**Semigroups, rings, and Markov chains**
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http://arxiv.org/abs/math/0006145v1
version of 20 June 2000 (arXiv preprint arXiv:math/0006145v1).
**Errata and addenda by Darij Grinberg**

I will refer to the results appearing in the paper "Semigroups, rings, and Markov chains" by the numbers under which they appear in this paper (specifically, in its version of 20 June 2000, which appears on arXiv as preprint arXiv:math/0006145v1).

# Errata

- **Page 1, §1:** "explcitly" $\to$ "explicitly".

- **Page 3, §1.4:** Replace "$q$-elements" by "$q$ elements". (The hyphen should not be there.)

- **Page 6, §2.1:** At the end of the first paragraph on this page (i.e., right after the sentence "Thus $S$ has the deletion property (D) stated in Section 1.1"), I suggest adding something like the following definitions:

  "When $S$ is a LRB, the lattice $L$ and the map supp in the above definition are determined uniquely up to isomorphism (this follows easily from the surjectivity of supp and the equivalence (3)). The lattice $L$ is called the *lattice of supports* (or the *support lattice*) of $S$. The surjection supp $: S \to L$ is called the *support map* of $S$. For any $s \in S$, the element supp $s$ of $L$ is called the *support* of $s$."

  (Of course, the purpose of these sentences is to introduce some notations that you use several times; you'll probably find a more succinct way to introduce them.)

- **Page 6, §2.1:** The last comma in "Sections 4, 5, and 6," probably should be a period.

- **Page 6, §2.2:** You write that $S_{\geq x}$ "is a LRB in its own right, the associated lattice being the interval $\left[X, \widehat{1}\right]$ in $L$, where $X = \operatorname{supp} x$". This is correct, but (in my opinion) not obvious enough to be left to the reader.

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[1] Here is a proof, for the sake of completeness:

- **Page 9, §4.2:** In "Let $\mathcal{G}_0 = \{G \in \mathcal{G} : \sigma_i(G) = +\}$", replace "$\sigma_i(G) = +$" by "$\sigma_i(G) = +$ for all $i \in J$".

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Let $x \in S$. Let $X = \text{supp}\, x$. The interval $\left[X, \widehat{1}\right]$ of the poset $L$ is a lattice (since $L$ is a lattice). Its join operation is the restriction of the join operation $\vee$ of the lattice $L$; therefore, we shall denote it by $\vee$ as well.

It remains to prove that $S_{\geq x}$ is a LRB with support lattice $\left[X, \widehat{1}\right]$ and support map $\text{supp}\,|_{S_{\geq x}}$. In order to do so, we need to prove the following three statements:

*Statement 1:* The set $S_{\geq x}$ is a subsemigroup of $S$ with identity $x$.

*Statement 2:* The map $\text{supp}\,|_{S_{\geq x}} \colon S_{\geq x} \to \left[X, \widehat{1}\right]$ is well-defined and surjective.

*Statement 3:* For any $a \in S_{\geq x}$ and $b \in S_{\geq x}$, we have $\left(\text{supp}\,|_{S_{\geq x}}\right)(ab) = \left(\left(\text{supp}\,|_{S_{\geq x}}\right)a\right) \vee \left(\left(\text{supp}\,|_{S_{\geq x}}\right)b\right)$.

*Statement 4:* Let $a \in S_{\geq x}$ and $b \in S_{\geq x}$. Then, $ab = a$ holds if and only if $\left(\text{supp}\,|_{S_{\geq x}}\right)b \leq \left(\text{supp}\,|_{S_{\geq x}}\right)a$.

(These four Statements are precisely the requirements in the definition of a LRB, except that we have renamed $x$ and $y$ as $a$ and $b$ since the letter $x$ is already taken for something else.)

*Proof of Statement 1:* Let $a \in S_{\geq x}$ and $b \in S_{\geq x}$. We have $a \geq x$ (since $a \in S_{\geq x}$). In other words, $x \leq a$. But (5) (applied to $y = a$) shows that $x \leq a \iff xa = a$. Thus, $xa = a$ (since $x \leq a$). But (5) (applied to $y = ab$) shows that $x \leq ab \iff xab = ab$. Hence, $x \leq ab$ (since $\underbrace{xa}_{=a}\, b = ab$). In other words, $ab \geq x$, so that $ab \in S_{\geq x}$.

