

# A Course in Combinatorial Optimization

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<http://homepages.cwi.nl/~lex/files/dict.pdf>

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## Errata and comments (Darij Grinberg)

I have read three chapters of these notes: 1, 2 and 10, as well as sporadic pieces of other chapters. Most of my comments on chapters 1 and 2 are nitpicks (these seem to have undergone some thorough error hunting already); some are probably misunderstandings on my side. I hope the comments on chapter 10 are more useful.

## Chapter 1

- **Proof of Theorem 1.3:** At the beginning of the proof, it wouldn't harm to add something like this (hopefully you can express it more succinctly): “Note that, for every vertex  $v \in V$ , the value  $f(v)$  never increases during the course of the algorithm; it can only stay constant or decrease. Besides, if  $u \in V$  is any vertex, then the value  $f(u)$  does not change during the  $u$ -iteration (because we cannot have  $f(u) > f(u) + l(a)$  for any arc  $a$ ). (Here, for any vertex  $p \in V$ , the  $p$ -iteration means the iteration of (3) in which the vertex  $p$  is chosen as  $u$ ; this is the iteration which results in the removal of  $u$  from  $U$ .) Consequently, if  $u \in V$  is any vertex, then, immediately after the  $u$ -iteration, we have  $f(v) \leq f(u) + l(a)$  for every  $a = (u, v) \in A$ .” These statements are obvious, but they are tacitly used below.

- **Proof of Theorem 1.3:** After “If  $i > 0$ , then (as  $v_{i-1} \in V \setminus U$ )”, add “the following inequality must have been valid directly after the  $v_{i-1}$ -iteration”. In fact, it isn't a priori clear that  $f(v_{i-1}) \leq f(v_i) + l(v_{i-1}, v_i)$  is true immediately after the  $u$ -iteration, but it is immediately clear that  $f(v_{i-1}) \leq f(v_i) + l(v_{i-1}, v_i)$  is true immediately after the  $v_{i-1}$ -iteration.

After (4), add the following sentence: “Since the value  $f(v_i)$  never increases during the course of the algorithm, this yields that  $f(v_i) \leq \text{dist}(v_i)$  also holds immediately after the  $u$ -iteration.”

- **Proof of Theorem 1.5:** In footnote <sup>3</sup>, replace “set” by “multiset”.
- **Proof of Theorem 1.6:** Replace “ $2i + 1 \leq j \leq 2i + 2$ ” by “ $2i \leq j \leq 2i + 1$ ”. Also, replace “While  $i > 0$ ” by “While  $i > 1$ ”, and replace “ $j := \left\lfloor \frac{i-1}{2} \right\rfloor$ ” by “ $j := \left\lfloor \frac{i}{2} \right\rfloor$ ”. (The indexing of your heap starts with 1, not with 0.)
- **Proof of Theorem 1.8:** In (13), replace “number of calls of make root =  $\text{decr}(f(u)) + \text{decr}(T)$ ” by “number of calls of make root  $\leq \text{decr}(f(u)) + \text{decr}(T)$ ”. In fact, while it is true that every call of make root immediately succeeds either a decrease of some  $f(u)$  or a decrease of  $T$ , it can happen that  $T$  decreases without make root being called (namely, this happens if we delete a  $u$  minimizing  $f(u)$ , but this  $u$  happens to belong to  $T$ ).

- **Proof of Theorem 1.8:** You write: “since we increase  $T$  at most once after we have decreased some  $f(u)$ ”. I think this could better be replaced by “since we increase  $T$  at most once after each decrease of some  $f(u)$ ”; otherwise it sounds as if, once  $f(u)$  has been decreased for the first time,  $T$  can’t increase twice anymore.
- **Proof of Theorem 1.8:** I don’t understand the steps of (14). Here is my estimation for the number of calls of repair, with some explanations that hopefully make it clear what I am doing:

First, let me make two conventions:

- When the set  $F$  decreases by  $k$  arcs (for  $k > 1$ ), we don’t consider this as one decrease of  $F$ , but as  $k$  decreases of  $F$ .
- When a  $u$  minimizing  $f(u)$  is being deleted, the root  $u$  is lost but every child of  $R$  becomes a root. We consider this as a decrease of  $R$  by 1 root followed by an increase of  $R$  by  $k$  roots (where  $k$  is the number of children of  $u$ ), but **not** as an increase of  $R$  by  $k - 1$  roots.

Here are the three possible ways how the algorithm can call repair:

- The algorithm calls repair when a vertex  $u$  minimizing  $f(u)$  and having at least one child is being deleted; in this case, repair is being called  $k$  times (where  $k$  is the number of children of  $u$ ). The set  $F$  decreases by  $k$  arcs (namely, the arcs connecting the now-deleted vertex  $u$  to its children).<sup>1</sup>
- The algorithm also can call repair from inside a repair subroutine. In this case, repair is being called directly after a root has been grafted on another root; thus, repair is being called directly after the set  $R$  of roots has decreased by 1 root (because the root that has been grafted on another root is no longer a root after this).
- The algorithm also can call repair from inside a make root subroutine. In this case, repair is being called directly after an arc has been removed; thus, repair is being called directly after the set  $F$  of arcs has decreased by 1 arc.

There are no other circumstances under which repair can be called in the algorithm. Thus, each calling of repair during the algorithm corresponds to a decrease

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<sup>1</sup>Also, the set  $R$  (the set of roots) decreases by 1 root (the root  $u$ ) and then increases by  $k$  roots, but we don’t care about this.

of  $F$  or a decrease of  $R$  (although this is not a 1-to-1 correspondence). Thus,

$$\begin{aligned}
 & \text{(number of calls of repair)} \\
 & \leq \underbrace{\text{decr}(R)}_{\leq \text{incr}(R)+p} + \text{decr}(F) \\
 & \leq \text{incr}(R) + p + \text{decr}(F) = \text{decr}(F) + p + \text{decr}(F) \\
 & \quad \left( \begin{array}{l} \text{because every increase of the set } R \text{ of roots immediately follows a decrease} \\ \text{of the set } F \text{ of arcs, and vice versa, so that } \text{incr}(R) = \text{decr}(F) \end{array} \right) \\
 & = 2 \text{decr}(F) + p \\
 & \leq 2(n(1 + 2 \log p) + (\text{number of calls of make root})) + p \\
 & \quad \left( \begin{array}{l} \text{because the set } F \text{ of arcs decreases by at most } 1 + 2 \log p \text{ elements every time we} \\ \text{delete a } u \text{ minimizing } f(u) \text{ (because we lose an arc for every child of } u, \text{ and} \\ \text{by Theorem 1.7 we know that } u \text{ has at most } 1 + 2 \log p \text{ children), and decreases} \\ \text{by at most 1 element every time we call the make root subroutine, and decreases} \\ \text{never else, so we have} \\ \text{decr}(F) \leq n(1 + 2 \log p) + (\text{number of calls of make root}) \end{array} \right) \\
 & = 2n(1 + 2 \log p) + 2 \underbrace{(\text{number of calls of make root})}_{\substack{\leq 2m+p \\ \text{(by (13))}}} + p \\
 & \leq 2n(1 + 2 \log p) + 2(2m + p) + p.
 \end{aligned}$$

This is almost the same as what you get, except for an additional  $2n$  addend. Have I made any mistakes here?

Note that I don't think we have  $(\text{number of calls of repair}) = \text{decr}(R) + \text{decr}(F)$  all the time. In fact, when we delete a  $u$  minimizing  $f(u)$ , and this  $u$  happens to have no children, then  $R$  decreases (since we lose the root  $u$ ) but we don't call repair.

- **Proof of Theorem 1.8:** I don't see what you mean by "deciding calling make root or repair". Don't you just want to say "calling make root or repair"?

(Anyway, why does a call of make root or repair take time  $O(1)$ ? Can we really look up an arbitrary value of the function  $b$  or  $l$  in  $O(1)$  time? And delete a vertex from a linked list in  $O(1)$  time?)

- **Application 1.4:** On page 16, replace "If activity  $j$  has been done after activity  $i$ " by "If activity  $j$  has to be done after activity  $i$ ". Or am I misunderstanding you?
- **Application 1.5:** In (22), what do you mean by "the maximum liter price in village  $Y$ "? Probably "the maximum liter price in village  $Y$  over all price equilibria"?
- **Exercise 1.3:** You use the word "distance" in two different meanings in this exercise. (You use it to mean both the length of the direct route and the length of the shortest path.)

- **Proof of Theorem 1.12:** A point that you miss in the analysis of the running time of the Dijkstra-Prim algorithm (and also of the Dijkstra algorithm in the proof of Corollary 1.8a) is the running time required to build the initial Fibonacci heap at the start of the algorithm. This isn't very difficult<sup>2</sup> and doesn't take very long<sup>3</sup>, so it doesn't change the end result, but I think it should be mentioned.
- **Paragraph on page 20 directly above Corollary 1.12a.** Replace “Choose an edge  $e_{k+1}$ ” by “Choose an edge  $e_{k+1} \notin \{e_1, e_2, \dots, e_k\}$ ”. I know, this can hardly be misunderstood either way, but for the sake of formal correctness I think the change should be done.
- **Exercise 1.9:** It might help to point out that “forest” means “spanning forest”, and  $|F|$  means the number of edges of  $F$ .

## Chapter 2

- **Second line of §2.1:** Replace “with any two points in  $C$ ” by “with any two points  $x, y$  in  $C$ ”.
- **Between (1) and (2):** Replace “A basic property of closed convex sets is” by “A basic property of closed convex sets  $C$  is”.
- **Proof of Theorem 2.2:** The Sufficiency part of this proof could be made a bit easier to follow if you would replace “there exist points  $x$  and  $y$  in  $P$  such that  $x \neq z \neq y$  and  $z = \frac{1}{2}(x + y)$ ” by “there exist points  $x$  and  $y$  in  $P$  and a  $\lambda$  with  $0 < \lambda < 1$  such that  $x \neq z$ ,  $y \neq z$  and  $z = \lambda x + (1 - \lambda)y$ ”, and replace “ $y - z = -(x - z)$ ” by “ $y - z = -\frac{\lambda}{1 - \lambda}(x - z)$  and  $-\frac{\lambda}{1 - \lambda} < 0$ ”. No reason to use midpoints here...
- **Between the proof of Theorem 2.2 and Theorem 2.3:** Some absolute nit-picking: When you write “ $A_z \neq A_{z'}$ ”, you actually mean “ $\{i \in \{1, 2, \dots, m\} \mid a_i z = b_i\} \neq \{i \in \{1, 2, \dots, m\} \mid a_i z' = b_i\}$ ”. (Though this is actually equivalent to  $A_z \neq A_{z'}$ , but the equivalence is not completely trivial.) Also, “collections of subrows of  $A$ ” should be “sets of rows of  $A$ ” (since “collection” could be misunderstood to mean “multiset”, and “subrow” is probably a typo for “row”).
- **Proof of Theorem 2.4:** You write: “To see the inclusion  $\supseteq$  in (25)”. You mean (27), not (25).
- **Proof of Theorem 2.4:** Replace “ $c = \mu_1 y_1 + \dots + \mu_s y_s$ ” by “ $c = \mu_1 y_1 + \dots + \mu_s y_s$ ”. (Actually, it would be better to write “ $c = \sum_{j=1}^s \mu_j y_j$ ” instead of “ $c = \mu_1 y_1 + \dots +$ ”.

<sup>2</sup>I think it is enough to start with a forest with no edges, and then call repair over and over again until we get a Fibonacci forest. We can't make use of the function  $b$  yet, but we can keep a list of roots  $u$  with their values of  $d^{\text{out}}(u)$ , sorted by the size of  $d^{\text{out}}(u)$ .

<sup>3</sup>I think it takes somewhere between  $O(|V|)$  and  $O(|V| \log |V|)$  in Corollary 1.8a and somewhere between  $O(|V|)$  and  $O(|V| \log |V|)$  in Theorem 1.12, at least if you start the Dijkstra-Prim algorithm with  $U_0 = \emptyset$  and all  $f(v) = \infty$  rather than with  $U_1 = \{v_1\}$

$\mu_s y_s$ ” here, and similarly write “ $\sum_{j=1}^s \mu_j = 1$ ” instead of “ $\mu_1 + \dots + \mu_s = 1$ ”; in fact, you use summation signs in the formula (28) below.)

- **Convex cones:** In (29), replace “ $\lambda_1 x_1 + \dots + \lambda_t x_t$ ” by “ $\lambda_1 x_1 + \dots + \lambda_t x_t$ ”.
- **Exercise 2.12:** I don’t see how to directly use the hint that you give to this exercise. In my opinion, cone  $X$  needs not be closed.

Here is a sketch of my solution to Exercise 2.12:

*Solution to Exercise 2.12 (sketched):*

$\implies$ : Let  $P = \{x \in \mathbb{R}^n \mid Ax \leq b\}$  be a polyhedron, with  $A$  being an  $m \times n$  matrix and  $b$  being a vector in  $\mathbb{R}^m$ . We want to write  $P$  in the form  $Q + C$  with  $Q$  being a polytope and  $C$  being a finitely generated cone.

First of all, we can WLOG assume that  $\text{Ker } A = 0$ .<sup>4</sup> Assume this.

Now, let  $x_1, x_2, \dots, x_t$  be the vertices of  $P$ . We claim that

$$A = \text{conv. hull} \{x_1, \dots, x_t\} + \{x \in \mathbb{R}^n \mid Ax \leq 0\}. \quad (1)$$

Once (1) is proven, it will clearly follow that  $P$  can be written in the form  $Q + C$  with  $Q$  being a polytope and  $C$  being a finitely generated cone (because  $\text{conv. hull} \{x_1, \dots, x_t\}$  is a polytope, and because  $\{x \in \mathbb{R}^n \mid Ax \leq 0\}$  is a finitely generated cone (according to Exercise 2.11)). Thus, in order to prove the  $\implies$  direction of Exercise 2.12, we only need to establish (1).

To prove (1), we need to show that

$$\text{if } z \in A \text{ then } z \in \text{conv. hull} \{x_1, \dots, x_t\} + \{x \in \mathbb{R}^n \mid Ax \leq 0\}. \quad (2)$$

To show this, we proceed as in the proof of Theorem 2.3 (thus, by induction over  $n - \text{rank}(A_z)$ ), with the following modification: The numbers

$$\mu_0 := \max \{\mu \mid z + \mu c \in P\} \quad \text{and} \quad \nu_0 := \max \{\nu \mid z - \nu c \in P\}$$

don’t necessarily both exist anymore, since  $P$  is no longer bounded. Thus, we must be in one of the following cases:

*Case 1:* Both numbers  $\mu_0$  and  $\nu_0$  exist. In this case, proceed exactly as in the proof of Theorem 2.3.

