

Semigroups, rings, and Markov chains

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Errata and addenda by Darij Grinberg

I will refer to the results appearing in the paper “Semigroups, rings, and Markov chains” by the numbers under which they appear in this paper (specifically, in its version of 20 June 2000, which appears on arXiv as preprint arXiv:math/0006145v1).

Errata

- **Page 1, §1:** “explicitly” → “explicitly”.
- **Page 3, §1.4:** Replace “ q -elements” by “ q elements”. (The hyphen should not be there.)
- **Page 4, Theorem 0:** “maximal elements (co-atoms)” should just say “co-atoms”. And the word “co-atom” should be defined: A *co-atom* of a lattice L is a maximal element of the poset $L \setminus \{\hat{1}\}$. (In contrast, the only maximal element of L – in the literal sense of this word – is $\hat{1}$. This is not what you want the sum to range over.)
- **Page 4, §1.6:** “the partial sums are eventually constant” → “the partial products are eventually constant”.
- **Page 4, §1.7:** “restarting” might be meant to be “restating”... though “restarting” also makes sense.
- **Page 6, §2.1:** At the end of the first paragraph on this page (i.e., right after the sentence “Thus S has the deletion property (D) stated in Section 1.1”), I suggest adding something like the following definitions:

“When S is a LRB, the lattice L and the map supp in the above definition are determined uniquely up to isomorphism (this follows easily from the surjectivity of supp and the equivalence

$$(xy = x) \iff (\text{supp } y \leq \text{supp } x), \quad (201)$$

which holds for all $x, y \in S$ ¹). The lattice L is called the *lattice of supports* (or the *support lattice*) of S . The surjection $\text{supp} : S \rightarrow L$ is called the *support map* of S . For any $s \in S$, the element $\text{supp } s$ of L is called the *support* of s .”

¹*Proof of (201):* Let $x, y \in S$. We must prove the equivalence $(xy = x) \iff (\text{supp } y \leq \text{supp } x)$. The “ \Leftarrow ” direction follows immediately from (3). The “ \Rightarrow ” direction can be proved as follows: From (2), we know that $\text{supp } (xy) = \text{supp } x \vee \text{supp } y \geq \text{supp } y$. Hence, $\text{supp } y \leq \text{supp } (xy) = \text{supp } x$ if $xy = x$. This proves the “ \Rightarrow ” direction of (201).

(Of course, the purpose of these sentences is to introduce some notations that you use several times; you'll probably find a more succinct way to introduce them.)

- **Page 6, §2.1:** The last comma in "Sections 4, 5, and 6," probably should be a period.
- **Page 6, §2.2:** You write that $S_{\geq x}$ "is a LRB in its own right, the associated lattice being the interval $[X, \hat{1}]$ in L , where $X = \text{supp } x$ ". This is correct, but (in my opinion) not obvious enough to be left to the reader.²

²Here is a proof, for the sake of completeness:

Let $x \in S$. Let $X = \text{supp } x$. The interval $[X, \hat{1}]$ of the poset L is a lattice (since L is a lattice). Its join operation is the restriction of the join operation \vee of the lattice L ; therefore, we shall denote it by \vee as well.

It remains to prove that $S_{\geq x}$ is a LRB with support lattice $[X, \hat{1}]$ and support map $\text{supp}|_{S_{\geq x}}$. In order to do so, we need to prove the following four statements:

Statement 1: The set $S_{\geq x}$ is a subsemigroup of S with identity x .

Statement 2: The map $\text{supp}|_{S_{\geq x}}: S_{\geq x} \rightarrow [X, \hat{1}]$ is well-defined and surjective.

Statement 3: For any $a \in S_{\geq x}$ and $b \in S_{\geq x}$, we have $(\text{supp}|_{S_{\geq x}})(ab) = ((\text{supp}|_{S_{\geq x}})a) \vee ((\text{supp}|_{S_{\geq x}})b)$.