Now, let us forget that we fixed $a$ and $b$. We thus have shown that $ab \in S_{\geq x}$ for every $a \in S_{\geq x}$ and $b \in S_{\geq x}$. In other words, $S_{\geq x}$ is a subsemigroup of $S$, although we do not yet know whether it has an identity.

Now, let $a \in S_{\geq x}$. Then, $a \geq x$ (since $a \in S_{\geq x}$). In other words, $x \leq a$. But (5) (applied to $y = a$) shows that $x \leq a \iff xa = a$. Thus, $xa = a$ (since $x \leq a$). Also, $\underbrace{a}_{=xa}\, x = xax = xa$ (by (4), applied to $y = a$), so that $ax = xa = a$.

Now, let us forget that we fixed $a$. We thus have shown that $ax = xa = a$ for every $a \in S_{\geq x}$. In other words, the subsemigroup $S_{\geq x}$ of $S$ has identity $a$. This finishes the proof of Statement 1.

*Proof of Statement 2:* Let $y \in S_{\geq x}$. Thus, $y \geq x$, so that $x \leq y$. Hence, $xy = y$ (by (5)). Thus, $\text{supp}\,\underbrace{y}_{=xy} = \text{supp}(xy) = \text{supp}\, x \vee \text{supp}\, y$ (by (2)), so that $\text{supp}\, y = \text{supp}\, x \vee \text{supp}\, y \geq \text{supp}\, x = X$. In other words, $\text{supp}\, y \in \left[X, \widehat{1}\right]$.

Let us now forget that we fixed $y$. We thus have shown that $\text{supp}\, y \in \left[X, \widehat{1}\right]$ for every $y \in S_{\geq x}$. Hence, $\text{supp}(S_{\geq x}) \subseteq \left[X, \widehat{1}\right]$. Therefore, the map $\text{supp}\,|_{S_{\geq x}} \colon S_{\geq x} \to \left[X, \widehat{1}\right]$ is well-defined.

We shall now show that this map is surjective. Indeed, let $Y \in \left[X, \widehat{1}\right]$. Thus, $Y \in L$ and $Y \geq X$. Since the map $\text{supp} \colon S \to L$ is surjective, we thus see that there exists some $y \in S$ satisfying $Y = \text{supp}\, y$. Consider such a $y$. We have $\text{supp}\, x = X \leq Y = \text{supp}\, y$. Hence, (3) (applied to $y$ and $x$ instead of $x$ and $y$) shows that $yx = y$.

But (5) (applied to $xy$ instead of $y$) shows that $x \leq xy \iff xxy = xy$. Hence, $x \leq xy$ (since $\underbrace{xx}_{=x}\, y = xy$). In other words, $xy \geq x$, so that $xy \in S_{\geq x}$. Now, (2) shows that $\text{supp}(xy) =$

- **Page 9, §4.2:** "faces in $G$" should be "faces in $\mathcal{G}$".

- **Page 13, §5.1, Remark:** "maximal elements" $\rightarrow$ "maximal elements ($\neq \widehat{1}$)".

- **Page 14:** In "where $c_X$ is the number of maximal chains in the interval $[X, V]$ in $L$", the lattice $L$ should be defined. (It is not the lattice of supports of $\overline{F}_{n,q}$, but rather the lattice of all subspaces of $V$, including those of dimension $n - 1$.)

- **Page 14:** "get a line $l_1$" $\rightarrow$ "get a line $\ell_1$".

- **Page 15, §6.1:** "the rank of any maximal" $\rightarrow$ "the size of any maximal".

- **Page 16:** "any fixed basis of $X$ to a basis" $\rightarrow$ "any fixed ordered basis of $X$ to an ordered basis".

- **Pages 24–25, §8.2:** Any appearance of "$A$" in §8.2 should be replaced by an "$R$". (There are three appearances on page 24, and six appearances on page 25, not counting the two "$A'$"s.)

- **Page 24, §8.2:** It would be useful to point out that $g(z)$ is understood to be an element of $A\left(\left(\frac{1}{z}\right)\right)$ (that is, of the ring of Laurent series in $\frac{1}{z}$ over $A$). Later it becomes clear that $g(z)$ is indeed a rational function in $z$ over $A$; but before this is shown, it is important to understand whether $g(z)$ is viewed as a Laurent series in $\frac{1}{z}$ (correct) or as a Laurent series in $z$ (incorrect).

  In my opinion, it is also useful to stress that you regard $k(z) \otimes A$ as a subring of $A\left(\left(\frac{1}{z}\right)\right)$, since $k(z)$ is canonically a subring of $k\left(\left(\frac{1}{z}\right)\right)$.