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<sup>4</sup>*Proof.* Assume that  $\text{Ker } A \neq 0$ . Let  $V$  be a subspace of  $\mathbb{R}^n$  which satisfies  $\mathbb{R}^n = (\text{Ker } A) \oplus V$ . Let  $P' = V \cap P$ . Then,  $P'$  is a polyhedron in  $V$ . Moreover, it is easy to see that  $P = P' + \text{Ker } A$  (since every  $p \in P$  and every  $w \in \text{Ker } A$  satisfy  $p + w \in P$ ). By an induction, we can assume that the  $\implies$  direction of Exercise 2.12 is already proven for the polyhedron  $P'$  (because  $P'$  lies in a vector space of smaller dimension than  $\mathbb{R}^n$ , so we can do an induction over the dimension). In other words, we can write  $P'$  in the form  $Q' + C'$  with  $Q'$  being a polytope in  $V$  and  $C'$  being a finitely generated cone in  $V$ . Thus,  $P = P' + \text{Ker } A$  becomes  $P = Q' + C' + \text{Ker } A$ . Since  $Q'$  is a polytope, while  $C' + \text{Ker } A$  is a finitely generated cone (in fact, if we write  $C'$  as cone  $\{c_1, c_2, \dots, c_i\}$  for some points  $c_1, c_2, \dots, c_i$ , and if we let  $(w_1, w_2, \dots, w_j)$  be a basis of the  $\mathbb{R}$ -vector space  $\text{Ker } A$ , then  $C' + \text{Ker } A = \text{cone} \{c_1, c_2, \dots, c_i, w_1, w_2, \dots, w_j, -w_1, -w_2, \dots, -w_j\}$ ), this shows that  $P$  can be written in the form  $Q + C$  with  $Q$  being a polytope and  $C$  being a finitely generated cone, qed.. In other words, the case when  $\text{Ker } A \neq 0$  has been reduced to a smaller case.

*Case 2:* The number  $\mu_0$  exists, but the number  $\nu_0$  doesn't. In this case, since  $\nu_0$  doesn't exist, there exist arbitrarily large  $\nu \in \mathbb{R}$  satisfying  $z - \nu c \in P$ . In other words, there exist arbitrarily large  $\nu \in \mathbb{R}$  satisfying  $A(z - \nu c) \leq b$ . This quickly yields that  $Ac \geq 0$ , hence  $A(-c) \leq 0$ , and thus  $-c \in \{x \in \mathbb{R}^n \mid Ax \leq 0\}$ , so that  $-\mu_0 c \in \{x \in \mathbb{R}^n \mid Ax \leq 0\}$  (since  $\mu_0 \geq 0$ ). Now, let  $x := z + \mu_0 c$ . Then,  $\text{rank}(A_x) > \text{rank}(A_z)$  (this is proven as in the proof of Theorem 2.3), thus  $x \in \text{conv.hull}\{x_1, \dots, x_t\} + \{x \in \mathbb{R}^n \mid Ax \leq 0\}$  (by the induction hypothesis), and thus

$$\begin{aligned} x + (-\mu_0 c) &\in \text{conv.hull}\{x_1, \dots, x_t\} + \{x \in \mathbb{R}^n \mid Ax \leq 0\} + \underbrace{(-\mu_0 c)}_{\in \{x \in \mathbb{R}^n \mid Ax \leq 0\}} \\ &\subseteq \text{conv.hull}\{x_1, \dots, x_t\} + \underbrace{\{x \in \mathbb{R}^n \mid Ax \leq 0\} + \{x \in \mathbb{R}^n \mid Ax \leq 0\}}_{\subseteq \{x \in \mathbb{R}^n \mid Ax \leq 0\}} \\ &\subseteq \text{conv.hull}\{x_1, \dots, x_t\} + \{x \in \mathbb{R}^n \mid Ax \leq 0\}. \end{aligned}$$

Since  $x + (-\mu_0 c) = x - \mu_0 c = z$  (since  $x = z + \mu_0 c$ ), this rewrites as

$$z \in \text{conv.hull}\{x_1, \dots, x_t\} + \{x \in \mathbb{R}^n \mid Ax \leq 0\},$$

and this completes our induction step.

*Case 3:* The number  $\nu_0$  exists, but the number  $\mu_0$  doesn't. This case is similar to Case 2.

*Case 4:* Neither of the numbers  $\mu_0$  and  $\nu_0$  exists. In this case, since  $\nu_0$  doesn't exist, there exist arbitrarily large  $\nu \in \mathbb{R}$  satisfying  $z - \nu c \in P$ . In other words, there exist arbitrarily large  $\nu \in \mathbb{R}$  satisfying  $A(z - \nu c) \leq b$ . This quickly yields that  $Ac \geq 0$ . Similarly, the non-existence of  $\mu_0$  yields  $Ac \leq 0$ . From  $Ac \geq 0$  and  $Ac \leq 0$ , it follows that  $Ac = 0$ , thus  $c \in \text{Ker } A = 0$ , contradicting  $c \neq 0$ . Hence, Case 4 cannot occur.

Altogether, in each case (except of Case 4 which cannot occur), we have shown that  $z \in \text{conv.hull}\{x_1, \dots, x_t\} + \{x \in \mathbb{R}^n \mid Ax \leq 0\}$ , so the induction is complete. The  $\implies$  part of Exercise 2.12 is thus proven.

$\Leftarrow$ : Let  $P$  be a subset of  $\mathbb{R}^n$  which can be written in the form  $P = Q + C$  with  $Q$  being a polytope and  $C$  being a finitely generated cone. We want to prove that  $P$  is a polyhedron.

Write  $Q$  in the form  $Q = \text{conv.hull}\{q_1, q_2, \dots, q_i\}$ . Write  $C$  in the form  $C = \text{cone}\{c_1, c_2, \dots, c_j\}$ . For any subset  $S$  of  $\mathbb{R}^n$  and any  $\lambda \in \mathbb{R}$ , let  $\begin{pmatrix} S \\ \lambda \end{pmatrix}$  denote the subset  $\left\{ \begin{pmatrix} s \\ \lambda \end{pmatrix} \in \mathbb{R}^{n+1} \mid s \in S \right\}$  of  $\mathbb{R}^{n+1}$ . Note that  $\begin{pmatrix} Q \\ 1 \end{pmatrix}$  is a convex set and  $\begin{pmatrix} C \\ 0 \end{pmatrix}$  is a convex cone. It is now easy to see that

$$\text{cone} \begin{pmatrix} Q \\ 1 \end{pmatrix} + \begin{pmatrix} C \\ 0 \end{pmatrix} = \text{cone} \left\{ \begin{pmatrix} q_1 \\ 1 \end{pmatrix}, \begin{pmatrix} q_2 \\ 1 \end{pmatrix}, \dots, \begin{pmatrix} q_i \\ 1 \end{pmatrix}, \begin{pmatrix} c_1 \\ 0 \end{pmatrix}, \begin{pmatrix} c_2 \\ 0 \end{pmatrix}, \dots, \begin{pmatrix} c_j \\ 0 \end{pmatrix} \right\}.$$

<sup>5</sup> Thus,  $\text{cone} \begin{pmatrix} Q \\ 1 \end{pmatrix} + \begin{pmatrix} C \\ 0 \end{pmatrix}$  is a finitely generated cone, hence (by Exercise 2.11) an intersection of finitely many linear halfspaces. In particular, this yields that  $\text{cone} \begin{pmatrix} Q \\ 1 \end{pmatrix} + \begin{pmatrix} C \\ 0 \end{pmatrix}$  is a polyhedron. Thus,

$$\left\{ x \in \text{cone} \begin{pmatrix} Q \\ 1 \end{pmatrix} + \begin{pmatrix} C \\ 0 \end{pmatrix} \mid (\text{the } (n+1)\text{-th coordinate of } x \text{ is } 1) \right\}$$

is also a polyhedron (because it is the intersection of the polyhedron  $\text{cone} \begin{pmatrix} Q \\ 1 \end{pmatrix} + \begin{pmatrix} C \\ 0 \end{pmatrix}$  with an affine hyperplane). But since it is easy to see that

$$\left\{ x \in \text{cone} \begin{pmatrix} Q \\ 1 \end{pmatrix} + \begin{pmatrix} C \\ 0 \end{pmatrix} \mid (\text{the } (n+1)\text{-th coordinate of } x \text{ is } 1) \right\} = \begin{pmatrix} Q+C \\ 1 \end{pmatrix},$$

this yields that  $\begin{pmatrix} Q+C \\ 1 \end{pmatrix}$  is a polyhedron. In other words,  $Q+C$  is a polyhedron. Since  $Q+C=P$ , this yields that  $P$  is a polyhedron. This solves the  $\Leftarrow$  direction of Exercise 2.12.

This was a nontrivial proof, and your hint helped me only in the  $\Leftarrow$  direction, and even that wasn't a simple application of the hint (since  $\text{cone} \begin{pmatrix} Q \\ 1 \end{pmatrix} + \begin{pmatrix} C \\ 0 \end{pmatrix}$  is not the same as your cone  $X$ ). Did I misunderstand something?

- **Exercise 2.13:** You could add (for the sake of clarity) a remark that if  $C$  is a convex cone, then  $C^* = \{y \in \mathbb{R}^n \mid x^T y \leq 0 \text{ for each } x \in C\}$ .
- **Exercises to §2.2:** I am somewhat surprised why you don't have this exercise:
  - 2.14a.** Prove that:
    - (i) If  $P$  is a polyhedron in  $\mathbb{R}^n$  and  $\phi$  is a linear map  $\mathbb{R}^n \rightarrow \mathbb{R}^m$ , then  $\phi(P)$  is a polyhedron.
    - (ii) If  $P$  is a polytope in  $\mathbb{R}^n$  and  $\phi$  is a linear map  $\mathbb{R}^n \rightarrow \mathbb{R}^m$ , then  $\phi(P)$  is a polytope.
    - (iii) If  $P$  is a polyhedron in  $\mathbb{R}^n$  and  $\phi$  is a linear map  $\mathbb{R}^m \rightarrow \mathbb{R}^n$ , then  $\phi^{-1}(P)$  is a polyhedron.
    - (iv) If  $P$  is a polyhedron in  $\mathbb{R}^n$ , and  $Q$  is a polyhedron in  $\mathbb{R}^m$ , then  $P \times Q$  is a polyhedron in  $\mathbb{R}^n \times \mathbb{R}^m \cong \mathbb{R}^{n+m}$ .
    - (v) If  $P$  is a polytope in  $\mathbb{R}^n$ , and  $Q$  is a polytope in  $\mathbb{R}^m$ , then  $P \times Q$  is a polytope in  $\mathbb{R}^n \times \mathbb{R}^m \cong \mathbb{R}^{n+m}$ .
    - (vi) If  $P$  and  $Q$  are two polyhedra in  $\mathbb{R}^n$ , then  $P+Q$  and  $P \cap Q$  are polyhedra.

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<sup>5</sup>To make this completely correct, we have to modify the definition of a cone so as to require that any cone contains 0. The only difference that this makes is that  $\emptyset$  no longer becomes a cone with this modified definition, so that  $\text{cone } \emptyset$  is no longer  $\emptyset$  but rather 0.

(vii) If  $P$  and  $Q$  are two polytopes in  $\mathbb{R}^n$ , then  $P + Q$  and  $P \cap Q$  are polytopes.

(viii) If  $P$  and  $Q$  are two polyhedra in  $\mathbb{R}^n$  and  $\mathbb{R}^m$ , respectively, then  $\{f \in \text{Hom}_{\mathbb{R}}(\mathbb{R}^n, \mathbb{R}^m) \mid f(P) \subseteq Q\}$  is a polyhedron in  $\text{Hom}_{\mathbb{R}}(\mathbb{R}^n, \mathbb{R}^m) \cong \mathbb{R}^{n \times m}$ .

(ix) If  $P$  and  $Q$  are two polytopes in  $\mathbb{R}^n$  and  $\mathbb{R}^m$ , respectively, then  $\text{conv. hull} \{p \otimes q \mid p \in P, q \in Q\}$  is a polytope in  $\mathbb{R}^n \otimes_{\mathbb{R}} \mathbb{R}^m \cong \mathbb{R}^{n \times m}$ .

(This exercise can be easily solved using Exercise 2.12. Note that the analogue of part (ix) for polyhedra doesn't directly hold – see <http://mathoverflow.net/questions/115677> .)

- **Proof of Corollary 2.5b:** Replace “system of linear inequalities” by “system of linear equations”.

- **Proof of Corollary 2.5b:** You write:

“According to Farkas’ lemma this implies that there exists a vector  $\begin{pmatrix} z \\ \mu \end{pmatrix}$  so that

$$\begin{pmatrix} A & b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} z \\ \mu \end{pmatrix} \geq \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \text{and} \quad (c^T \quad \delta) \begin{pmatrix} z \\ \mu \end{pmatrix} < 0.$$

”

In my opinion, you are going a bit too fast here.<sup>6</sup>

- **Geometric interpretation of the Duality theorem:** On page 35, you write: “Without loss of generality we may assume that the first  $k$  rows of  $A$  belong to the matrix  $A_{x^*}$ .” You are not very precise here. What you really want to assume is that the first  $k$  rows of  $A$ , **and no others**, belong to the matrix  $A_{x^*}$ .