Statement 4: Let $a \in S_{\geq x}$ and $b \in S_{\geq x}$. Then, $ab = a$ if $(\text{supp}|_{S_{\geq x}})b \leq (\text{supp}|_{S_{\geq x}})a$.

(These four Statements are precisely the requirements in the definition of a LRB, except that we have renamed x and y as a and b since the letter x is already taken for something else.)

Proof of Statement 1: Let $a \in S_{\geq x}$ and $b \in S_{\geq x}$. We have $a \geq x$ (since $a \in S_{\geq x}$). In other words, $x \leq a$. But (5) (applied to $y = a$) shows that $x \leq a \iff xa = a$. Thus, $xa = a$ (since $x \leq a$). But (5) (applied to $y = ab$) shows that $x \leq ab \iff xab = ab$. Hence, $x \leq ab$ (since $\underbrace{xa}_=a b = ab$). In other words, $ab \geq x$, so that $ab \in S_{\geq x}$.

Now, let us forget that we fixed a and b . We thus have shown that $ab \in S_{\geq x}$ for every $a \in S_{\geq x}$ and $b \in S_{\geq x}$. In other words, $S_{\geq x}$ is a subsemigroup of S , although we do not yet know whether it has an identity.

Now, let $a \in S_{\geq x}$. Then, $a \geq x$ (since $a \in S_{\geq x}$). In other words, $x \leq a$. But (5) (applied to $y = a$) shows that $x \leq a \iff xa = a$. Thus, $xa = a$ (since $x \leq a$). Also, $\underbrace{ax}_=xa = xax = xa$ (by

(4), applied to $y = a$), so that $ax = xa = a$.

Now, let us forget that we fixed a . We thus have shown that $ax = xa = a$ for every $a \in S_{\geq x}$. In other words, the subsemigroup $S_{\geq x}$ of S has identity x . This finishes the proof of Statement 1. ■

Proof of Statement 2: Let $y \in S_{\geq x}$. Thus, $y \geq x$, so that $x \leq y$. Hence, $xy = y$ (by (5)). Thus, $\underbrace{\text{supp } y}_=xy = \text{supp}(xy) = \text{supp } x \vee \text{supp } y$ (by (2)), so that $\text{supp } y = \text{supp } x \vee \text{supp } y \geq$

$\text{supp } x = X$. In other words, $\text{supp } y \in [X, \hat{1}]$.

- **Page 9, §4.2:** In "Let $\mathcal{G}_0 = \{G \in \mathcal{G} : \sigma_i(G) = +\}$ ", replace " $\sigma_i(G) = +$ " by " $\sigma_i(G) = +$ for all $i \in J$ ".
- **Page 9, §4.2:** "faces in G " should be "faces in \mathcal{G} ".
- **Page 13, §5.1, Remark:** "maximal elements" \rightarrow "co-atoms". (See the correction to Theorem 0 for a precise definition of "co-atom".)
- **Page 14:** In "where c_X is the number of maximal chains in the interval $[X, V]$ in L ", the lattice L should be defined. (It is not the lattice of supports of $\overline{F}_{n,q}$, but rather the lattice of all subspaces of V , including those of dimension $n - 1$.)
- **Page 14:** "get a line l_1 " \rightarrow "get a line ℓ_1 ".
- **Page 15, §6.1:** "the rank of any maximal" \rightarrow "the size of any maximal".
- **Page 16:** "any fixed basis of X to a basis" \rightarrow "any fixed ordered basis of X to an ordered basis".

Let us now forget that we fixed y . We thus have shown that $\text{supp } y \in [X, \widehat{1}]$ for every $y \in S_{\geq x}$. Hence, $\text{supp } (S_{\geq x}) \subseteq [X, \widehat{1}]$. Therefore, the map $\text{supp } |_{S_{\geq x}}: S_{\geq x} \rightarrow [X, \widehat{1}]$ is well-defined.

