G \to \operatorname{GL}(W)$ is the action of G on W). We will use one specific automorphism of G, which we shall call γ from now on: namely, the automorphism sending each $A \in G$ to $\left(A^{-1}\right)^T$. It is well-known that $V^\star \cong V \circ \gamma$ as G-modules.[4]

Hence, $\nabla^{\lambda^\circ} V^* \cong \nabla^{\lambda^\circ}(V \circ \gamma) \cong \left(\nabla^{\lambda^\circ} V\right) \circ \gamma$. Thus, we just need to show that

$$\nabla^\lambda V \cong \left(\left(\nabla^{\lambda^\circ} V\right) \circ \gamma\right) \otimes (\det V)^{\otimes s}. \tag{12}$$

We shall do so by constructing an explicit isomorphism. We define a K-algebra endomorphism Ω of $K[Z]_{\det Z}$ by requiring that

$$\Omega\left(z_{i,j}\right) = \left(Z^{-1}\right)_{j,d+1-i} \qquad \text{for all } i, j \in [d].$$

This endomorphism Ω is well-defined (by the universal property of the localization $K[Z]_{\det Z}$, because the K-algebra morphism $K[Z] \to K[Z]_{\det Z}$ that sends each $z_{i,j}$

[4] This is essentially the definition of the G-action on $V^\star$.

to $\left(Z^{-1}\right)_{j,d+1-i}$ sends $\det Z$ to $\pm\det\left(Z^{-1}\right) = \pm\left(\det Z\right)^{-1}$, which is an invertible element of $K\left[Z\right]_{\det Z}$).

This endomorphism Ω is an automorphism of $K\left[Z\right]_{\det Z}$ (actually, $\Omega^4 = \mathrm{id}$ if I am not wrong, but either way Ω is a composition of some rather well-known involutions: one that "sends" Z to Z^T, another that "sends" Z to Z^{-1}, and another that sends $z_{i,j}$ to $z_{d+1-i,j}$).

It is not hard to see that the map Ω is a G-module isomorphism from $K\left[Z\right]_{\det Z}$ to $K\left[Z\right]_{\det Z} \circ \gamma$. Indeed, this boils down to proving the equality

$$\Omega\left(A \cdot f\right) = \gamma\left(A\right) \cdot \Omega\left(f\right) \qquad \text{for all } A \in G \text{ and } f \in K\left[Z\right]_{\det Z}.$$

To prove this equality, we can WLOG assume that $f \in K\left[Z\right]$. Since Ω is a K-algebra morphism, this equality needs to be proved for $f = z_{i,j}$ only. But this is rather straightforward: Setting $f = z_{i,j}$, we can compare

$$\begin{aligned}
\Omega\left(A \cdot f\right) &= \Omega\left(A \cdot z_{i,j}\right) = \Omega\left(\sum_{k=1}^{d} z_{i,k} A_{k,j}\right) = \sum_{k=1}^{d} \underbrace{\Omega\left(z_{i,k}\right)}_{=\left(Z^{-1}\right)_{k,d+1-i}} \underbrace{A_{k,j}}_{=\left(A^T\right)_{j,k}} \\
&= \sum_{k=1}^{d} \left(Z^{-1}\right)_{k,d+1-i} \left(A^T\right)_{j,k} = \left(A^T Z^{-1}\right)_{j,d+1-i}
\end{aligned}$$

with

$$\begin{aligned}
\gamma\left(A\right) \cdot \Omega\left(f\right) &= \underbrace{\gamma\left(A\right)}_{=\left(A^{-1}\right)^T} \cdot \underbrace{\Omega\left(z_{i,j}\right)}_{=\left(Z^{-1}\right)_{j,d+1-i}} = \left(A^{-1}\right)^T \cdot \left(Z^{-1}\right)_{j,d+1-i} \\
&= \left(\left(Z\left(A^{-1}\right)^T\right)^{-1}\right)_{j,d+1-i} = \left(A^T Z^{-1}\right)_{j,d+1-i},
\end{aligned}$$

and get precisely this claim.

So we conclude that Ω is a G-module isomorphism from $K\left[Z\right]_{\det Z}$ to $K\left[Z\right]_{\det Z} \circ \gamma$.