- **Proof of Corollary 2.6a:** There is an alternative, simpler proof of Corollary 2.6a, which proceeds by applying Theorem 2.6 to  $n, m, -A^T, -c$  and  $-b$  instead of  $m, n, A, b$  and  $c$ . I am absolutely not suggesting that you should replace your proof of Corollary 2.6a by this one, because your proof shows a deeper idea that

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<sup>6</sup>Here is how I would argue (immediately after (38)):

“In other words, the following system of linear equations

$$\begin{pmatrix} A & b \\ 0 & 1 \end{pmatrix}^T \begin{pmatrix} y \\ \lambda \end{pmatrix} = \begin{pmatrix} c \\ \delta \end{pmatrix}$$

in the variables  $y$  and  $\lambda$  has no nonnegative solution  $\begin{pmatrix} y \\ \lambda \end{pmatrix}$ . According to Theorem 2.5, this implies

that there exists a vector  $\begin{pmatrix} z \\ \mu \end{pmatrix}$  such that

$$\begin{pmatrix} z \\ \mu \end{pmatrix}^T \begin{pmatrix} A & b \\ 0 & 1 \end{pmatrix}^T \geq \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} z \\ \mu \end{pmatrix}^T \begin{pmatrix} c \\ \delta \end{pmatrix} < 0.$$

Consider this vector. Then,  $\begin{pmatrix} z \\ \mu \end{pmatrix}^T \begin{pmatrix} A & b \\ 0 & 1 \end{pmatrix}^T \geq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$  rewrites as  $\begin{pmatrix} A & b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} z \\ \mu \end{pmatrix} \geq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ ,

so that  $\mu \geq 0$  and  $Az + b\mu \geq 0$ . Also,  $\begin{pmatrix} z \\ \mu \end{pmatrix}^T \begin{pmatrix} c \\ \delta \end{pmatrix} < 0$  rewrites as  $(c^T \quad \delta) \begin{pmatrix} z \\ \mu \end{pmatrix} < 0$ .

is used in solving the exercises, whereas my proof only shows the “idea” that min and max can be interchanged at the cost of introducing some minus signs. But I wouldn’t consider it a bad thing if you would demonstrate the idea that you use for proving Corollary 2.6a on Exercise 2.23 rather than leaving it as an exercise for the reader. In fact, it seems to me that most of the time when you apply linear programming duality to combinatorial proofs in your text, you are not using Theorem 2.6, but rather you are using Exercise 2.23, so it seems that Exercise 2.23 is the more relevant form of linear programming duality for combinatorics, and deserves a better place than an exercise. But this is a subjective opinion of mine.

## Chapter 4

- **Proof of Theorem 4.1:** After “there exists for each  $c \in U \cup \{v\}$  a  $c - T$  path”, add “ $Q_c$ ”.
- **Proof of Theorem 4.1:** Replace “pairwise disjoint” by “pairwise vertex-disjoint”.
- **§4.1:** You write: “They can be derived from the directed case by replacing each undirected edge  $uw$  by two opposite arcs  $(u, w)$  and  $(w, u)$ .”

This is not literally true – at least not for the directed arc-disjoint version of Menger’s theorem (Corollary 4.1b). Indeed, the analogue of Corollary 4.1b for undirected graphs does not directly from Corollary 4.1b. If  $G$  is an undirected graph, and if  $D$  is the directed graph obtained from  $G$  by replacing each undirected edge  $uw$  by two opposite arcs  $(u, w)$  and  $(w, u)$ , then  $k$  arc-disjoint paths in  $D$  **need not** directly “project down to”  $k$  edge-disjoint paths in  $G$ , because it could happen that one of these paths uses an arc  $(u, w)$  while another uses the opposite arc  $(w, u)$ . There might well be a fix for this problem, but it is not obvious to me.

- **§4.3:** Here you write: “Moreover, for the sake of exposition we assume that for no arc  $(u, v)$ , also  $(v, u)$  is an arc”.

I understand why you make this assumption: It is needed for the flow augmenting algorithm<sup>7</sup>. However, let me point out an alternative way to make the flow augmenting algorithm work, without making this assumption:

Consider Case 1 in the flow augmenting algorithm.

For each  $i \in \{1, 2, \dots, k\}$ , we color the arc  $a_i \in A_f$  either in red or in blue (not in both colors simultaneously), in such a way that the following holds:

- If  $a_i$  is colored red, then  $a_i \in A$  and  $c(a_i) - f(a_i) > 0$  holds.
- If  $a_i$  is colored blue, then  $a_i^{-1} \in A$  and  $f(a_i^{-1}) > 0$  holds.

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<sup>7</sup>Indeed, if you do not make this assumption, then it can happen that a single  $i \in \{1, 2, \dots, k\}$  satisfies **both** (14) (i) and (14) (ii) at the same time; in this case, the definition of the  $\sigma_1, \sigma_2, \dots, \sigma_k$  in (14) becomes ambiguous (because it gives two different (non-equivalent) definitions of  $\sigma_i$ ), and the definition (15) of the map  $f' : A \rightarrow \mathbb{R}_+$  fails to guarantee that  $f'$  satisfies the flow conservation law.

(Such a coloring clearly exists, because for each  $i \in \{1, 2, \dots, k\}$ , at least one of the statements  $(a_i \in A \text{ and } c(a_i) - f(a_i) > 0)$  and  $(a_i^{-1} \in A \text{ and } f(a_i^{-1}) > 0)$  holds<sup>8</sup>. If only one of them holds, then the color of  $a_i$  is thus uniquely determined; but if both of them hold, it can be chosen at will. But we must choose it once and consider it as fixed from then on.)

Now, the definition of the numbers  $\sigma_1, \sigma_2, \dots, \sigma_k$  has to be modified as follows:

$$\begin{aligned} \sigma_i &= \begin{cases} c(a_i) - f(a_i), & \text{if } a_i \text{ is colored red;} \\ f(a_i^{-1}), & \text{if } a_i \text{ is colored blue} \end{cases} \\ &= \begin{cases} \text{the } \sigma_i \text{ from (14) (i),} & \text{if } a_i \text{ is colored red;} \\ \text{the } \sigma_i \text{ from (14) (ii),} & \text{if } a_i \text{ is colored blue.} \end{cases} \end{aligned}$$

Furthermore, the definition of  $f' : A \rightarrow \mathbb{R}_+$  has to be modified as follows:

$$f'(a) := \begin{cases} f(a) + \alpha, & \text{if } a \text{ is an arc of } P \text{ colored red;} \\ f(a) - \alpha, & \text{if } a^{-1} \text{ is an arc of } P \text{ colored blue;} \\ f(a), & \text{otherwise.} \end{cases}$$

for each  $a \in A$ .

It is not hard to see that this  $f'$  is indeed an  $s - t$  flow under  $c$ .

This modification of the flow augmenting algorithm makes your assumption that “for no arc  $(u, v)$ , also  $(v, u)$  is an arc” unnecessary in §4.3. (However, this assumption remains necessary for §4.6, where it is used (for example) in the formula (35).)

- **Proof of Theorem 4.4:** There are some steps missing here (although not too hard to fill in). You are claiming without proof that  $|\alpha(D_f)| \leq |A|$  for each  $f$ . In order to prove this, we need the following lemma:

**Lemma 4.4b.** Let  $D = (V, A)$  and  $s, t \in V$ . Then,  $\alpha(D) \cap \alpha(D)^{-1} = \emptyset$ .

This lemma is easy to check (you have essentially proven it inside the proof of Proposition 3), but I think it should be stated explicitly. From this lemma (applied to  $D_f$  instead of  $D$ ), we can conclude that  $\alpha(D_f)$  is a subset of  $A \cup A^{-1}$  having at most half the size of  $A \cup A^{-1}$  (because it can never contain both  $a$  and  $a^{-1}$  for any given  $a \in A$ ); therefore its size is  $|\alpha(D_f)| \leq \frac{1}{2}|A \cup A^{-1}| = |A| - \frac{1}{2}|A \cap A'| \leq |A|$ .

## Chapter 5

- **Corollary 5.6c:** The formula (31) would become clearer if you would replace “ $U \subseteq V$ ” by “ $U \in \mathcal{P}_{\text{odd}}(V)$ ”.

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<sup>8</sup>This is because  $a_i \in A_f$ .

## Chapter 8

- **§8.1:** On page 133, replace “*LP*-problem” by “LP-problem” (there is no reason to put “LP” in math mode here).
- **Exercise 8.2:** This is correct, but I am wondering whether the problem is as difficult as it seems to me. Below is an outline of my solution, which you probably should ignore due to its exorbitant length and ugliness; I have merely written it up to convince myself that it is true (but I cannot say I am very much convinced). Exercise 8.2 is surrounded by some of the easiest exercises I have seen in your notes, including the Exercise 8.1 which is completely trivial (all that one needs to do in that exercise is noticing that  $P$  is bounded and thus  $P \cap \mathbb{Z}^n$  is finite, right?), so I am very surprised that Exercise 8.2 did not give in to most of my attacks.

Before I sketch my solution, let me add a new exercise for Chapter 2:

**2.6a.** Let  $n \in \mathbb{N}$ . Let  $A$  and  $B$  be two subsets of  $\mathbb{R}^n$  such that  $A + B \subseteq A$ . Prove that

$$\text{conv} . \text{hull } A + \text{cone } B = \text{conv} . \text{hull } A.$$

(I’m not claiming this exercise is interesting for its own sake, but I’ll use it in my solution.) The solution to Exercise 2.6a is really straightforward.

*Solution to Exercise 8.2 (sketched):*

The matrix  $A$  is rational. Thus, we can WLOG assume that  $A$  is integral (since otherwise, we can multiply both  $A$  and  $b$  with the (positive) common denominator of the fractions which occur in  $A$ ). Assume this.

We can WLOG assume that  $b$  is an integer (else, just replace  $b$  by  $\lfloor b \rfloor$ , and nothing changes). Assume this.

We can rewrite the two directions of Exercise 2.12 as two lemmas:

*Lemma U1:* Let  $\mathfrak{P}$  be a polyhedron in  $\mathbb{R}^n$ . Then, there exist a finite subset  $Q$  of  $\mathbb{R}^n$ , an integer  $k \in \mathbb{N}$  and some vectors  $c_1, c_2, \dots, c_k$  in  $\mathbb{R}^n$  such that  $\mathfrak{P} = \text{conv} . \text{hull } Q + \text{cone } \{c_1, c_2, \dots, c_k\}$ .

*Lemma U2:* Let  $\mathfrak{Q}$  be a polytope in  $\mathbb{R}^n$ . Let  $\mathfrak{C}$  be a finitely generated convex cone in  $\mathbb{R}^n$ . Then,  $\mathfrak{Q} + \mathfrak{C}$  is a polyhedron in  $\mathbb{R}^n$ .

Now, let us define a couple of notions related to rational numbers.

(a) A subset  $H$  of  $\mathbb{Q}^n$  will be called a  $\mathbb{Q}$ -halfspace if there exist a vector  $c \in \mathbb{Q}^n$  with  $c \neq 0$  and a  $\delta \in \mathbb{Q}$  such that  $H = \{x \in \mathbb{Q}^n \mid c^T x \leq \delta\}$ .

(b) A  $\mathbb{Q}$ -polyhedron in  $\mathbb{Q}^n$  will mean an intersection of finitely many  $\mathbb{Q}$ -halfspaces in  $\mathbb{Q}^n$ .

(c) A subset  $C$  of  $\mathbb{Q}^n$  is called  $\mathbb{Q}$ -convex if for all  $x$  and  $y$  in  $C$  and any  $\lambda \in \mathbb{Q}$  satisfying  $0 \leq \lambda \leq 1$ , the vector  $\lambda x + (1 - \lambda)y$  belongs to  $C$ .

(d) If  $X$  is a subset of  $\mathbb{Q}^n$ , then the  $\mathbb{Q}$ -convex hull of  $X$  will mean the subset

$$\{\lambda_1 x_1 + \lambda_2 x_2 + \dots + \lambda_t x_t \mid t \in \mathbb{N}, x_1, x_2, \dots, x_t \in X; \lambda_1, \lambda_2, \dots, \lambda_t \in \mathbb{Q}; \lambda_1, \lambda_2, \dots, \lambda_t \geq 0; \lambda_1 + \lambda_2 + \dots + \lambda_t = 1\}$$

of  $\mathbb{Q}^n$ . We denote this  $\mathbb{Q}$ -convex hull by  $\text{conv} . \text{hull}_{\mathbb{Q}} X$ . It is the smallest  $\mathbb{Q}$ -convex set which contains  $X$  as a subset.

(e) A subset  $C$  of  $\mathbb{Q}^n$  is called a  $\mathbb{Q}$ -convex cone if it satisfies  $0 \in C$  and for any  $x \in C$  and  $y \in C$  and any  $\lambda \in \mathbb{Q}$  and  $\mu \in \mathbb{Q}$  such that  $\lambda \geq 0$  and  $\mu \geq 0$ , one has  $\lambda x + \mu y \in C$ .

(f) For any  $X \subseteq \mathbb{Q}^n$ , we define  $\text{cone}_{\mathbb{Q}} X$  to be the subset

$$\{\lambda_1 x_1 + \lambda_2 x_2 + \dots + \lambda_t x_t \mid t \in \mathbb{N}, x_1, x_2, \dots, x_t \in X; \lambda_1, \lambda_2, \dots, \lambda_t \in \mathbb{Q}; \lambda_1, \lambda_2, \dots, \lambda_t \geq 0\}$$

of  $\mathbb{Q}^n$ . It is the smallest  $\mathbb{Q}$ -convex cone which contains  $X$  as a subset.

It is clear that these notions of a  $\mathbb{Q}$ -halfspace, a  $\mathbb{Q}$ -polyhedron, the  $\mathbb{Q}$ -convex hull  $\text{conv} . \text{hull}_{\mathbb{Q}} X$  etc. are defined exactly in the same way as you defined the notions of a halfspace, a polyhedron, the convex hull  $\text{conv} . \text{hull} X$  etc. (except for minor differences in the wording of the definitions) but using  $\mathbb{Q}$  as the ground field instead of  $\mathbb{R}$ .