We shall now show that this map is surjective. Indeed, let $Y \in [X, \widehat{1}]$. Thus, $Y \in L$ and $Y \geq X$. Since the map $\text{supp} : S \rightarrow L$ is surjective, we thus see that there exists some $y \in S$ satisfying $Y = \text{supp } y$. Consider such a y . We have $\text{supp } x = X \leq Y = \text{supp } y$. Hence, (3) (applied to y and x instead of x and y) shows that $yx = y$.

But (5) (applied to xy instead of y) shows that $x \leq xy \iff xxy = xy$. Hence, $x \leq xy$ (since $\underbrace{xx}_=x y = xy$). In other words, $xy \geq x$, so that $xy \in S_{\geq x}$. Now, (2) shows that $\text{supp } (xy) =$

$\underbrace{\text{supp } x}_=X \vee \underbrace{\text{supp } y}_=Y = X \vee Y = Y$ (since $X \leq Y$). Hence, $Y = \text{supp} \left(\underbrace{xy}_{\in S_{\geq x}} \right) \in \text{supp } (S_{\geq x}) = (\text{supp } |_{S_{\geq x}}) (S_{\geq x})$.

Now, let us forget that we fixed Y . We thus have shown that $Y \in (\text{supp } |_{S_{\geq x}}) (S_{\geq x})$ for every $Y \in [X, \widehat{1}]$. In other words, $[X, \widehat{1}] \subseteq (\text{supp } |_{S_{\geq x}}) (S_{\geq x})$. In other words, the map $\text{supp } |_{S_{\geq x}}: S_{\geq x} \rightarrow [X, \widehat{1}]$ is surjective. This finishes the proof of Statement 2. ■

Proof of Statement 3: Let $a \in S_{\geq x}$ and $b \in S_{\geq x}$. Since $S_{\geq x}$ is a subsemigroup of S (by Statement 1), this shows that $ab \in S_{\geq x}$. Now, (2) (applied to a and b instead of x and y) shows that $\text{supp } (ab) = \text{supp } a \vee \text{supp } b$. This rewrites as $(\text{supp } |_{S_{\geq x}}) (ab) = ((\text{supp } |_{S_{\geq x}}) a) \vee ((\text{supp } |_{S_{\geq x}}) b)$ (since $(\text{supp } |_{S_{\geq x}}) (ab) = \text{supp } (ab)$, $(\text{supp } |_{S_{\geq x}}) a = \text{supp } a$ and $(\text{supp } |_{S_{\geq x}}) b = \text{supp } b$). This proves Statement 3. ■

Proof of Statement 4: Assume that $(\text{supp } |_{S_{\geq x}}) b \leq (\text{supp } |_{S_{\geq x}}) a$. This rewrites as $\text{supp } b \leq \text{supp } a$ (since $(\text{supp } |_{S_{\geq x}}) a = \text{supp } a$ and $(\text{supp } |_{S_{\geq x}}) b = \text{supp } b$). Hence, (3) (applied to a and b instead of x and y) yields $ab = a$. This proves Statement 4. ■

Now, all four Statements are proven, and the proof is complete. ■

- **Page 18:** "the first e_i such that $\{e, e_1, \dots, e_i\}$ contains a cycle" \rightarrow "the first e_i such that the list (e, e_1, \dots, e_i) contains a cycle or two equal elements".
- **Pages 24–25, §8.2:** Any appearance of "A" in §8.2 should be replaced by an "R". (There are three appearances on page 24, and six appearances on page 25, not counting the two "A'"s.)
- **Page 24, §8.2:** It would be useful to point out that $g(z)$ is understood to be an element of $R \left(\left(\frac{1}{z} \right) \right)$ (that is, of the ring of Laurent series in $\frac{1}{z}$ over R). It can indeed be shown that $g(z)$ is a rational function in z over R , in an appropriate sense³; but before this is shown, it is important to know whether $g(z)$ is viewed as a Laurent series in $\frac{1}{z}$ (correct) or as a Laurent series in z (incorrect).
- **Page 24, proof of Proposition 2:** I think this proof is missing a few steps. You leave the following statements unproven:

Statement 1: Suppose that R is split semisimple with primitive idempotents $(e_i)_{i \in I}$, and write $a = \sum_i \lambda_i e_i$ with $\lambda_i \in k$. Then, the λ_i (for $i \in I$) are distinct.