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$\underbrace{\text{supp } x}_{=X} \vee \underbrace{\text{supp } y}_{=Y} = X \vee Y = Y$ (since $X \leq Y$). Hence, $Y = \text{supp}\left(\underbrace{xy}_{\in S_{\geq x}}\right) \in \text{supp}(S_{\geq x}) =$ $(\text{supp}\,|_{S_{\geq x}})(S_{\geq x})$.

Now, let us forget that we fixed $y$. We thus have shown that $Y \in (\text{supp}\,|_{S_{\geq x}})(S_{\geq x})$ for every $Y \in \left[X, \widehat{1}\right]$. In other words, $\left[X, \widehat{1}\right] \subseteq (\text{supp}\,|_{S_{\geq x}})(S_{\geq x})$. In other words, the map $\text{supp}\,|_{S_{\geq x}}: S_{\geq x} \rightarrow \left[X, \widehat{1}\right]$ is surjective. This finishes the proof of Statement 2.

*Proof of Statement 3:* Let $a \in S_{\geq x}$ and $b \in S_{\geq x}$. Since $S_{\geq x}$ is a subsemigroup of $S$ (by Statement 1), this shows that $ab \in S_{\geq x}$. Now, (2) (applied to $a$ and $b$ instead of $x$ and $y$) shows that $\text{supp}(ab) = \text{supp}\,a \vee \text{supp}\,b$. This rewrites as $(\text{supp}\,|_{S_{\geq x}})(ab) = ((\text{supp}\,|_{S_{\geq x}})\,a) \vee ((\text{supp}\,|_{S_{\geq x}})\,b)$ (since $(\text{supp}\,|_{S_{\geq x}})(ab) = \text{supp}(ab)$, $(\text{supp}\,|_{S_{\geq x}})\,a = \text{supp}\,a$ and $(\text{supp}\,|_{S_{\geq x}})\,b = \text{supp}\,b$). This proves Statement 3.

*Proof of Statement 4:* Assume that $(\text{supp}\,|_{S_{\geq x}})\,b \leq (\text{supp}\,|_{S_{\geq x}})\,a$. This rewrites as $\text{supp}\,b \leq \text{supp}\,a$ (since $(\text{supp}\,|_{S_{\geq x}})\,a = \text{supp}\,a$ and $(\text{supp}\,|_{S_{\geq x}})\,b = \text{supp}\,b$). Hence, (3) (applied to $a$ and $b$ instead of $x$ and $y$) yields $ab = a$. This proves Statement 4.

Now, all four Statements are proven, and the proof is complete.

This explains how exactly you can make sense of the statement that $g(z)$ is a rational function (namely, this statement means that $g(z)$ lies in the subring $k(z) \otimes A$ of $A\left(\left(\dfrac{1}{z}\right)\right)$).

- **Page 24, proof of Proposition 2:** I think this proof is missing a few steps. You leave the following statements unproven:

  *Statement 1:* Suppose that $R$ is split semisimple with primitive idempotents $(e_i)_{i \in I}$, and write $a = \sum\limits_i \lambda_i e_i$ with $\lambda_i \in K$. Then, the $\lambda_i$ (for $i \in I$) are distinct.

  *Statement 2:* Up to relabelling, there is only one choice of a set $I$ and two families $(e_i)_{i \in I} \in R^I$ and $(\lambda_i)_{i \in I} \in K^I$ satisfying (18).

  (Statement 1 is needed to prove the first sentence of Proposition 2. Statement 2 is needed to prove the second sentence.)

  *Proof of Statement 1:* Let $j$ and $j'$ be two distinct elements of $I$.

  Let $G$ be the subset $\left\{ \sum\limits_i \mu_i e_i \;\mid\; (\mu_i)_{i \in I} \in K^I \text{ and } \mu_j = \mu_{j'} \right\}$ of $R$. Then, $G$ is a $k$-subalgebra of $R$ (this is easy to see) and satisfies $G \neq R$ (since $e_j \notin G$). But every $k$-subalgebra of $R$ which contains $a$ must be $R$ itself (since $a$ generates the $k$-algebra $R$). Hence, if we had $a \in G$, then we would have $G = R$ (since $G$ would be a $k$-subalgebra of $R$ which contains $a$), which would contradict $G \neq R$. Thus, we cannot have $a \in G$. In other words, we have $a \notin G$. In other words, $\lambda_j \neq \lambda_{j'}$.