You spent most of Chapter 2 discussing linear programming and polyhedra over the field  $\mathbb{R}$ . However, most results from Chapter 2 (more precisely, Theorem 2.2, Theorem 2.3, Corollary 2.3a, Theorem 2.4, Theorem 2.5, Corollary 2.5a, Corollary 2.5b, Lemma 2.1, Theorem 2.6 and Corollary 2.6a, as well as most exercises) still hold when the underlying field  $\mathbb{R}$  is replaced by  $\mathbb{Q}$  (or any other ordered field), as long as the notions used therein (e. g., the notions of convexity, polytopes, polyhedra, hyperplanes, etc.) are replaced by the corresponding notions which are based on  $\mathbb{Q}$  instead of  $\mathbb{R}$  as the ground field (for example, the notion of a halfspace must be replaced by the notion of a  $\mathbb{Q}$ -halfspace, the notion of  $\text{conv} . \text{hull} X$  for a set  $X \subseteq \mathbb{R}^n$  must be replaced by the notion of  $\text{conv} . \text{hull}_{\mathbb{Q}} X$  for a set  $X \subseteq \mathbb{Q}^n$ , etc.).<sup>9</sup> In particular, Lemma U1 (being a consequence of Exercise 2.12) still holds when the underlying field  $\mathbb{R}$  is replaced by  $\mathbb{Q}$ , as long as the appropriate replacements are done. In other words, the following lemma holds:

*Lemma U3:* Let  $\mathfrak{P}$  be a  $\mathbb{Q}$ -polyhedron in  $\mathbb{Q}^n$ . Then, there exist a finite subset  $Q$  of  $\mathbb{Q}^n$ , an integer  $k \in \mathbb{N}$  and some vectors  $c_1, c_2, \dots, c_k$  in  $\mathbb{Q}^n$  such that  $\mathfrak{P} = \text{conv} . \text{hull}_{\mathbb{Q}} Q + \text{cone}_{\mathbb{Q}} \{c_1, c_2, \dots, c_k\}$ .

But now, let us get back to the solution of Exercise 8.2.

We know that  $A$  is integral and  $b$  is an integer. Thus,

$$\underbrace{P}_{=\{x \mid Ax \leq b\}} \cap \mathbb{Q}^n = \{x \mid Ax \leq b\} \cap \mathbb{Q}^n = \{x \in \mathbb{Q}^n \mid Ax \leq b\}$$

is a  $\mathbb{Q}$ -polyhedron in  $\mathbb{Q}^n$ . Denote this  $\mathbb{Q}$ -polyhedron by  $P'$ . Clearly,

$$\underbrace{P'}_{=P \cap \mathbb{Q}^n} \cap \mathbb{Z}^n = P \cap \underbrace{\mathbb{Q}^n \cap \mathbb{Z}^n}_{=\mathbb{Z}^n} = P \cap \mathbb{Z}^n.$$

Applied to  $\mathfrak{P} = P'$ , Lemma U3 yields that there exist a finite subset  $Q$  of  $\mathbb{Q}^n$ , an integer  $k \in \mathbb{N}$  and some vectors  $c_1, c_2, \dots, c_k$  in  $\mathbb{Q}^n$  such that  $P' = \text{conv} . \text{hull}_{\mathbb{Q}} Q +$

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<sup>9</sup>That said, the proofs you give don't necessarily work over  $\mathbb{Q}$ ...

$\text{cone}_{\mathbb{Q}} \{c_1, c_2, \dots, c_k\}$ . Consider this  $Q$ , this  $k$  and these  $c_1, c_2, \dots, c_k$ . WLOG assume that  $c_1, c_2, \dots, c_k$  belong to  $\mathbb{Z}^n$  (because otherwise, we can multiply the vectors  $c_1, c_2, \dots, c_k$  by their common denominator).

Let  $\mathfrak{C} = \text{cone}_{\mathbb{Q}} \{c_1, c_2, \dots, c_k\}$ . Then,  $\mathfrak{C}$  is a finitely generated  $\mathbb{Q}$ -convex cone.

Let  $\mathfrak{Q} = \text{conv. hull}_{\mathbb{Q}} Q$ . Then,  $\mathfrak{Q}$  is a  $\mathbb{Q}$ -polytope.

We have

$$P' = \underbrace{\text{conv. hull}_{\mathbb{Q}} Q}_{=\mathfrak{Q}} + \underbrace{\text{cone}_{\mathbb{Q}} \{c_1, c_2, \dots, c_k\}}_{=\mathfrak{C}} = \mathfrak{Q} + \mathfrak{C}.$$

Let  $\mathfrak{K}$  be the subset  $\sum_{i=1}^k \text{conv. hull}_{\mathbb{Q}} \{0, c_i\}$  of  $\mathbb{Q}^n$  (where the sum sign is based on

the Minkowski sum of subsets of  $\mathbb{Q}^n$ ). Then,  $\mathfrak{Q} + \mathfrak{K} = \mathfrak{Q} + \sum_{i=1}^k \text{conv. hull}_{\mathbb{Q}} \{0, c_i\}$  is a sum of finitely many  $\mathbb{Q}$ -polytopes, hence  $\mathfrak{Q} + \mathfrak{K}$  itself is a bounded set. Thus,  $(\mathfrak{Q} + \mathfrak{K}) \cap \mathbb{Z}^n$  is a finite set.

From the definitions of  $\mathfrak{K}$  and  $\mathfrak{C}$ , it follows readily that  $\mathfrak{K} \subseteq \mathfrak{C}$ , so that

$$\mathfrak{Q} + \underbrace{\mathfrak{K}}_{\subseteq \mathfrak{C}} \subseteq \mathfrak{Q} + \mathfrak{C} = P'. \quad (3)$$

We will now prove that

$$\text{conv. hull}(P' \cap \mathbb{Z}^n) = \text{conv. hull}((\mathfrak{Q} + \mathfrak{K}) \cap \mathbb{Z}^n) + \text{cone} \{c_1, c_2, \dots, c_k\}. \quad (4)$$

(Note that, in this equation, we are again working over the ground field  $\mathbb{R}$ , not over  $\mathbb{Q}$ , even if  $\mathfrak{Q}$  and  $\mathfrak{K}$  are defined over  $\mathbb{Q}$ .)

*Proof of (4):* Let  $x \in P' \cap \mathbb{Z}^n$  be arbitrary. Then,  $x \in P'$  and  $x \in \mathbb{Z}^n$ . Since  $x \in P' = \mathfrak{Q} + \mathfrak{C}$ , there exist two elements  $q \in \mathfrak{Q}$  and  $c \in \mathfrak{C}$  such that  $x = q + c$ . Consider these  $q$  and  $c$ .

Since  $c \in \mathfrak{C} = \text{cone}_{\mathbb{Q}} \{c_1, c_2, \dots, c_k\}$ , there exists a  $k$ -tuple  $(\lambda_1, \lambda_2, \dots, \lambda_k)$  of nonnegative elements of  $\mathbb{Q}$  such that  $c = \sum_{i=1}^k \lambda_i c_i$ . Consider this  $k$ -tuple  $(\lambda_1, \lambda_2, \dots, \lambda_k)$ .

We have

$$\begin{aligned} c &= \sum_{i=1}^k \underbrace{\lambda_i}_{=[\lambda_i] + (\lambda_i - [\lambda_i])} c_i = \sum_{i=1}^k ([\lambda_i] + (\lambda_i - [\lambda_i])) c_i \\ &= \sum_{i=1}^k [\lambda_i] c_i + \sum_{i=1}^k \underbrace{(\lambda_i - [\lambda_i]) c_i}_{\substack{\in \text{conv. hull}_{\mathbb{Q}} \{0, c_i\} \\ (\text{since } \lambda_i - [\lambda_i] \in [0, 1])}} \\ &\in \sum_{i=1}^k [\lambda_i] c_i + \underbrace{\sum_{i=1}^k \text{conv. hull}_{\mathbb{Q}} \{0, c_i\}}_{=\mathfrak{K}} = \sum_{i=1}^k [\lambda_i] c_i + \mathfrak{K}, \end{aligned}$$

so that

$$x = \underbrace{q}_{\in \mathfrak{Q}} + \underbrace{c}_{\in \sum_{i=1}^k [\lambda_i] c_i + \mathfrak{K}} \in \mathfrak{Q} + \sum_{i=1}^k [\lambda_i] c_i + \mathfrak{K} = \sum_{i=1}^k [\lambda_i] c_i + \mathfrak{Q} + \mathfrak{K}.$$

Hence,

$$x - \sum_{i=1}^k [\lambda_i] c_i \in \mathfrak{Q} + \mathfrak{K}.$$

Combined with

$$\underbrace{x}_{\in \mathbb{Z}^n} - \sum_{i=1}^k \underbrace{[\lambda_i]}_{\in \mathbb{Z}} \underbrace{c_i}_{\in \mathbb{Z}^n} \in \mathbb{Z}^n - \sum_{i=1}^k \mathbb{Z} \cdot \mathbb{Z}^n \subseteq \mathbb{Z}^n,$$

this yields

$$x - \sum_{i=1}^k [\lambda_i] c_i \in (\mathfrak{Q} + \mathfrak{K}) \cap \mathbb{Z}^n \subseteq \text{conv} . \text{hull} ((\mathfrak{Q} + \mathfrak{K}) \cap \mathbb{Z}^n).$$

But every  $i \in \{1, 2, \dots, k\}$  satisfies  $\lambda_i \geq 0$  and thus  $[\lambda_i] \geq 0$ . Hence,

$$\sum_{i=1}^k [\lambda_i] c_i \in \text{cone} \{c_1, c_2, \dots, c_k\}.$$

Now,

$$\begin{aligned} x &= \underbrace{\left( x - \sum_{i=1}^k [\lambda_i] c_i \right)}_{\in \text{conv} . \text{hull} ((\mathfrak{Q} + \mathfrak{K}) \cap \mathbb{Z}^n)} + \underbrace{\sum_{i=1}^k [\lambda_i] c_i}_{\in \text{cone} \{c_1, c_2, \dots, c_k\}} \\ &\in \text{conv} . \text{hull} ((\mathfrak{Q} + \mathfrak{K}) \cap \mathbb{Z}^n) + \text{cone} \{c_1, c_2, \dots, c_k\}. \end{aligned}$$

Now forget that we fixed  $x$ . We thus have proven that every  $x \in P' \cap \mathbb{Z}^n$  satisfies  $x \in \text{conv} . \text{hull} ((\mathfrak{Q} + \mathfrak{K}) \cap \mathbb{Z}^n) + \text{cone} \{c_1, c_2, \dots, c_k\}$ . In other words,

$$P' \cap \mathbb{Z}^n \subseteq \text{conv} . \text{hull} ((\mathfrak{Q} + \mathfrak{K}) \cap \mathbb{Z}^n) + \text{cone} \{c_1, c_2, \dots, c_k\}. \quad (5)$$

Since set  $\text{conv} . \text{hull} ((\mathfrak{Q} + \mathfrak{K}) \cap \mathbb{Z}^n) + \text{cone} \{c_1, c_2, \dots, c_k\}$  is convex, this yields

$$\text{conv} . \text{hull} (P' \cap \mathbb{Z}^n) \subseteq \text{conv} . \text{hull} ((\mathfrak{Q} + \mathfrak{K}) \cap \mathbb{Z}^n) + \text{cone} \{c_1, c_2, \dots, c_k\}. \quad (6)$$

We will now prove that

$$\text{conv} . \text{hull} ((\mathfrak{Q} + \mathfrak{K}) \cap \mathbb{Z}^n) + \text{cone} \{c_1, c_2, \dots, c_k\} \subseteq \text{conv} . \text{hull} (P' \cap \mathbb{Z}^n). \quad (7)$$

Indeed, let us set  $\mathfrak{A} = P' \cap \mathbb{Z}^n$  and  $\mathfrak{B} = \{c_1, c_2, \dots, c_k\}$ . Then, it is very easy to see that  $\mathfrak{A} + \mathfrak{B} \subseteq \mathfrak{A}$ <sup>10</sup>. Thus, Exercise 2.6a (applied to  $\mathfrak{A}$  and  $\mathfrak{B}$  instead of  $A$  and  $B$ ) yields that

$$\text{conv} \cdot \text{hull } \mathfrak{A} + \text{cone } \mathfrak{B} = \text{conv} \cdot \text{hull } \mathfrak{A}.$$

Now,

$$\begin{aligned} & \text{conv} \cdot \text{hull} \left( \underbrace{(\mathfrak{Q} + \mathfrak{K}) \cap \mathbb{Z}^n}_{\substack{\subseteq P' \\ \text{(by (3))}}} \right) + \text{cone} \underbrace{\{c_1, c_2, \dots, c_k\}}_{=\mathfrak{B}} \\ & \subseteq \text{conv} \cdot \text{hull} \underbrace{(P' \cap \mathbb{Z}^n)}_{=\mathfrak{A}} + \text{cone } \mathfrak{B} = \text{conv} \cdot \text{hull } \mathfrak{A} + \text{cone } \mathfrak{B} = \text{conv} \cdot \text{hull} \underbrace{\mathfrak{A}}_{=P' \cap \mathbb{Z}^n} \\ & = \text{conv} \cdot \text{hull} (P' \cap \mathbb{Z}^n). \end{aligned}$$

This proves (7).

Combining (7) with (6), we obtain

$$\text{conv} \cdot \text{hull} (P' \cap \mathbb{Z}^n) = \text{conv} \cdot \text{hull} ((\mathfrak{Q} + \mathfrak{K}) \cap \mathbb{Z}^n) + \text{cone} \{c_1, c_2, \dots, c_k\}.$$

Thus, (4) is proven.

Since  $P' \cap \mathbb{Z}^n = P \cap \mathbb{Z}^n$ , we can rewrite (4) as

$$\text{conv} \cdot \text{hull} (P \cap \mathbb{Z}^n) = \text{conv} \cdot \text{hull} ((\mathfrak{Q} + \mathfrak{K}) \cap \mathbb{Z}^n) + \text{cone} \{c_1, c_2, \dots, c_k\}. \quad (8)$$

Notice that  $\text{conv} \cdot \text{hull} ((\mathfrak{Q} + \mathfrak{K}) \cap \mathbb{Z}^n)$  is a polytope (since  $(\mathfrak{Q} + \mathfrak{K}) \cap \mathbb{Z}^n$  is a finite set) and that  $\text{cone} \{c_1, c_2, \dots, c_k\}$  is a finitely generated convex cone. Therefore, Lemma U2 (applied to  $\text{conv} \cdot \text{hull} ((\mathfrak{Q} + \mathfrak{K}) \cap \mathbb{Z}^n)$  and  $\text{cone} \{c_1, c_2, \dots, c_k\}$  instead of  $\mathfrak{Q}$  and  $\mathfrak{C}$ ) yields that  $\text{conv} \cdot \text{hull} ((\mathfrak{Q} + \mathfrak{K}) \cap \mathbb{Z}^n) + \text{cone} \{c_1, c_2, \dots, c_k\}$  is a polyhedron in  $\mathbb{R}^n$ . Since  $\text{conv} \cdot \text{hull} ((\mathfrak{Q} + \mathfrak{K}) \cap \mathbb{Z}^n) + \text{cone} \{c_1, c_2, \dots, c_k\} = \text{conv} \cdot \text{hull} (P \cap \mathbb{Z}^n)$  (by (8)), this rewrites as follows:  $\text{conv} \cdot \text{hull} (P \cap \mathbb{Z}^n)$  is a polyhedron in  $\mathbb{R}^n$ . This solves Exercise 8.2.