Statement 2: Up to relabelling, there is only one choice of a set I and two families $(e_i)_{i \in I} \in R^I$ and $(\lambda_i)_{i \in I} \in k^I$ satisfying (18) (with the λ_i being distinct elements of k and the e_i being nonzero elements of R).

(Statement 1 is needed to prove the first sentence of Proposition 2. Statement 2 is needed to prove the second sentence.)

Proof of Statement 1. Let j and j' be two distinct elements of I .

Let G be the subset $\left\{ \sum_i \mu_i e_i \mid (\mu_i)_{i \in I} \in k^I \text{ and } \mu_j = \mu_{j'} \right\}$ of R . Then, G is a k -subalgebra of R (this is easy to see) and satisfies $G \neq R$ (since $e_j \notin G$). But

³Namely, $g(z)$ can be written as $\frac{1}{p(z)}q(z)$ for some nonzero polynomial $p(z) \in k[z]$ and some polynomial $q(z) \in R[z]$. The easiest way to see this is as follows:

The powers a^0, a^1, a^2, \dots of a are k -linearly dependent (since they lie in the finite-dimensional k -algebra R). That is, some nontrivial k -linear combination of these powers is 0. In other words, there exists a nonzero polynomial $p(z) \in k[z]$ such that $p(a) = 0$. Consider this p . Thus, a is a root of p in the commutative ring R . Hence, in the polynomial ring $R[z]$, we have $z - a \mid p(z)$. Thus, $p(z)$ can be written as $(z - a) \cdot q(z)$ for some $q(z) \in R[z]$.

Consider this $q(z)$. Now, from $p(z) = (z - a) \cdot q(z)$, we obtain $\frac{1}{z - a} = \frac{1}{p(z)}q(z)$. That is,

$g(z) = \frac{1}{p(z)}q(z)$ (since $g(z) = \frac{1}{z - a}$), qed.

every k -subalgebra of R which contains a must be R itself (since a generates the k -algebra R). Hence, if we had $a \in G$, then we would have $G = R$ (since G would be a k -subalgebra of R which contains a), which would contradict $G \neq R$. Thus, we cannot have $a \in G$. In other words, we have $a \notin G$. In other words, $\lambda_j \neq \lambda_{j'}$.

Now, let us forget that we fixed j and j' . We thus have shown that $\lambda_j \neq \lambda_{j'}$ for any two distinct elements j and j' of I . In other words, the λ_i (for $i \in I$) are distinct. This proves Statement 1. \square

First proof of Statement 2. Consider a set I and two families $(e_i)_{i \in I} \in R^I$ and $(\lambda_i)_{i \in I} \in k^I$ satisfying (18) (with the λ_i being distinct elements of k and the e_i being nonzero elements of R). We shall show that the λ_i and the corresponding e_i can be reconstructed from $g(z)$ (up to labelling). This will clearly prove Statement 2.

The function $g(z)$ is a rational function in z (in the sense that $g(z)$ can be written as $\frac{1}{p(z)}q(z)$ for some nonzero polynomial $p(z) \in k[z]$ and some polynomial $q(z) \in R[z]$), and thus⁴ can be expanded as a Laurent series in $z - \mu$ for each $\mu \in k$. Hence, for every $\mu \in k$, an element $\text{Res}_\mu g(z)$ of R is well-defined (namely, it is defined as the coefficient of $(z - \mu)^{-1}$ when $g(z)$ is expanded as a Laurent series in $z - \mu$). Now, (18) shows that the elements λ_i of k are the poles of $g(z)$ (that is, the elements $\mu \in k$ for which $\text{Res}_\mu g(z) \neq 0$ ⁵), and the elements e_i of R are their corresponding residues (i.e., we have $e_i = \text{Res}_{\lambda_i} g(z)$ for every $i \in I$). Thus, the λ_i and the corresponding e_i can be reconstructed from $g(z)$ (up to labelling). This proves Statement 2. \square