If U is a k-element subset of $[d]$, then the definition of Ω yields

$$\begin{aligned}
\Omega\left(\det\left(Z_{[k],U}\right)\right) &= \det\left(\left(Z^{-1}\right)_{U,\{d-k+1,d-k+2,\ldots,d\}}\right) \\
&= \pm\frac{\det\left(Z_{[d-k],\widetilde{U}}\right)}{\det Z} \qquad \text{(by (7))}.
\end{aligned}$$

Since Ω is a K-algebra morphism, we therefore have

$$\Omega\left(\prod_{j=1}^{s} \det\left(Z_{\left[\lambda_j'\right],U_j}\right)\right) = \pm\frac{1}{\left(\det Z\right)^s} \prod_{j=1}^{s} \det\left(Z_{\left[d-\lambda_j'\right],\widetilde{U}_j}\right)$$

whenever $U_1, U_2, \ldots, U_s$ are subsets of $[d]$ of sizes $\lambda'_1, \lambda'_2, \ldots, \lambda'_s$. In view of (10) and (11), this entails $\Omega\left(\nabla^\lambda V\right) = \dfrac{1}{(\det Z)^s} \cdot \nabla^{\lambda^\circ} V$. In other words, $\Omega\left(\nabla^\lambda V\right) \cdot (\det Z)^s = \nabla^{\lambda^\circ} V$. Thus, multiplication by $(\det Z)^s$ is a K-module isomorphism from $\Omega\left(\nabla^\lambda V\right)$ to $\nabla^{\lambda^\circ} V$. This isomorphism is furthermore G-equivariant as a map from $\left(\Omega\left(\nabla^\lambda V\right)\right) \otimes (\det V)^{\otimes s}$ to $\nabla^{\lambda^\circ} V$, because any $A \in G$ and any $f \in \Omega\left(\nabla^\lambda V\right)$ satisfy

$$A \cdot \left(f \cdot (\det Z)^s\right) = (A \cdot f) \cdot \left(\underbrace{\underbrace{A \cdot \det Z}_{=\det(ZA)}}_{=\det Z \cdot \det A} \right)^s = (A \cdot f) \cdot (\det Z \cdot \det A)^s$$
$$= (\det A)^s \cdot (A \cdot f) \cdot (\det Z)^s .$$

Hence, we conclude that

$$\left(\Omega\left(\nabla^\lambda V\right)\right) \otimes (\det V)^{\otimes s} \cong \nabla^{\lambda^\circ} V \qquad \text{as } G\text{-modules.} \tag{13}$$

Applying this to λ° instead of λ, we obtain

$$\left(\Omega\left(\nabla^{\lambda^\circ} V\right)\right) \otimes (\det V)^{\otimes s} \cong \nabla^{\lambda} V \qquad \text{as } G\text{-modules} \tag{14}$$

(since $(\lambda^\circ)^\circ = \lambda$).

However, $\left(\Omega\left(\nabla^\lambda V\right)\right) \circ \gamma \cong \nabla^\lambda V$ (since Ω is a G-module isomorphism from $K[Z]_{\det Z}$ to $K[Z]_{\det Z} \circ \gamma$). Since γ is an involution, this entails

$$\Omega\left(\nabla^\lambda V\right) \cong \underbrace{\left(\Omega\left(\nabla^\lambda V\right)\right) \circ \gamma}_{\cong \nabla^\lambda V} \circ \gamma \cong \left(\nabla^\lambda V\right) \circ \gamma .$$

Applying this to λ° instead of λ, we obtain $\Omega\left(\nabla^{\lambda^\circ} V\right) \cong \left(\nabla^{\lambda^\circ} V\right) \circ \gamma$. Thus, (14) rewrites as

$$\left(\left(\nabla^{\lambda^\circ} V\right) \circ \gamma\right) \otimes (\det V)^{\otimes s} \cong \nabla^{\lambda} V \qquad \text{as } G\text{-modules.}$$

This proves (12) and thus proves Theorem 1.2.

References

[Ful97] William Fulton, *Young Tableaux With Applications to Representation Theory and Geometry*, Cambridge University Press, reprint 1999.