- **Exercise 8.5:** Replace “ $P$ ” by “ $P = \{x \mid Ax \leq b\}$ ”.
- **Exercise 8.16:** In (43), add a plus sign before “ $\lambda_m P_m$ ”.
- **Exercise 8.16:** After “A matrix  $M$ ”, add “with nonnegative entries”.

<sup>10</sup> *Proof.* Let  $\xi \in \mathfrak{A} + \mathfrak{B}$ . Then, there exist  $\alpha \in \mathfrak{A}$  and  $\beta \in \mathfrak{B}$  such that  $\xi = \alpha + \beta$ . Consider these  $\alpha$  and  $\beta$ .

Since  $\alpha \in \mathfrak{A} = P' \cap \mathbb{Z}^n \subseteq \mathbb{Z}^n$  and  $\beta \in \mathfrak{B} = \{c_1, c_2, \dots, c_k\} \subseteq \mathbb{Z}^n$  (since  $c_1, c_2, \dots, c_k$  belong to  $\mathbb{Z}^n$ ), we have  $\alpha + \beta \in \mathbb{Z}^n + \mathbb{Z}^n \subseteq \mathbb{Z}^n$ . Thus,  $\xi = \alpha + \beta \in \mathbb{Z}^n$ . On the other hand, from  $\alpha \in \mathfrak{A} = P' \cap \mathbb{Z}^n \subseteq P' = \mathfrak{Q} + \mathfrak{C}$  and  $\beta \in \mathfrak{B} = \{c_1, c_2, \dots, c_k\} \subseteq \text{cone}_{\mathbb{Q}} \{c_1, c_2, \dots, c_k\} = \mathfrak{C}$ , we obtain

$$\alpha + \beta \in (\mathfrak{Q} + \mathfrak{C}) + \mathfrak{C} = \mathfrak{Q} + \underbrace{\mathfrak{C} + \mathfrak{C}}_{\substack{\subseteq \mathfrak{C} \\ \text{(since } \mathfrak{C} \text{ is a cone)}}} \subseteq \mathfrak{Q} + \mathfrak{C} = P'.$$

Thus,  $\xi = \alpha + \beta \in P'$ . Combined with  $\xi \in \mathbb{Z}^n$ , this yields  $\xi \in P' \cap \mathbb{Z}^n = \mathfrak{A}$ .

Now, forget that we fixed  $\xi$ . We thus have proven that every  $\xi \in \mathfrak{A} + \mathfrak{B}$  satisfies  $\xi \in \mathfrak{A}$ . In other words,  $\mathfrak{A} + \mathfrak{B} \subseteq \mathfrak{A}$ , qed.

## Chapter 9

- **§9.1:** In the condition (5), it wouldn't harm to state that the demand of the arc  $(s_i, t_i)$  should be understood as  $d_i$ . (You only ever refer to the  $d_1, d_2, \dots, d_k$  as 'demands', never actually linking them to the arcs  $(s_1, t_1), (s_2, t_2), \dots, (s_k, t_k)$ .)
- **Proof of Theorem 9.1:** On page 153, you write: "for any  $x \in R^A$ ". I assume the " $R$ " should be an " $\mathbb{R}$ " here.

- **Proof of Theorem 9.1:** On page 153, you write: "This can be seen by extending the undirected graph  $G$  by adding two new vertices  $s'$  and  $t'$  and four new edges  $\{s', s_1\}, \{t_1, t'\}$  (both with capacity  $d_1$ ) and  $\{s', s_2\}, \{t_2, t'\}$  (both with capacity  $d_2$ ) as in Figure 9.3." I am not convinced that it's really an undirected graph that you want here; first of all, I don't remember you ever formulating the max-flow-min-cut theorem for undirected graphs; moreover, I do think you want the edges  $\{s', s_1\}, \{t_1, t'\}, \{s', s_2\}, \{t_2, t'\}$  to be directed. So I would rather write:

"To see this, we define the *directed double cover*  $\widehat{G}$  of  $G$  as the directed graph obtained from  $G$  by replacing every edge  $\{v, w\}$  of  $G$  by two arcs  $(v, w)$  and  $(w, v)$ . Extend this directed graph  $\widehat{G}$  by adding two new vertices  $s'$  and  $t'$  and four new arcs  $(s', s_1), (t_1, t')$  (both with capacity  $d_1$ ) and  $(s', s_2), (t_2, t')$  (both with capacity  $d_2$ ) as in Figure 9.3."

(Figure 9.3 now should have arrows on the four new arcs, and a  $\widehat{G}$  in lieu of  $G$ .)

- **Proof of Theorem 9.1:** On page 154, you write: "To see this we extend  $G$  with vertices  $s''$  and  $t''$  and edges  $\{s'', s_1\}, \{t_1, t''\}$  (both with capacity  $d_1$ ) and  $\{s'', t_2\}, \{s_2, t''\}$  (both with capacity  $d_2$ ) (cf. Figure 9.4)."

Similarly to the my previous comment on this proof, this should be replaced by:

"To see this we extend the directed graph  $\widehat{G}$  by adding two vertices  $s''$  and  $t''$  and four arcs  $(s'', s_1), (t_1, t'')$  (both with capacity  $d_1$ ) and  $(s'', t_2), (s_2, t'')$  (both with capacity  $d_2$ ) (cf. Figure 9.4)."

(Figure 9.4 also needs arrows on the four new arcs, and a  $\widehat{G}$  in lieu of  $G$ .)

- **Proof of Theorem 9.2:** On the last line of this proof, in " $x_2 = \frac{1}{2}(x' - x)$ ", the " $'$ " should be a mathmode " $'$ ".
- **Proof of Theorem 9.3:** On the second line of page 158, replace " $v_{i', j_{i'}} \neq s_{i'}$ " by " $v_{i', j_{i'}} \neq t_{i'}$ ".
- **§9.6:** On page 169, you write: "Since for each fixed  $i = 1, \dots, k$ , a solution  $x_i$  to (43) is an  $s_i - t_i$  flow, we can decompose  $x_i$  as a nonnegative combination of  $s_i - t_i$  paths." Why? (I don't think a flow can be literally decomposed into paths; what I think can be done is decomposing a  $s_i - t_i$  flow as a nonnegative combination of  $s_i - t_i$  paths plus a nonnegative circulation. I guess you're throwing out the circulation since it does not change the value of the flow nor does subtracting it break the capacity constraints.)
- **§9.6:** In (44) (i), replace " $x_j(e)$ " by " $x_i(e)$ ".

## Chapter 10

- **Definition of “basis” on page 174:** You write: “That is, for any set  $Z \in \mathcal{I}$  with  $B \subseteq Z \subseteq Y$  one has  $Z = B$ ”. Maybe it would be better to clarify this, e. g., as follows: “That is,  $B$  is a basis if and only if for any set  $Z \in \mathcal{I}$  with  $B \subseteq Z \subseteq Y$  one has  $Z = B$ ”.
- **Description of the greedy algorithm on page 174:** You write: “We stop if no  $y$  satisfying (5)(i) exist, that is, if  $\{y_1, \dots, y_k\}$  is a basis.” I would add here: “Then, we output  $\{y_1, \dots, y_k\}$ .”
- **Exercise 10.2:** You use the notion “circuit” here, but you don’t define it until §10.2.
- **§10.2:** On page 176, replace “minimal” by “minimal”.
- **§10.2:** On page 176, you write: “Finally,  $r$  arises in this way if and only if  
(9) (i)  $r(\emptyset) = 0$ ,  
(ii) if  $Z \subseteq Y \subseteq X$  then  $r(Z) \leq r(Y)$ .”  
This claim is wrong. Fortunately, you only use the “only if” direction, which is correct indeed (and trivial). The “if” direction can be easily refuted, e. g., by the counterexample  $r(Y) = \max(0, |Y| - 1)$  (where  $X$  is any finite set with at least two elements). Fortunately, if we add the condition (10) to (9)(i) and (9)(ii), **then** any  $r$  satisfying these three conditions must indeed arise by (8) from a nonempty down-monotone collection  $\mathcal{I}$  of subsets of  $X$  (and, in fact, from a matroid  $(X, \mathcal{I})$ ). But without (10), the  $\mathcal{I}$  needs not exist.
- **§10.2:** In the footnote <sup>22</sup> on page 176, it would be better to replace “no two of which are contained in each other” by “none of which is contained in another” (or else it sounds as if no two sets can be **mutually** contained in each other **at the same time**, i. e., no two sets can be equal).
- **Proof of Theorem 10.2:** In the proof of the (iii) $\implies$ (i) implication, replace “ $(B' \setminus \{y\}) \cup \{x\}$ ” by “ $(B' \setminus \{y\}) \cup \{x\} \in \mathcal{B}$ ”.
- **Proof of Theorem 10.2:** In the proof of the (iv) $\implies$ (i) implication, replace “choose  $y \in F \setminus F'$ ” by “choose  $x \in F \setminus F''$ ”.
- **Proof of Theorem 10.2:** In the proof of the (i) $\implies$ (vi) implication, replace “ $F \subseteq F' \subseteq Y \cup Z$ ” by “ $F \subseteq F' \subseteq Y \cup Z$ ”.
- **Proof of Theorem 10.2:** In the proof of the (vi) $\implies$ (i) implication, replace “ $x \in F' \setminus F \cup U$ ” by “ $x \in F' \setminus (F \cup U)$ ” (for the sake of disambiguation of the ambiguous expression  $F' \setminus F \cup U$ ).
- **§10.2, definition of “basis”:** On page 178, replace “any in  $\mathcal{B}$  is called a *basis*” by “any set in  $\mathcal{B}$  is called a *basis*”.

- **§10.2, example for contraction:** On page 179, you use the notion of the “cycle matroid”, but you only define these words in §10.3 (although the cycle matroid itself, without its name, already appears in §10.1).
- **Exercise 10.4:** Here you use the notions of graphic and cographic matroids; these notions are not defined until §10.3.
- **Exercise 10.6:** I guess you can just as well add the following exercise somewhere here:  
If  $(X, \mathcal{I})$  and  $(Y, \mathcal{J})$  are two matroids such that  $X \cap Y = \emptyset$ , then  $(X \cup Y, \{U \cup V \mid U \in \mathcal{I} \text{ and } V \in \mathcal{J}\})$  is also a matroid.  
(This, of course, is a particular case of Exercise 10.27 (iii).)
- **Exercise 10.7:** You have never defined what a “loop” is. (A *loop* of a matroid  $(X, \mathcal{I})$  is an element  $x \in X$  such that  $\{x\} \notin \mathcal{I}$ .)
- **Exercise 10.7:** In part (iii) of this exercise, replace “ $Y \subseteq X$ ” by “ $Y \subseteq X \setminus \{x\}$ ”.
- **Exercise 10.8:** This exercise has no (i) part but two (ii) parts. (I assume the first (ii) part should be (i).)
- **Exercise 10.7-10.8:** I am somewhat surprised about the order in which you give these exercises. In my opinion, it is easiest to first solve Exercise 10.8 (ii) (by repeated application of Theorem 10.3), then conclude Theorem 10.8 (i) (since any subsets  $B, Y$  and  $U$  of  $X$  such that  $B \subseteq Y \subseteq X$  and  $U \subseteq X \setminus Y$  satisfy
 
$$r_M(U \cup Y) - r_M(Y) \leq r_M(U \cup B) - r_M(B)$$
 (by (10), applied to  $U \cup B$  and  $Y$  instead of  $Y$  and  $Z$ )), and then derive Exercise 10.7.
- **Exercise 10.9:** I would add to this exercise the condition that “ $Y \cap Z = \emptyset$ ” (otherwise, the statement of the exercise doesn’t make much sense).
- **§10.3, I:** I would add somewhere in the definition of graphic matroids (or in Section 10.1) that the graph  $G$  is allowed to have double edges (otherwise, Exercise 10.11 would be wrong) and loops (else, Exercise 10.18 would sound strange :)).
- **§10.3, II:** Replace the period at the end of (19) by a comma.
- **§10.3, II:** Replace “where, for each graph  $H$ , let  $\kappa(H)$ ” by “where, for each graph  $H$ , we let  $\kappa(H)$ ”.
- **§10.3, II:** Replace “By definition, a subset  $C$  of  $E$  is a circuit of  $M^*(G)$  if” by “By definition, a subset  $C$  of  $E$  is a circuit of  $M^*(G)$  if and only if”.
- **§10.3, III:** On page 181, you write: “Note that the rank  $r_M(Y)$  of any subset  $Y$  of  $X$  is equal to the rank of the matrix formed by the columns indexed by  $Y$ .” Either replace  $M$  by  $(X, \mathcal{I})$  here, or define  $M$  to be  $(X, \mathcal{I})$ .

- **§10.3, IV:** On page 181, replace “A set  $Y = \{y_1, \dots, y_n\}$  is called a *partial transversal*” by “A set  $Y = \{y_1, \dots, y_n\}$  (with  $y_1, \dots, y_n$  pairwise distinct) is called a *partial transversal*”.
- **Proof of Theorem 10.5:** On page 182, replace “Let  $N$  and  $N'$  denote the edges” by “Let  $N$  and  $N'$  denote the sets of edges”.
- **Exercise 10.12:** I think “ $M = (V, \mathcal{I})$ ” should be “ $M = (X, \mathcal{I})$ ” here.
- **Exercise 10.13:** Replace “ $F$ ” by “ $E$ ” (or by “ $(V, E)$ ”) here.
- **Proof of Lemma 10.1:** You write:

“By König’s matching theorem there exist a subset  $S$  of  $Y \setminus Z$  and a subset  $S'$  of  $Z \setminus Y$  such that for each edge  $\{y, z\}$  of  $H(M, Y)$  satisfying  $z \in S'$  one has  $y \in S$  and such that  $|S| < |S'|$ .”

There are a few things that I’d like to change about this sentence.