Second proof of Statement 2. Here is a more pedestrian proof of Statement 2:

⁴Here we are using the fact that a rational function in z can be expanded as a Laurent series in $z - \mu$ for each $\mu \in k$. The way to do this is straightforward: Write the rational function as $\frac{1}{p(z)}q(z)$ for some nonzero polynomial $p(z) \in k[z]$ and some polynomial $q(z) \in R[z]$; then compute $\frac{1}{p(w + \mu)}q(w + \mu)$ in the Laurent series ring $R((w))$; this will be the desired expansion in $z - \mu$ (of course, w stands for $z - \mu$). A quick verification is required to show that this result is well-defined (the polynomial $p(w + \mu)$ is nonzero since $p(z)$ is nonzero) and does not depend on the exact choice of p and q (because if $\frac{1}{p(z)}q(z) = \frac{1}{\tilde{p}(z)}\tilde{q}(z)$, then $\tilde{p}(z)q(z) = p(z)\tilde{q}(z)$ in $R[z]$, and thus $\tilde{p}(w + \mu)q(w + \mu) = p(w + \mu)\tilde{q}(w + \mu)$, whence $\frac{1}{p(w + \mu)}q(w + \mu) = \frac{1}{\tilde{p}(w + \mu)}\tilde{q}(w + \mu)$).

⁵This is not the general definition of a pole, but in our case it does its job just fine, since all our poles are simple. (Non-simple poles could have residue equal to 0.)

We WLOG assume (for the sake of simplicity) that the family $(\lambda_i)_{i \in I}$ is self-indexing, i.e., that each λ_i equals the corresponding index i . This assumption can be made, since the λ_i are required to be distinct. So we must prove the following claim: There is only one choice of a finite subset I of k and a family $(e_\lambda)_{\lambda \in I} \in R^I$ of nonzero elements $e_\lambda \in R$ satisfying $g(z) = \sum_{\lambda \in I} \frac{e_\lambda}{z - \lambda}$.

In order to prove this, we must show that any such two choices of I and $(e_\lambda)_{\lambda \in I}$ are equal.

So let us consider two such choices: I and $(e_\lambda)_{\lambda \in I}$ on one side, and J and $(f_\lambda)_{\lambda \in J}$ on another. So I and J are two finite subsets of k , whereas $(e_\lambda)_{\lambda \in I}$ and $(f_\lambda)_{\lambda \in J}$ are two families of nonzero elements of R satisfying $g(z) = \sum_{\lambda \in I} \frac{e_\lambda}{z - \lambda}$ and $g(z) = \sum_{\lambda \in J} \frac{f_\lambda}{z - \lambda}$. We must show that $I = J$ and $(e_\lambda)_{\lambda \in I} = (f_\lambda)_{\lambda \in J}$.

Set $U := I \cup J$; this is again a finite subset of k . Extend the family $(e_\lambda)_{\lambda \in I} \in R^I$ to a family $(e_\lambda)_{\lambda \in U} \in R^U$ by setting $e_\lambda := 0$ for each $\lambda \notin I$. Likewise, extend the family $(f_\lambda)_{\lambda \in J} \in R^J$ to a family $(f_\lambda)_{\lambda \in U} \in R^U$ by setting $f_\lambda := 0$ for each $\lambda \notin J$. Note that

$$I = \{\lambda \in U \mid e_\lambda \neq 0\} \quad (202)$$

(since the original e_λ 's for $\lambda \in I$ were nonzero by assumption, whereas the new e_λ 's for $\lambda \notin I$ are zero by definition) and

$$J = \{\lambda \in U \mid f_\lambda \neq 0\} \quad (203)$$

(similarly).