  Now, let us forget that we fixed $j$ and $j'$. We thus have shown that $\lambda_j \neq \lambda_{j'}$ for any two distinct elements $j$ and $j'$ of $I$. In other words, the $\lambda_i$ (for $i \in I$) are distinct. This proves Statement 1.

  *Proof of Statement 2:* Consider a set $I$ and two families $(e_i)_{i \in I} \in R^I$ and $(\lambda_i)_{i \in I} \in K^I$ satisfying (18). We shall show that the $\lambda_i$ and the corresponding $e_i$ can be reconstructed from $g(z)$ (up to labelling). This will clearly prove Statement 2.

  The function $g(z)$ is a rational function in $z$ (in the sense that $g(z) \in k(z) \otimes A$). Hence, for every $\mu \in k$, an element $\mathrm{Res}_\mu \, g(z)$ of $A$ is well-defined (namely, it is defined as the coefficient of $(z - \mu)^{-1}$ when $g(z)$ is expanded as a Laurent series in $z - \mu$). Now, (18) shows that the elements $\lambda_i$ of $k$ are the poles of $g(z)$ (that is, the elements $\mu \in k$ for which $\mathrm{Res}_\mu \, g(z) \neq 0$), and the elements $e_i$ of $A$ are their corresponding residues (i.e., we have $e_i = \mathrm{Res}_{\lambda_i} \, g(z)$ for every $i \in I$). Thus, the $\lambda_i$ and the corresponding $e_i$ can be reconstructed from $g(z)$ (up to labelling). This proves Statement 2.

- **Page 28, §8.6:** Replace "(Section 2.3)" by "(Section 5.1)".

- **Page 32, Proposition 4:** Why does "the smallest face $F \leq C'$ such that $FC = C'$" exist?

- **Page 37, §A.1:** Replace "and is denoted $\sigma(F)$" by "and is denoted $\sigma(F) = (\sigma_i(F))_{i \in I}$". (This way, the notation $\sigma_i(F)$ is also defined.)

- **Page 37, §A.1:** Somewhere here you should probably define $\mathcal{C}$ as the set of all chambers of $\mathcal{A}$. (You use this notation in §A.7, and maybe earlier.)

- **Page 37, §A.2:** The notation "$F$ is a *face* of $G$" is not defined. (It means "$F \leq G$".)

- **Page 41, §B.2:** Replace "If $S$ has an identity e" by "If $S$ has an identity $e$" (the "$e$" should be in mathmode).

- **Page 41, §B.2:** Replace "it is is a lattice" by "it is a lattice".

- **Page 41, proof of Proposition 9:** Replace "$cx = x$" by "$cx = c$" (twice).

- **Page 42, §C.1:** The paragraph starting with "In case $L$ is a geometric lattice" has confused me for a while until I resolved the ambiguities. The problem is that the $m_X$ in (14) are defined not for $X \in L$ but for $X \in \bar{L}$, so the correct version of the claim "$d\left(\left[\widehat{X}, 1\right]\right) = m_X$" should be "$d\left(\left[\widehat{X}, 1\right]\right) =$
$\begin{cases} m_X, & \text{if } \operatorname{rank}(X) \neq n-1; \\ 0, & \text{if } \operatorname{rank}(X) = n-1 \end{cases}$ (where the $m_X$ are those of (14), not those of (13))". In particular, I find it important to point out that the $m_X$ are not those of (13); at the first sight they would seem to be the numbers more directly connected to the $d\left(\left[\widehat{X}, 1\right]\right)$.

  You might also want to replace "random walk" by "random walk on the chambers of $\bar{S}$" to get this point across again.

- **Page 43, Proposition 10:** You should say that a "cover of $X$" means an element $Y \in L$ such that $Y$ covers $X$.

- **Page 45, §C.3:** The equality (46) holds only for $n > 0$.

- **Page 45, §C.3:** Replace "a flag $X_1 < X_2 < \cdots < X_l$" by "a flag $\widehat{0} < X_1 < X_2 < \cdots < X_l < \widehat{1}$".

- **Page 46, Proposition 11:** Again, (50) holds only for $n > 0$.

- **Page 46, proof of Proposition 11:** You say that $(-1)^n h_{[n-1]}(L) = \mu_L\left(\widehat{0}, \widehat{1}\right)$ holds "by (49)". I do not know enough topology to understand this, but it might be worth pointing out that $(-1)^n h_{[n-1]}(L) = \mu_L\left(\widehat{0}, \widehat{1}\right)$ also follows from (47) using Hall's formula for the Möbius function.