First, “for each edge  $\{y, z\}$  of  $H(M, Y)$ ” should be “for each edge  $\{y, z\}$  of  $\tilde{H}$ ”, where  $\tilde{H}$  should be defined (probably before this sentence) as the induced subgraph of  $H(M, Y)$  on the vertex set  $Y \Delta Z$ .

Second, while your claim indeed can be derived from König’s matching theorem, it is easier to conclude it from Hall’s marriage theorem. (Hall’s marriage theorem and König’s matching theorem are known to be equivalent, but your claim is closer to Hall’s than to König’s.)

Third, it would probably better to explain which bipartite graph you are applying König’s matching theorem (or Hall’s marriage theorem) to. You are applying it to the bipartite graph  $\tilde{H}$  that I defined above.

- **Proof of Lemma 10.1:** Replace “for some  $x \notin S$ ” by “for some  $x \in (Y \setminus Z) \setminus S$ ” (because just having  $x \notin S$  is not enough for what you want to do). Maybe it would also be useful to explain why  $U = (Y \setminus \{x\}) \cup \{z\}$  for some  $x \in (Y \setminus Z) \setminus S$ .

11

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<sup>11</sup>The explanation isn’t very long:

$$\begin{aligned}
 \text{We have } T &= \underbrace{(Y \cap Z)}_{\subseteq Y} \cup \underbrace{S}_{\subseteq Y \setminus Z \subseteq Y} \cup \{z\} \subseteq \underbrace{Y \cup Y}_{=Y} \cup \{z\} = Y \cup \{z\} \text{ and therefore } U \subseteq \underbrace{T}_{\subseteq Y \cup \{z\}} \cup Y \subseteq \\
 Y \cup \{z\} \cup Y &= Y \cup \{z\}. \text{ Since } |U| = |Y|, \text{ this yields that } U = (Y \cup \{z\}) \setminus \{t\} \text{ for some } t \in Y \cup \{z\}. \\
 \text{We will now show that } t &\in (Y \setminus Z) \setminus S \text{ and that } U = (Y \setminus \{t\}) \cup \{z\}. \\
 \text{Since } t \notin (Y \cup \{z\}) \setminus \{t\} &= U \text{ and } T \subseteq U, \text{ we have } t \notin T. \text{ Combined with } t \in Y \cup \{z\}, \text{ this yields} \\
 t \in (Y \cup \{z\}) \setminus T &= (Y \setminus T) \cup \underbrace{(\{z\} \setminus T)}_{=\emptyset} = Y \setminus \underbrace{T}_{\substack{=(Y \cap Z) \cup S \cup \{z\} \\ \supseteq (Y \cap Z) \cup S}} \subseteq Y \setminus ((Y \cap Z) \cup S) \\
 &= \underbrace{(Y \setminus (Y \cap Z))}_{=Y \setminus Z} \setminus S = (Y \setminus Z) \setminus S.
 \end{aligned}$$

Also, since  $t \notin T$  but  $z \in \{z\} \subseteq (Y \cap Z) \cup S \cup \{z\} = T$ , we have  $t \neq z$ , so that  $(Y \setminus \{t\}) \cup \{z\} = (Y \cup \{z\}) \setminus \{t\} = U$ .

Thus, we have proven  $t \in (Y \setminus Z) \setminus S$  and that  $U = (Y \setminus \{t\}) \cup \{z\}$ . Hence,  $U = (Y \setminus \{x\}) \cup \{z\}$  for some  $x \in (Y \setminus Z) \setminus S$  (namely, for  $x = t$ ), qed.

I guess you can simplify this proof further.

- **Proof of Lemma 10.2:** You write:

“By the unicity of  $N$  there exists an edge  $\{y, z\} \in N$ , with  $y \in Y \setminus Z$  and  $z \in Z \setminus Y$ , with the property that there is no  $z' \in Z \setminus Y$  such that  $z' \neq z$  and  $\{y, z'\}$  is an edge of  $H(M, Y)$ .”

How do you prove this?<sup>12</sup>

- **Proof of Lemma 10.2:** You write:

“There exists an  $S \in \mathcal{I}$  such that  $Z' \setminus \{y\} \subseteq S \subseteq (Y \setminus \{y\}) \cup \{z\}$  and  $|S| = |Y|$  (since  $(Y \setminus \{y\}) \cup Z = (Y \setminus \{y\}) \cup \{z\} \cup Z'$  and since  $(Y \setminus \{y\}) \cup \{z\}$  belongs to  $\mathcal{I}$ ).

There seems to be a typo in this sentence (the equality  $(Y \setminus \{y\}) \cup Z = (Y \setminus \{y\}) \cup \{z\} \cup Z'$  does not hold, since  $y$  belongs to the right hand side but not to the left hand side), but more importantly I find this sentence too compressed.<sup>13</sup>

<sup>12</sup>The only proof I see requires a lemma from graph theory:

**Lemma 10.2a.** Let  $G$  be a bipartite graph with the two parts  $A$  and  $B$  (this means that the sets  $A$  and  $B$  are disjoint, their union is the set of all vertices of  $G$ , and every edge of  $G$  connects a vertex in  $A$  with a vertex in  $B$ ). Assume that  $G$  has a perfect matching from  $A$  to  $B$ . Then, there exists a vertex  $v \in A$  such that for every edge  $e$  of  $G$  satisfying  $v \in e$ , there exists a perfect matching of  $G$  which contains the edge  $e$ .

Lemma 10.2a seems to be folklore, although I can't find a good reference for it. Two proofs of Lemma 10.2a have been given at

<http://www.artofproblemsolving.com/Forum/viewtopic.php?f=42&t=289103> .

We will need the following corollary of Lemma 10.2a:

**Corollary 10.2b.** Let  $G$  be a bipartite graph with the two parts  $A$  and  $B$  (this means that the sets  $A$  and  $B$  are disjoint, their union is the set of all vertices of  $G$ , and every edge of  $G$  connects a vertex in  $A$  with a vertex in  $B$ ). Assume that  $G$  has a unique perfect matching from  $A$  to  $B$ . Then, there exists a vertex  $v \in A$  which has exactly one neighbor in  $G$ .

Corollary 10.2b follows very quickly from Lemma 10.2a.

Now, let us prove that (in the proof of Lemma 10.2) there exists an edge  $\{y, z\} \in N$ , with  $y \in Y \setminus Z$  and  $z \in Z \setminus Y$ , with the property that there is no  $z' \in Z \setminus Y$  such that  $z' \neq z$  and  $\{y, z'\}$  is an edge of  $H(M, Y)$ . In fact, let  $\tilde{H}$  be the induced subgraph of  $H(M, Y)$  on the vertex set  $Y \triangle Z$ . Then (by the conditions of Lemma 10.2) the bipartite graph  $\tilde{H}$  has a unique perfect matching. Hence, Corollary 10.2b (applied to  $\tilde{H}$ ,  $Y \setminus Z$  and  $Z \setminus Y$ ) yields that there exists a vertex  $v \in Y \setminus Z$  which has exactly one neighbor in  $\tilde{H}$ . Let  $y$  be this  $v$ . Then,  $y$  has exactly one neighbor in  $\tilde{H}$ . Since  $N$  is a perfect matching of  $\tilde{H}$  (by the conditions of Lemma 10.2), there exists an edge  $\{y, z\} \in N$  with  $z \in Z \setminus Y$ . If there would be a  $z' \in Z \setminus Y$  such that  $z' \neq z$  and  $\{y, z'\}$  is an edge of  $H(M, Y)$ , then  $y$  would have at least two neighbors in  $\tilde{H}$  (namely,  $z$  and  $z'$ ), which would contradict the fact that  $y$  has exactly one neighbor in  $\tilde{H}$ . Hence, there is no  $z' \in Z \setminus Y$  such that  $z' \neq z$  and  $\{y, z'\}$  is an edge of  $H(M, Y)$ .

We thus have proven that there exists an edge  $\{y, z\} \in N$ , with  $y \in Y \setminus Z$  and  $z \in Z \setminus Y$ , with the property that there is no  $z' \in Z \setminus Y$  such that  $z' \neq z$  and  $\{y, z'\}$  is an edge of  $H(M, Y)$ .

<sup>13</sup>I would replace this sentence by the following paragraph (which is probably too detailed):

Since  $\{y, z\}$  is an edge of  $H(M, Y)$ , we have  $(Y \setminus \{y\}) \cup \{z\} \in \mathcal{I}$ . Combined with  $Z' \setminus \{y\} \in \mathcal{I}$  (since  $Z' \in \mathcal{I}$ ), this yields that there exists an  $S \in \mathcal{I}$  such that  $Z' \setminus \{y\} \subseteq S \subseteq (Z' \setminus \{y\}) \cup ((Y \setminus \{y\}) \cup \{z\})$  and  $|S| = |(Y \setminus \{y\}) \cup \{z\}|$  (by the axioms of a matroid). Consider this  $S$ . We have

$$\begin{aligned} S &\subseteq \left( \underbrace{Z'}_{=(Z \setminus \{z\}) \cup \{y\}} \setminus \{y\} \right) \cup ((Y \setminus \{y\}) \cup \{z\}) = \underbrace{(((Z \setminus \{z\}) \cup \{y\}) \setminus \{y\})}_{\subseteq Z \setminus \{z\} \subseteq Z} \cup ((Y \setminus \{y\}) \cup \{z\}) \\ &\subseteq Z \cup (Y \setminus \{y\}) \cup \{z\} = (Y \setminus \{y\}) \cup \underbrace{Z \cup \{z\}}_{=Z} = (Y \setminus \{y\}) \cup Z \\ &\hspace{10em} \text{(since } z \in Z) \end{aligned}$$

- **Proof of Lemma 10.2:** You write: “Assuming  $Z \notin \mathcal{I}$ , we know  $z \notin S$ ”. Why we have  $z \notin S$  took me quite a while to realize; I wish you would spend some more words explaining this.<sup>14</sup>

- **Proof of Lemma 10.2:** You write: “and hence  $r((Y \cup Z') \setminus \{y\}) = |Y|$ ”. Again, why is this true?

This time, I actually don’t think it is true (although I don’t have a counterexample). What is definitely true is that  $r((Y \cup Z') \setminus \{y\}) \geq |Y|$  (and this is all we need later). Actually, I wouldn’t even speak about  $r((Y \cup Z') \setminus \{y\})$ , but instead just say that  $S \subseteq (Y \cup Z') \setminus \{y\}$ <sup>15</sup>. And this is all that we need to prove that there exists an  $z' \in Z' \setminus Y$  such that  $(Y \setminus \{y\}) \cup \{z'\}$  belongs to  $\mathcal{I}$ .<sup>16</sup>

- **Exercise 10.20:** In part (i), replace “to  $X' := (Z \setminus \delta(y)) \cup \{y\}$  and  $X'' := (Z \setminus \delta(y)) \cup (Y \setminus \{y\})$ ” by “to the sets  $X' := (Z \setminus \delta(y)) \cup \{y\}$  and  $X'' := (Z \setminus \delta(y)) \cup (Y \setminus \{y\})$  in lieu of  $Y$  and  $Z$ ”.

- **Example 10.5c:** It might be useful to say here that the “underlying undirected graph” is not to be understood as a graph in the usual sense (e. g., as a pair  $(V, E)$  with  $E$  being a collection of 2-element sets of  $V$ ), but it must be implemented as an “edge-aware” graph.

Here, by an *edge-aware* graph, I mean a triple  $(V, E, \text{end})$  with  $V$  and  $E$  being two sets, and  $\text{end} : E \rightarrow \mathcal{P}_2(V)$  being a map. (If we allow loops, then  $\text{end}$  should be a map  $E \rightarrow \mathcal{P}_1(V) \cup \mathcal{P}_2(V)$  instead of a map  $E \rightarrow \mathcal{P}_2(V)$ .) The elements of  $V$  are called the *vertices* of the edge-aware graph  $(V, E, \text{end})$ . The elements of  $E$  are called the *edges* of the edge-aware graph  $(V, E, \text{end})$ . If  $e$  is an edge of an

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and  $|S| = |(Y \setminus \{y\}) \cup \{z\}| = (|Y| - 1) + 1 = |Y|$ .

<sup>14</sup>For what it’s worth, here is my argument:

We have  $Z \neq S$  (since  $Z \notin \mathcal{I}$  but  $S \in \mathcal{I}$ ). Assume that  $z \in S$ . Then,  $\{z\} \subseteq S$ . On the other hand, since  $Z' = (Z \setminus \{z\}) \cup \{y\}$ , we have  $Z' \setminus \{y\} = ((Z \setminus \{z\}) \cup \{y\}) \setminus \{y\} = Z \setminus \{z\}$  (since  $y \notin Z \setminus \{z\}$ ), thus  $(Z' \setminus \{y\}) \cup \{z\} = (Z \setminus \{z\}) \cup \{z\} = Z$  (since  $z \in Z$ ). Hence,  $Z = (Z' \setminus \{y\}) \cup \{z\} \subseteq S$  (since  $Z' \setminus \{y\} \subseteq S$  and  $\{z\} \subseteq S$ ). Combined with  $|Z| = |Y| = |S|$ , this yields  $Z = S$ , contradicting  $Z \neq S$ . Hence, our assumption (that  $z \in S$ ) was wrong. Thus,  $z \notin S$ .

<sup>15</sup>because combining  $S \subseteq (Y \setminus \{y\}) \cup Z$  and  $z \notin S$ , we get

$$\begin{aligned} S &\subseteq ((Y \setminus \{y\}) \cup Z) \setminus \{z\} = \underbrace{((Y \setminus \{y\}) \setminus \{z\})}_{\subseteq Y \setminus \{y\}} \cup (Z \setminus \{z\}) \subseteq (Y \setminus \{y\}) \cup \underbrace{(Z \setminus \{z\})}_{\substack{= ((Z \setminus \{z\}) \cup \{y\}) \setminus \{y\} \\ \text{(since } y \notin Z, \text{ thus } y \notin Z \setminus \{z\})}} \\ &= (Y \setminus \{y\}) \cup \left( \underbrace{((Z \setminus \{z\}) \cup \{y\}) \setminus \{y\}}_{= Z'} \right) = (Y \setminus \{y\}) \cup (Z' \setminus \{y\}) = (Y \cup Z') \setminus \{y\} \end{aligned}$$

<sup>16</sup>In fact, the existence of this  $z'$  can now be shown as follows: We have  $Y \setminus \{y\} \in \mathcal{I}$  (since  $Y \in \mathcal{I}$ ) and  $S \in \mathcal{I}$  and  $|Y \setminus \{y\}| = |Y| - 1 < |Y| = |S|$ . Thus, there must exist a  $w \in S \setminus (Y \setminus \{y\})$  such that  $(Y \setminus \{y\}) \cup \{w\}$  belongs to  $\mathcal{I}$  (by the axioms of a matroid). Consider this  $w$ . We have

$$\begin{aligned} w \in \underbrace{S}_{\subseteq (Y \cup Z') \setminus \{y\}} \setminus (Y \setminus \{y\}) &\subseteq ((Y \cup Z') \setminus \{y\}) \setminus (Y \setminus \{y\}) \\ &\subseteq (Y \cup Z') \setminus Y = Z' \setminus Y. \end{aligned}$$

Thus, there exists an  $z' \in Z' \setminus Y$  such that  $(Y \setminus \{y\}) \cup \{z'\}$  belongs to  $\mathcal{I}$  (namely,  $z' = w$ ).

edge-aware graph  $(V, E, \text{end})$ , then the elements of  $\text{end } e$  are called the *endpoints* of  $e$  and are said to *lie* on  $e$  (or *belong* to  $e$ ).