The sum $\sum_{\lambda \in U} \frac{e_\lambda}{z - \lambda}$ equals the sum $\sum_{\lambda \in I} \frac{e_\lambda}{z - \lambda}$ (since the former sum differs from the latter only in the presence of some addends with $\lambda \notin I$, but all these addends vanish because of $e_\lambda = 0$). Thus,

$$\sum_{\lambda \in U} \frac{e_\lambda}{z - \lambda} = \sum_{\lambda \in I} \frac{e_\lambda}{z - \lambda} = g(z).$$

Similarly, $\sum_{\lambda \in U} \frac{f_\lambda}{z - \lambda} = g(z)$. Comparing these two equalities,

$$\sum_{\lambda \in U} \frac{e_\lambda}{z - \lambda} = \sum_{\lambda \in U} \frac{f_\lambda}{z - \lambda}.$$

Subtracting the left-hand side from the right-hand side, we obtain

$$0 = \sum_{\lambda \in U} \frac{f_\lambda}{z - \lambda} - \sum_{\lambda \in U} \frac{e_\lambda}{z - \lambda} = \sum_{\lambda \in U} \frac{f_\lambda - e_\lambda}{z - \lambda}.$$

Multiplying this equality by $\prod_{\mu \in U} (z - \mu)$, we find

$$0 = \sum_{\lambda \in U} (f_\lambda - e_\lambda) \prod_{\mu \in U \setminus \{\lambda\}} (z - \mu). \quad (204)$$

This is an equality in the polynomial ring $R[z]$ (since $R[z]$ is a subring of $R\left(\left(\frac{1}{z}\right)\right)$). Hence, for each $v \in U$, we can substitute $z = v$ in (204), and obtain

$$\begin{aligned} 0 &= \sum_{\lambda \in U} (f_\lambda - e_\lambda) \underbrace{\prod_{\mu \in U \setminus \{\lambda\}} (v - \mu)}_{\substack{=0 \text{ whenever } \lambda \neq v \\ \text{(because if } \lambda \neq v, \text{ then } v \in U \setminus \{\lambda\}, \text{ and thus} \\ \text{this product contains the factor } v - \mu = 0)}} \\ &= \sum_{\substack{\lambda \in U; \\ \lambda = v}} (f_\lambda - e_\lambda) \prod_{\mu \in U \setminus \{\lambda\}} (v - \mu) \\ &= (f_v - e_v) \prod_{\mu \in U \setminus \{v\}} (v - \mu). \end{aligned}$$

Dividing this equality by the nonzero scalar $\prod_{\mu \in U \setminus \{v\}} (v - \mu) \in k$ (which is nonzero because it is a product of the nonzero scalars $v - \mu \in k$), we obtain $0 = f_v - e_v$, so that $e_v = f_v$.

Thus, we have shown that

$$e_v = f_v \quad \text{for each } v \in U. \quad (205)$$

Thus, the right-hand sides of the equalities (202) and (203) agree. Hence, the left-hand sides agree, too. That is, $I = J$. Therefore, (205) yields $(e_\lambda)_{\lambda \in I} = (f_\lambda)_{\lambda \in J}$, and thus our proof is complete. \square

- **Page 24, proof of Proposition 2:** Here is an alternative proof of the "if" direction of Proposition 2:

Assume that $g(z)$ has the form (18) where the λ_i are distinct elements of k and where the e_i are nonzero elements of R . We must show that R is split semisimple.

From $g(z) = (1/z)f(1/z)$, we obtain $zg(z) = f(1/z)$. Substituting $z = 1/t$ in this, we obtain $(1/t)g(1/t) = f(t)$.