Given an edge-aware graph  $(V, E, \text{end})$ , we can canonically obtain a graph in the usual sense (namely,  $(V, \text{end}(E))$ ), but by doing this, we are losing some information (namely, we are losing the “labeling” of the edges and their multiplicities). Even the multigraph  $(V, (\text{multiset image of } E \text{ under } \text{end}))$  contains less information than the edge-aware graph  $(V, E, \text{end})$  since equal elements in a multiset are indistinguishable.

So, when  $D = (V, A)$  is a directed graph, I define the *underlying undirected graph* of  $D$  as the edge-aware graph  $(V, A, \text{end})$ , where the map  $\text{end} : A \rightarrow V$  is defined by

$$(\text{end}(a) = \{\text{the source and the sink of } a\} \quad \text{for every } a \in A).$$

This makes your example correct. If we would define the underlying undirected graph of  $D$  as a graph, or even a multigraph, in the usual sense, then its set (or multiset) of edges would not in general be  $A$  (at best, it would be a multiset with the same cardinality as  $A$ , but we cannot define a matroid on a multiset).

- **§10.5, directly after Example 10.5d:** You write: “In this section we describe an algorithm for finding a maximum-cardinality common independent sets in two given matroids.” Replace “sets” by “set” here.
- **§10.5, Case 2 in the description of the algorithm:** Replace “vertex vertex” by “vertex”.
- **Proof of Theorem 10.6:** You write: “and that, as  $(Y' \setminus \{y_0\}) \cap X_1 = \emptyset$ ,  $r_{M_1}((Y \cup Y') \setminus \{y_0\}) = |Y|$ ”.

I think neither the claim that  $(Y' \setminus \{y_0\}) \cap X_1 = \emptyset$  nor the claim that  $r_{M_1}((Y \cup Y') \setminus \{y_0\}) = |Y|$  is very obvious.<sup>17</sup>

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<sup>17</sup>Let me give my proofs for these claims, though I can bet you have shorter ones:

*Proof of  $(Y' \setminus \{y_0\}) \cap X_1 = \emptyset$ :* Let  $\xi \in (Y' \setminus \{y_0\}) \cap X_1$  be arbitrary. Then,  $\xi \in Y' \setminus \{y_0\}$  and  $\xi \in X_1$ . Clearly,

$$\begin{aligned} \xi \in Y' \setminus \{y_0\} &\subseteq (Y \cup \{y_0, y_1, \dots, y_m\}) \setminus \{y_0\} \\ &\left( \text{since (25) yields } Y' = \underbrace{(Y \setminus \{z_1, z_2, \dots, z_m\})}_{\subseteq Y} \cup \{y_0, y_1, \dots, y_m\} \subseteq Y \cup \{y_0, y_1, \dots, y_m\} \right) \\ &= \underbrace{(Y \setminus \{y_0\})}_{\subseteq Y} \cup \underbrace{(\{y_0, y_1, \dots, y_m\} \setminus \{y_0\})}_{=\{y_1, y_2, \dots, y_m\}} \\ &\subseteq Y \cup \{y_1, y_2, \dots, y_m\}. \end{aligned}$$

But  $\xi \in X_1 \subseteq X \setminus Y$ , so that  $\xi \notin Y$ . Combining  $\xi \in Y \cup \{y_1, y_2, \dots, y_m\}$  with  $\xi \notin Y$ , we obtain

$$\xi \in (Y \cup \{y_1, y_2, \dots, y_m\}) \setminus Y \subseteq \{y_1, y_2, \dots, y_m\}.$$

Thus, there exists a  $j \in \{1, 2, \dots, m\}$  such that  $\xi = y_j$ . Consider this  $j$ . Then,  $y_j = \xi \in X_1$ . Hence, by taking the segment of the path  $P$  beginning with the vertex  $y_j$  and ending with the vertex  $y_m$  (the last vertex of  $P$ ), we get a path from some vertex in  $X_1$  to some vertex in  $X_2$  (since  $y_m \in X_2$ ). But

- **Proof of Theorem 10.7:** You write:

“In the second alternative,  $(y, x)$  is an arc of  $H(M_1, M_2, Y)$  entering  $U$ . This contradicts the definition of  $U$  (as  $y \notin U$  and  $x \in U$ ).”

I think this should rather be:

“In the second alternative,  $(y, x)$  is an arc of  $H(M_1, M_2, Y)$  entering  $U$  (as  $y \notin U$  and  $x \in U$ ). This contradicts the definition of  $U$ .”

- **Proof of Theorem 10.7:** You write:

“Similarly we have that  $r_{M_2}(X \setminus U) = |Y \setminus U|$ .”

I wish to see some more details on what the “Similarly” here means.<sup>18</sup>

- **Between the proof of Theorem 10.7 and Theorem 10.8:** You write: “The algorithm clearly has polynomially bounded running time, since we can construct the auxiliary directed graph  $H(M_1, M_2, Y)$  and find the path  $P$  (if it exists), in polynomial time.” The second comma in this sentence makes no sense.
- **Proof of Theorem 10.9:** Maybe worth noticing: When you say “the algorithm” in this proof, you don’t literally mean the cardinality common independent set augmenting algorithm, but you mean the algorithm “set  $Y := \emptyset$ , and apply repeatedly the cardinality common independent set augmenting algorithm, replacing  $Y$  by  $Y'$  after each finished iteration, until the algorithm no longer outputs a  $Y'$ ”.
- **Exercise 10.23:** For the sake of precision, it would be better to replace “be subsets” by “be sequences of subsets”.

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this new path is shorter than  $P$ . This contradicts the fact that  $P$  was chosen to be a shortest path from some vertex in  $X_1$  to some vertex in  $X_2$ .

Now, forget that we fixed  $\xi$ . We thus have proven that every  $\xi \in (Y' \setminus \{y_0\}) \cap X_1$  satisfies a contradiction. In other words, there exists no  $\xi \in (Y' \setminus \{y_0\}) \cap X_1$ . Thus,  $(Y' \setminus \{y_0\}) \cap X_1 = \emptyset$ , qed.

*Proof of  $r_{M_1}((Y \cup Y') \setminus \{y_0\}) = |Y|$ :* We have  $y_0 \notin Y$  (since  $y_0 \in X \setminus Y$ ). Since  $Y \in \mathcal{I}_1$  and  $Y \subseteq (Y \cup Y') \setminus \{y_0\}$  (because  $Y \subseteq Y \cup Y'$  and  $y_0 \notin Y$ ), we have  $r_{M_1}((Y \cup Y') \setminus \{y_0\}) \geq |Y|$ .

Assume now that  $r_{M_1}((Y \cup Y') \setminus \{y_0\}) > |Y|$ . Since  $Y \in \mathcal{I}_1$  and  $Y \subseteq (Y \cup Y') \setminus \{y_0\}$ , this yields that there exists some  $\eta \in ((Y \cup Y') \setminus \{y_0\}) \setminus Y$  such that  $Y \cup \{\eta\} \in \mathcal{I}_1$ . Consider this  $\eta$ .

We have

$$\eta \in ((Y \cup Y') \setminus \{y_0\}) \setminus Y = \underbrace{((Y \cup Y') \setminus Y)}_{\subseteq Y'} \setminus \{y_0\} \subseteq Y' \setminus \{y_0\}.$$

On the other hand,  $\eta \in \underbrace{((Y \cup Y') \setminus \{y_0\})}_{\subseteq X} \setminus Y \subseteq X \setminus Y$  and  $Y \cup \{\eta\} \in \mathcal{I}_1$ , so that  $\eta \in X_1$  (by the

definition of  $X_1$ ). Combining  $\eta \in Y' \setminus \{y_0\}$  with  $\eta \in X_1$ , we obtain  $\eta \in (Y' \setminus \{y_0\}) \cap X_1 = \emptyset$ , which is absurd. Thus, we have obtained a contradiction. This contradiction shows that our assumption (that  $r_{M_1}((Y \cup Y') \setminus \{y_0\}) > |Y|$ ) was wrong. Hence,  $r_{M_1}((Y \cup Y') \setminus \{y_0\}) \leq |Y|$ . Combined with  $r_{M_1}((Y \cup Y') \setminus \{y_0\}) \geq |Y|$ , this yields  $r_{M_1}((Y \cup Y') \setminus \{y_0\}) = |Y|$ , qed.

<sup>18</sup>Here is what I think it means: We know that  $U \supseteq X_2$  and  $U \cap X_1 = \emptyset$ . In other words,  $X \setminus U \supseteq X_1$  and  $(X \setminus U) \cap X_2 = \emptyset$ . Hence, the same argument that we used to prove (27) can be modified to become a proof of  $r_{M_2}(X \setminus U) = |Y \cap (X \setminus U)|$ . (The modifications necessary to obtain a proof  $r_{M_2}(X \setminus U) = |Y \cap (X \setminus U)|$  are very simple: Replace every appearance of  $U$  by  $X \setminus U$ ; replace every appearance of  $X_1$  by  $X_2$ ; replace every appearance of  $M_1$  by  $M_2$ ; replace every appearance of  $\mathcal{I}_1$  by  $\mathcal{I}_2$ ; replace every appearance of  $(y, x)$  by  $(x, y)$ .)

Thus, we have shown that  $r_{M_2}(X \setminus U) = |Y \cap (X \setminus U)|$ . Since  $Y \cap (X \setminus U) = Y \setminus U$ , this rewrites as  $r_{M_2}(X \setminus U) = |Y \setminus U|$ .

- **Exercise 10.31:** In part (i), replace “ $k(r_M(X) - r_M(U)) \geq |X \setminus U|$ ” by “ $k(r_M(X) - r_M(U)) \leq |X \setminus U|$ ”.
- **Exercise 10.32:** Replace “into  $t$  classes” by “into  $t$  nonempty classes”.
- **Exercise 10.34:** Replace “ $Y_1 \cup Z_1 \cup Z_2$ ” by “ $Y_1 \cup Z_2$ ”. Also, why do you require  $B$  and  $B'$  to be disjoint? The problem would still be true if you didn't (and the solution wouldn't be much harder: one can WLOG assume that  $B_1 \cap B_2 = \emptyset$  by means of contracting  $B_1 \cap B_2$ ).
- **Exercise 10.35:** First, I would prefer to see a definition of what  $b(U)$  means in this exercise. (Namely,  $b(U) := \sum_{u \in U} b(u)$  for every  $U \subseteq V$ .)

Second, I think this exercise needs the additional requirement that  $b(v) > 0$  for every  $v \in V$ . (From a look at Exercise 4.5 I am pretty sure that you define  $\mathbb{Z}_+$  as  $\{0, 1, 2, \dots\}$  rather than as  $\{1, 2, 3, \dots\}$ , so this requirement is not a tautology. By the way, what do you think about explicitly defining your  $\mathbb{Z}_+$  at the beginning of the notes?) I think a counterexample to the claim of this exercise without the requirement that  $b(v) > 0$  for every  $v \in V$  would be the following pair  $(G, b)$ : The graph  $G$  consists of three vertices  $x, y$  and  $z$  with edges  $xz$  and  $yz$ , and the function  $b$  is given by  $b(x) = b(y) = 1$  and  $b(z) = 0$ .

- **§10.6:** On the third line from below on page 190, replace “if matroid” by “if matroids”.
- **§10.6, description of the weighted common independent set augmenting algorithm:** Two lines below (36), remove “counting multiplicities”, since a path does not traverse any vertex twice. Also, add the following definition:

“The *length* of a circuit  $C$ , denoted by  $l(C)$ , is defined as  $\sum_{c \in VC} \#_C(c)$ , where  $\#_C(c)$  denotes the number of edges belonging to the circuit  $C$  which are outgoing from  $c$ .”

Note that I wouldn't try to reformulate the definition of the length of a path to make it also cover the case of a circuit, because it is not clear what the multiplicities are for a circuit.

- **Theorem 10.10:** Replace “circuit” by “cycle”.

(I am assuming that the difference between a circuit and a cycle is that a circuit may contain repeated vertices (except from its first and last vertex), whereas a cycle cannot. In this case, I don't know whether Theorem 10.10 holds for arbitrary circuits, but at least I am fairly sure that the proof you are giving does not.

Actually, I am no longer sure that the meanings of the words “circuit” and “cycle” are the ones I believe. For example, Exercise 4.15 probably uses the word “circuit” in the meaning of “cycle with no repeated vertices”. Maybe it would be better to disambiguate these two terms and define them somewhere in Chapter 1.)