But substituting $z = 1/t$ in (18) and multiplying by $1/t$, we obtain

$$(1/t)g(1/t) = (1/t) \sum_{i \in I} \frac{e_i}{1/t - \lambda_i} = \sum_{i \in I} \frac{e_i}{1 - \lambda_i t} = \sum_{i \in I} \sum_{m \geq 0} e_i (\lambda_i t)^m$$

in the ring $R[[t]]$. Comparing this with $(1/t)g(1/t) = f(t) = \frac{1}{1-at} = \sum_{m \geq 0} (at)^m$, we obtain

$$\sum_{m \geq 0} (at)^m = \sum_{i \in I} \sum_{m \geq 0} e_i (\lambda_i t)^m.$$

Comparing the t^m -coefficients on both sides of this equality, we find

$$a^m = \sum_{i \in I} e_i \lambda_i^m \quad \text{for each } m \geq 0. \quad (206)$$

Hence, for each polynomial $h(t) \in k[t]$, we have

$$h(a) = \sum_{i \in I} e_i h(\lambda_i) \quad (207)$$

(because (206) shows that this is true whenever $h(t)$ is a monomial t^m , and thus, by linearity, this is also true for all polynomials h).

Now, let $h(t) \in k[t]$ be the polynomial $\prod_{i \in I} (t - \lambda_i)$. Then, $h(t)$ is a product of distinct monic linear factors. Its roots are precisely the λ_i . In particular, $h(\lambda_i) = 0$ for all $i \in I$. Hence, (207) simplifies to $h(a) = 0$. In other words, a is a root of h .

Now, let $m(t) \in k[t]$ be the minimal polynomial of a over k . Then, $m(t)$ divides any polynomial in $k[t]$ that has a as a root. In particular, $m(t)$ divides $h(t)$ (since h has a as a root). Hence, $m(t) \mid h(t) = \prod_{i \in I} (t - \lambda_i)$.

Therefore,

$$m(t) = \prod_{i \in J} (t - \lambda_i) \quad \text{for some subset } J \text{ of } I$$

(since $m(t)$ is monic).

We have $m(a) = 0$ (since $m(t)$ is the minimal polynomial of a). Hence, the canonical k -algebra homomorphism

$$\begin{aligned} k[t] &\rightarrow R, \\ t &\mapsto a \end{aligned}$$

has $m(t)$ in its kernel, and thus factors via the quotient ring $k[t]/(m(t))$. We thus obtain a k -algebra homomorphism

$$\begin{aligned} k[t]/(m(t)) &\rightarrow R, \\ \bar{t} &\mapsto a \end{aligned}$$

(which sends each $\overline{p(t)} \in k[t]/(m(t))$ to $p(a) \in R$). This homomorphism is surjective (since R is generated by a) and injective (since $m(t)$ is the

minimal polynomial of a , so that every nonzero polynomial of smaller degree than $m(t)$ will take a nonzero value at a); thus, it is invertible, i.e., a k -algebra isomorphism. Therefore, as a k -algebra, we have

$$\begin{aligned} R &\cong k[t] / (m(t)) = k[t] / \left(\prod_{i \in J} (t - \lambda_i) \right) && \left(\text{since } m(t) = \prod_{i \in J} (t - \lambda_i) \right) \\ &\cong \prod_{i \in J} \underbrace{(k[t] / (t - \lambda_i))}_{\cong k} && \left(\begin{array}{l} \text{by the Chinese Remainder Theorem,} \\ \text{since the } \lambda_i \text{ are distinct} \end{array} \right) \\ &\cong \prod_{i \in J} k = k^J. \end{aligned}$$

This shows that R is split semisimple.

This proof also shows that Proposition 2 holds for any field k , not just for subfields of \mathbb{C} . (The original proof might also show this, but it would take more thought.)