- **Proof of Theorem 10.10:** On page 191, replace “by Lemma 10.2 a matching” by “by Lemma 10.2 a perfect matching”.
- **Proof of Theorem 10.10:** On page 191, replace “a directed circuit  $C_1$ ” by “a directed cycle  $C_1$ ”, and replace “into directed circuits” by “into directed cycles”. [It might generally be good to handle the distinction between circuits and cycles with the same care that you take for paths vs. walks.]
- **Proof of Theorem 10.10:** The ending of this proof is very hard to understand and possibly wrong. First, it has typos (one should replace every “ $l(VC_j)$ ” by “ $l(C_j)$ ”, replace every “ $l(VC)$ ” by “ $l(C)$ ”, and replace every “proposition” by “theorem”). Second, I think there is too much going on at the same time in this argument, and the claim that “ $l(VC_j) \leq l(VC)$  for all  $j \leq k$ ” at the end might not be true (I have not looked for a counterexample, but I don’t see a reason why it should hold).<sup>19</sup>

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<sup>19</sup>I would replace the part of the proof that begins with “If, say  $VC_k = VC$ ” and ends with the end of the proof by the following (again, my hope is that you can do much better):

“From (37), we see that  $l(C_1) + l(C_2) + \dots + l(C_k) = 2l(C)$ . If some  $j \in \{1, 2, \dots, k\}$  such that  $VC_j \neq VC$  satisfies  $l(C_j) < 0$ , then Theorem 10.10 is proven (by taking  $C' = C_j$ ). Hence, for the rest of this proof, we can assume WLOG that no such  $j$  exists. Assume this. Thus, every  $j \in \{1, 2, \dots, k\}$  such that  $VC_j \neq VC$  satisfies  $l(C_j) \geq 0$ .

Note that every of the cycles  $C_1, C_2, \dots, C_k$  can be regarded as a cycle in the directed graph  $H(M_1, M_2, Y)$ , because it uses each of the doubled arcs in  $N_2$  at most once. (But the resulting  $k$  cycles in  $H(M_1, M_2, Y)$  won’t be arc-disjoint.)

We know that  $t$  (like every vertex in  $VC$ ) is left by exactly two arcs of  $D$ . Let  $e$  and  $f$  be these two arcs. Since the cycles  $C_1, C_2, \dots, C_k$  form a decomposition of  $A$ , each of the two arcs  $e$  and  $f$  belongs to one of the cycles  $C_1, C_2, \dots, C_k$ . That is, there exist  $u \in \{1, 2, \dots, k\}$  and  $v \in \{1, 2, \dots, k\}$  such that  $e \in C_u$  and  $f \in C_v$ . Consider these  $u$  and  $v$ . Since  $e$  and  $f$  are two distinct arcs of  $D$  leaving the vertex  $t$ , we have  $u \neq v$  (because no cycle may contain two distinct arcs leaving the same vertex). Since  $e \in C_u$  and  $t \in e$ , we have  $t \in VC_u$ .

Recall that  $VC_j = VC$  for at most one  $j$ . Hence, we must be in one of the following two cases:

*Case A:* There exists exactly one  $j$  satisfying  $VC_j = VC$ .

*Case B:* There exists no  $j$  satisfying  $VC_j = VC$ .

Let us consider Case A first. In this case, there exists exactly one  $j$  satisfying  $VC_j = VC$ . Hence, we can WLOG assume that  $VC_k = VC$ , whereas every  $j \neq k$  satisfies  $VC_j \neq VC$ . Thus, every  $j \neq k$  satisfies  $l(C_j) \geq 0$  (since we know that every  $j \in \{1, 2, \dots, k\}$  such that  $VC_j \neq VC$  satisfies  $l(C_j) \geq 0$ ). Consequently, every  $j \neq k$  satisfies

$$\begin{aligned} l(C_j) &\leq l(C_1) + l(C_2) + \dots + l(C_{k-1}) = \underbrace{(l(C_1) + l(C_2) + \dots + l(C_k))}_{=2l(C)} - \underbrace{l(C_k)}_{=l(C)} \\ &= 2l(C) - l(C) = l(C). \end{aligned}$$

(since  $VC_k = VC$  and since  $C_k$  and  $C$  are cycles)

But since  $u \neq v$ , it is not possible that both  $u$  and  $v$  equal  $k$ . Hence, at least one of the numbers  $u$  and  $v$  does not equal  $k$ . Let us WLOG assume that  $u \neq k$ . Thus,  $VC_u \neq VC$  (since every  $j \neq k$  satisfies  $VC_j \neq VC$ ) and  $l(C_u) \leq l(C)$  (since every  $j \neq k$  satisfies  $l(C_j) \leq l(C)$ ) and also  $t \in VC_u$  (as we know). Thus, Theorem 10.10 is proven (by taking  $C' = C_u$ ) in Case A.

Now, let us consider Case B. In this case, there exists no  $j$  satisfying  $VC_j = VC$ . Hence, every  $j \in \{1, 2, \dots, k\}$  satisfies  $VC_j \neq VC$ . Thus, every  $j \in \{1, 2, \dots, k\}$  satisfies  $l(C_j) \geq 0$  (since we know that every  $j \in \{1, 2, \dots, k\}$  such that  $VC_j \neq VC$  satisfies  $l(C_j) \geq 0$ ). As a consequence, using  $u = v$  we obtain

$$l(C_u) + l(C_v) \leq l(C_1) + l(C_2) + \dots + l(C_k) = 2l(C).$$

Hence, at least one of the two inequalities  $l(C_u) \leq l(C)$  and  $l(C_v) \leq l(C)$  must hold (because

- **Proof of Theorem 10.11:** I think that for the sake of clarity, it would be better to replace “To see necessity, suppose  $H(M_1, M_2, Y)$  has a cycle  $C$  of negative length” by “To see necessity, suppose that  $Y$  is extreme but  $H(M_1, M_2, Y)$  has a cycle  $C$  of negative length”. Similarly, I believe that it would be better to replace “To see sufficiency, consider a  $Z \in \mathcal{I}_1 \cap \mathcal{I}_2$  with  $|Z| = |Y|$ ” by “To see sufficiency, assume that  $H(M_1, M_2, Y)$  has no directed cycle of negative length, and consider a  $Z \in \mathcal{I}_1 \cap \mathcal{I}_2$  with  $|Z| = |Y|$ ”. Note that this might be just my personal problem. (When I see words like “necessity” and “sufficiency” in relation to an if-and-only-if assertion, it is not clear to me what they mean: which of the two sides is meant to be necessary/sufficient? But maybe there is a common convention how to understand this. I prefer subdividing proofs of equivalences into “ $\implies$ ” and “ $\impliedby$ ” parts, though.)
- **Proof of Theorem 10.11:** Replace “Proposition 10.10” by “Theorem 10.10”.
- **Between Theorem 10.11 and Theorem 10.12:** You write: “This theorem implies that we can find in the algorithm a shortest path  $P$  in polynomial time (with the Bellman-Ford method).” Here you are using the fact that the Bellman-Ford method (as described in the last paragraph of page 13) computes not just a minimum-length  $s - t$  path, but actually a minimum-length  $s - t$  path which has a minimum number of arcs among all minimum-length  $s - t$  paths. Maybe this claim could be added to §1.3 as an exercise?

Also, there is a further little difference between the Bellman-Ford method presented in Chapter 1 and the algorithm you are using here: In Chapter 1, the Bellman-Ford method was constructed for a directed graph with a length function on arcs, while here you have a directed graph with a length function on vertices. Fortunately, it is easy to reduce the latter situation to the former (by adding a new vertex  $s$  along with an arc from every vertex in  $X_2$  to  $s$ , and then defining the length of any arc to be the length of its source, and searching for a shortest path from  $X_1$  to  $s$  with respect to this lengths of arcs).

- **Proof of Theorem 10.12:** On page 192, replace “adding arcs from each vertex in  $X_1$  to  $t$ , and from  $t$  to each vertex in  $X_2$ ” by “adding arcs from each vertex in  $X_2$  to  $t$ , and from  $t$  to each vertex in  $X_1$ ”.
- **Proof of Theorem 10.12:** On page 193, replace “So  $Z = (Y + t) \triangle VC$ ” by “Set  $Z = (Y + t) \triangle VC$ ”.
- **Proof of Theorem 10.12:** On page 193, replace “Proposition 10.10” by “Theorem 10.10”.

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otherwise we would have  $l(C_u) > l(C)$  and  $l(C_v) > l(C)$ , and therefore  $\underbrace{l(C_u)}_{>l(C)} + \underbrace{l(C_v)}_{>l(C)} > l(C) + l(C) = 2l(C)$ , contradicting  $l(C_u) + l(C_v) \leq 2l(C)$ ). Let us WLOG assume that  $l(C_u) \leq l(C)$  holds. Thus,  $VC_u \neq VC$  (since every  $j \in \{1, 2, \dots, k\}$  satisfies  $VC_j \neq VC$ ) and  $l(C_u) \leq l(C)$  and also  $t \in VC_u$  (proven above). Thus, Theorem 10.10 is proven (by taking  $C' = C_u$ ) in Case B.

Hence, Theorem 10.10 is proven in both cases A and B. Since we know that cases A and B cover all possibilities, this yields that we have proven Theorem 10.10.”

- **Proof of Theorem 10.12:** At the end of this proof, add the sentence: “Since  $Z = Y'$ , this yields that  $Y'$  is extreme.”
- **Proof of Theorem 10.13:** Replace “we have an extreme” by “we have a maximum-weight”.
- **Exercise 10.38:** Replace “rooted tree” by “rooted tree (with vertex set  $V$  and arc set  $\subseteq A$ )”.
- **Proof of Theorem 10.14:** On the second line of this proof, replace “... $w(y_m)$ ” by “...  $\geq w(y_m)$ ”.
- **§10.7:** On page 195, you write: “So system (41) is totally dual integral.” This is the first time the words “totally dual integral” appear in your text. Since you apparently never use total dual integrality, it probably isn’t necessary to define what it means, but it is still weird to suddenly use this notion without ever introducing it.
- **Proof of Theorem 10.15:** Here, you write: “Since  $B$  is a maximum-weight basis,  $H(M_1, M_2, B)$  has no directed circuits of negative length.” Here, it might be useful to add “(by Exercise 10.39)”.
- **Proof of Theorem 10.15:** You write: “Hence there exists a function  $\phi : X \rightarrow \mathbb{Z}$  so that  $\phi(y) - \phi(x) \leq l(y)$  for each arc  $(x, y)$  of  $H(M_1, M_2, B)$ .” It took me a while to understand why this is true; what you are using here is the following (easy but not exactly trivial) fact:  
**Lemma 10.15a.** Let  $G = (V, D)$  be a directed graph. Let  $l : V \rightarrow \mathbb{Z}$  be a function. For every  $x \in V$ , denote  $l(x)$  as the *length of  $x$* . The *length* of a path  $P$  in  $G$ , denoted by  $l(P)$ , will mean the sum of the lengths of the vertices traversed by  $P$ , counting multiplicities. Assume that the graph  $G$  has no directed circuit of negative length. Then, there exists a function  $\phi : V \rightarrow \mathbb{Z}$  such that  $\phi(y) - \phi(x) \leq l(y)$  for each arc  $(x, y)$  of  $G$ .  
 Couldn’t this be a nice exercise for Chapter 1?
- **Proof of Theorem 10.15:** In (47), replace “ $\leq w(x)$ ” by “ $\leq w(y)$ ”.
- **Proof of Theorem 10.16:** Between (49) and (50), you write: “and  $M_2 := (X \cup U, \mathcal{J}_2)$ ”. The “ $M_2$ ” here should be an “ $M'_2$ ”.
- **Proof of Theorem 10.16:** Between (49) and (50), you write: “Define  $\tilde{w} : X \rightarrow \mathbb{Z}$ ”. The “ $X$ ” here should be an “ $X \cup U$ ”.
- **Proof of Theorem 10.16:** You write: “In fact,  $Y \cup W$  is a maximum-weight common basis”. I would replace the words “In fact” by “Moreover” here, since “In fact” sounds as if you are proving the claim in the preceding sentence.
- **Proof of Theorem 10.16:** On page 196, replace “ $\tilde{w}_1, \tilde{w}_2 : X \rightarrow \mathbb{Z}$ ” by “ $\tilde{w}_1, \tilde{w}_2 : X \cup U \rightarrow \mathbb{Z}$ ”.
- **Proof of Theorem 10.16:** On the first line of page 197, replace “subtracting” by “subtract”.

- **§10.7:** Two lines above (51), you write: “By Theorem 10.14 the intersection”. I think you want to cite Corollary 10.14a rather than Theorem 10.14.
- **Proof of Theorem 10.17:** Replace “By Theorem 10.15” by “By Theorem 10.16”.
- **Proof of Theorem 10.17:** The proof of optimality is a bit of a mess (you use some  $Z$  which you haven’t introduced, and it’s not very clear what is being done). Here is how I would correct it:

“Rename  $Y$  as  $Z$  (so that the letter  $Y$  is free again and can be used as a summation index). Denote by  $\tilde{z}$  the vector  $\chi^Z \in \mathbb{R}^X$ . Since  $Z$  is a maximum-weight independent set in  $M_1$  with respect to  $w_1$ , we know that  $\tilde{z} = \chi^Z$  is an optimum solution of the problem (42) with respect to  $M_1, w_1$ . As a consequence,

$$\begin{aligned}
& w_1^T \tilde{z} \\
&= \max \left\{ w_1^T z \mid z(x) \geq 0 \text{ for all } x \in X, \text{ and } z(Y) \leq r_{M_1}(Y) \text{ for all } Y \subseteq X \right\} \\
&= \min \left\{ \sum_{Y \subseteq X} y_Y r_{M_1}(Y) \mid y_Y \geq 0 \text{ for all } Y \subseteq X, \text{ and } \sum_{Y \subseteq X, x \in Y} y_Y \geq w(x) \text{ for all } x \in X \right\} \\
&= \sum_{Y \subseteq X} y'_Y r_{M_1}(Y)
\end{aligned}$$

(since  $y'$  is an optimum solution for problem (43) with respect to  $M_1, w_1$ ). Similarly,  $w_2^T \tilde{z} = \sum_{Y \subseteq X} y''_Y r_{M_2}(Y)$ . Now, since  $w = w_1 + w_2$ , we have

$$\begin{aligned}
w^T \tilde{z} &= \underbrace{w_1^T \tilde{z}}_{= \sum_{Y \subseteq X} y'_Y r_{M_1}(Y)} + \underbrace{w_2^T \tilde{z}}_{= \sum_{Y \subseteq X} y''_Y r_{M_2}(Y)} = \sum_{Y \subseteq X} y'_Y r_{M_1}(Y) + \sum_{Y \subseteq X} y''_Y r_{M_2}(Y) \\
&= \sum_{Y \subseteq X} (y'_Y r_{M_1}(Y) + y''_Y r_{M_2}(Y)).
\end{aligned}$$

Thus,  $\tilde{z}$  is an integer optimum solution to (52), and  $(y'_Y, y''_Y)_{Y \subseteq X}$  is an integer optimum solution to (53), qed.”