- **Page 26:** After "in terms of the function $g(z)$ of Section 8.2", add "(for $R = \mathbb{R}[w]$ and $a = w$)".
- **Page 27, §8.4:** After "We have therefore proved", add "(using Proposition 2)".
- **Page 28, §8.6:** Replace "(Section 2.3)" by "(Section 5.1)".
- **Page 32, Proposition 4:** Why does "the smallest face $F \leq C'$ such that $FC = C'$ " exist?
- **Page 36, §9.6:** Replace " $k[w]$ " by " $k[p]$ ". Also, this claim about constructing primitive idempotents is tacitly understood to cover the case when k is a field of characteristic 0, since otherwise it is not clear what "generic" means.
- **Page 36, Theorem 8:** "the isomorphism of Theorem 7" should be "the anti-isomorphism of Theorem 7".
- **Page 37, §9.7:** Remove the comma at the end of "Then E_0, \dots, E_{n-2}, E_n ".
- **Page 37, §A.1:** Replace "and is denoted $\sigma(F)$ " by "and is denoted $\sigma(F) = (\sigma_i(F))_{i \in I}$ ". (This way, the notation $\sigma_i(F)$ is also defined.)
- **Page 37, §A.1:** Somewhere here you should probably define \mathcal{C} as the set of all chambers of \mathcal{A} . (You use this notation in §A.7, and maybe earlier.)
- **Page 37, §A.2:** The notation " F is a face of G " is not defined. (It means " $F \leq G$ ".)

- **Page 40:** “stated as in exercise” should be “stated as an exercise”.
- **Page 41, §B.2:** Replace “If S has an identity e ” by “If S has an identity e ” (the “ e ” should be in mathmode).
- **Page 41, §B.2:** Replace “it is is a lattice” by “it is a lattice”.
- **Page 41, proof of Proposition 9:** Replace “ $cx = x$ ” by “ $cx = c$ ” in the second sentence of this proof (but not in the third sentence).
- **Page 42, §C.1:** The paragraph starting with “In case L is a geometric lattice” has confused me for a while until I resolved the ambiguities. The problem is that the m_X in (14) are defined only for $X \in \bar{L}$, not for all $X \in L$, so the correct version of the claim “ $d\left(\left[X, \hat{1}\right]\right) = m_X$ ” should be “ $d\left(\left[X, \hat{1}\right]\right) = \begin{cases} m_X, & \text{if } \text{rank}(X) \neq n-1; \\ 0, & \text{if } \text{rank}(X) = n-1 \end{cases}$ for all $X \in L$ (where the m_X are those of (14), not those of (13))”. In particular, I find it important to point out that the m_X are not those of (13); at first sight they would seem to be the numbers more directly connected to the $d\left(\left[X, \hat{1}\right]\right)$.
You might also want to replace “random walk” by “random walk on the chambers of \bar{S} ” to get this point across again.
- **Page 43, Proposition 10:** Since the audience of this work is unlikely to consist entirely of combinatorialists, it is worth explaining that a “cover of X ” means an element $Y \in L$ such that Y covers X .
- **Page 45, §C.3:** The equality (46) holds only for $n > 0$.
- **Page 45, §C.3:** In the sentence “For $J \subseteq [n-1]$, let $f_J(L)$ be the number of flags in L of type J , where the *type* of a flag $X_1 < X_2 < \dots < X_l$ is the set $\{\text{rank } X_i\}_{1 \leq i \leq l}$ ”, the word “flag” should probably be replaced by “chain” both times. (At least, the word “chain” is more standard for this usage.)
- **Page 46, Proposition 11:** Again, (50) holds only for $n > 0$.
- **Page 46, proof of Proposition 11:** You say that $(-1)^n h_{[n-1]}(L) = \mu_L(\hat{0}, \hat{1})$ holds “by (49)”. I do not know enough topology to understand this, but it might be worth pointing out that $(-1)^n h_{[n-1]}(L) = \mu_L(\hat{0}, \hat{1})$ also follows from (47) using Hall’s formula for the Möbius function.
- **Page 49, reference [38]:** The reference needs volume and issue: Amer. J. Math. 57 (1935), no. 3, 509–533.